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HIGHER MATHEMATICS IN PROBLEMS AND EXERCISES

PART 2

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About the book

In this two-volume textbook the authors tried to reveal the essence of the main notions and theorems encountered in courses of higher mathematics by presenting and solving specially selected problems and exercises.
Part 2 covers the following themes: multiple, line and surface integrals; series; differential equations (ordinary and partial); elements of the probability theory; elements of the theory of functions of a complex variable; elements of operational calculus; calculation methods.
Each section begins with a brief theoretical introduction. Typical problems are followed by detailed solutions.
The book is intended for engineering students.

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Contents

Chapter 1

Double and Triple Integrals

1.1. Double Integral in Rectangular Coordinates

Suppose the function $f(x, y)$ is defined in a bounded closed domain D of the plane xOy. Let us partition the domain D, in an arbitrary way, into n subdomains with areas $\Delta\sigma_1, \Delta\sigma_2, \ldots, \Delta\sigma_n$ and diameters $d_1, d_2, \ldots, d_n$ (the *diameter of a domain* is the greatest distance between two points belonging to the boundary of the domain). Let us choose an arbitrary point $P_k(\xi_k; \eta_k)$ in each subdomain and multiply the value of the function at the point P_k by the area of that subdomain.

The *integral sum* of the function $f(x, y)$ over the domain D is the sum of the form

$$\sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta\sigma_k = f(\xi_1, \eta_1)\Delta\sigma_1 + f(\xi_2, \eta_2)\Delta\sigma_2 + \ldots + f(\xi_n, \eta_n)\Delta\sigma_n.$$

The *Double integral* of the function $f(x, y)$ over the domain D is the limit of the integral sum, provided that the greatest of the diameters of the subdomains tends to zero:

$$\iint_D f(x, y)\,d\sigma = \lim_{\max d_k \to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta\sigma_k.$$

If the function $f(x, y)$ is continuous in the closed domain D, then the limit of the integral sum exists and does not depend on the manner of partitioning the domain D into subdomains or on the selection of the points P_k (*theorem on the existence* of a double integral).

If $f(x, y) > 0$ in the domain D, then the double integral $\iint_D f(x, y)\,d\sigma$ is equal to the *volume of the cylindrical body* bounded above by the surface $z = f(x, y)$, on the side by a cylindrical surface with generators parallel to the Oz axis, and below by the domain D of the xOy plane.

Main properties of a double integral

1°. $\iint_D [f_1(x, y) \pm f_2(x, y)]\,d\sigma = \iint_D f_1(x, y)\,d\sigma \pm \iint_D f_2(x, y)\,d\sigma.$

2°. $\iint_D c \cdot f(x, y)\,d\sigma = c\iint_D f(x, y)\,d\sigma$, where c is a constant.

3°. If the domain of integration D is divided into two subdomains D_1 and D_2, then

$$\iint\limits_{D} f(x, y)\,d\sigma = \iint\limits_{D_1} f(x, y)\,d\sigma + \iint\limits_{D_2} f(x, y)\,d\sigma.$$

In Cartesian coordinates the double integral is usually written as $\iint\limits_{D} f(x, y)\,dx\,dy$.

Rules for calculating double integrals

There are two main forms of the domain of integration.

1. The domain of integration D is bounded on the left and on the right by the straight lines $x = a$ and $x = b$ $(a < b)$, and below and above by the continuous curves $y = \varphi_1(x)$ and $y = \varphi_2(x)$ $[\varphi_1(x) \leqslant \varphi_2(x)]$, each of which is intersected by a vertical straight line only at one point (Fig. 1).

For such a domain, the double integral can be calculated by the formula

$$\iint\limits_{D} f(x, y)\,dx\,dy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy.$$

In this case, the first thing is to calculate the integral $\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy$, in which x is considered to be constant.

2. The integration domain D is bounded below and above by the straight lines $y = c$ and $y = d$ $(c < d)$, and on the left and on the right by the continuous curves $x = \psi_1(y)$ and $x = \psi_2(y)$ $(\psi_1(y) \leqslant \psi_2(y))$, each of which is intersected by a horizontal straight line only at one point (Fig. 2).

For such a domain, the double integral is calculated by the formula

$$\iint\limits_{D} f(x, y)\,dx\,dy = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\,dx.$$

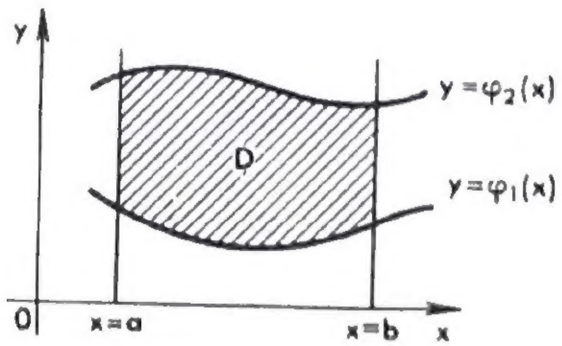


Fig. 1

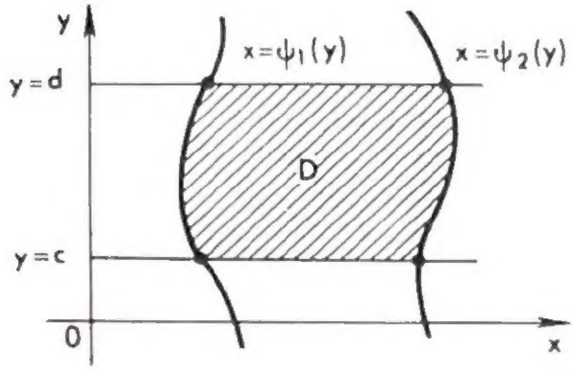


Fig. 2

First the integral $\int\limits_{\psi_1(x)}^{\psi_2(x)} f(x, y)\,dx$ is calculated in which y is taken to be constant.

The right-hand sides of the indicated formulas are known as *double-iterated* (or *repeated*) integrals.

In a more general case, the integration domain can be reduced to the main cases by means of partitioning.

1. Calculate $\iint\limits_D x \ln y \, dx\, dy$, if the domain D is a rectangle $0 \leqslant x \leqslant 4$, $1 \leqslant y \leqslant e$.

Solution. We have

$$\iint\limits_D x \ln y \, dx\, dy = \int\limits_0^4 x\, dx \int\limits_1^e \ln y\, dy$$

$$= \left[\frac{x^2}{2}\right]_0^4 \cdot [y \ln y - y]_1^e = 8 \cdot (e - e + 1) = 8.$$

2. Calculate $\iint\limits_D (\cos^2 x + \sin^2 y)\,dx\,dy$, if the domain D is a square $0 \leqslant x \leqslant \pi/4$, $0 \leqslant y \leqslant \pi/4$.

Solution. We find

$$\iint\limits_D (\cos^2 x + \sin^2 y)\,dx\,dy = \int\limits_0^{\pi/4} dx \int\limits_0^{\pi/4} (\cos^2 x + \sin^2 y)\,dy$$

$$= \int\limits_0^{\pi/4} \left[y\cos^2 x + \frac{y}{2} - \frac{1}{4}\sin 2y\right]_0^{\pi/4} dx = \int\limits_0^{\pi/4} \left(\frac{\pi}{4}\cos^2 x + \frac{\pi}{8} - \frac{1}{4}\right) dx$$

$$= \left[\frac{\pi}{8}\left(x + \frac{1}{2}\sin 2x\right) + \left(\frac{\pi}{8} - \frac{1}{4}\right)x\right]_0^{\pi/4}$$

$$= \frac{\pi}{8}\left(\frac{\pi}{4} + \frac{1}{2}\right) + \left(\frac{\pi}{8} - \frac{1}{4}\right)\frac{\pi}{4} = \frac{\pi^2}{16}.$$

3. Calculate $I = \int\limits_1^2 dx \int\limits_x^{x^2} (2x - y)\, dy.$

Solution. We have

$$I = \int\limits_1^2 \left[2xy - \frac{1}{2}y^2\right]_x^{x^2} dx = \int\limits_1^2 \left(2x^3 - \frac{1}{2}x^4 - 2x^2 + \frac{1}{2}x^2\right) dx$$

$$= \left[\frac{1}{2}x^4 - \frac{1}{10}x^5 - \frac{1}{2}x^3\right]_1^2 = 0.9.$$

4. Calculate $\iint\limits_D (x - y)\, dx\, dy$, if the domain D is bounded by the lines $y = 2 - x^2$, $y = 2x - 1$.

Solution. We construct the domain D. The first line is a parabola, with vertex at the point (0; 2), symmetric about the Oy axis. The second line is a straight line. Solving simultaneously the equations $y = 2 - x^2$ and $y = 2x - 1$, we find the coordinates of the intersection points: $A(-3; -7)$, $B(1; 1)$ (Fig. 3).

The domain of integration is of the first kind. We find

$$\iint\limits_D (x - y)\, dx\, dy = \int\limits_{-3}^{1} dx \int\limits_{2x-1}^{2-x^2} (x - y)\, dy$$

$$= \int\limits_{-3}^{1} \left[xy - \frac{1}{2}y^2\right]_{2x-1}^{2-x^2} dx = \int\limits_{-3}^{1} \Big(2x - x^3 - 2 + 2x^2 -$$

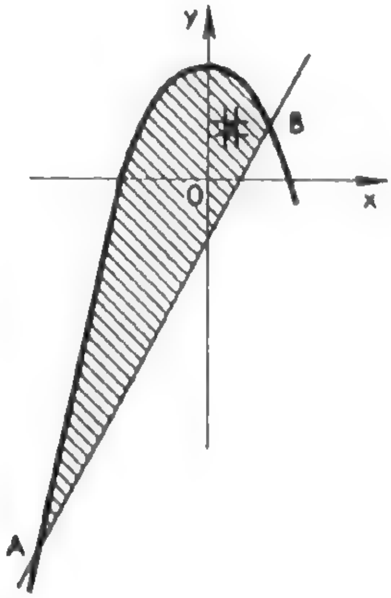


Fig. 3

$$- \frac{1}{2}x^4 - 2x^2 + x + 2x^2 - 2x + \frac{1}{2}\Big) dx$$

$$= \int_{-3}^{1} \left(-\frac{1}{2}x^4 - x^3 + 2x^2 + x - \frac{3}{2}\right) dx$$

$$= \left[-\frac{1}{10}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 - \frac{3}{2}x\right]_{-3}^{1} = 4\frac{4}{15}.$$

5. Calculate $\iint\limits_D (x + 2y)\,dx\,dy$, if the domain D is bounded by the straight lines $y = x, y = 2x, x = 2, x = 3$.

Solution. We find

$$\iint\limits_D (x + 2y)\,dx\,dy = \int_2^3 dx \int_x^{2x} (x + 2y)\,dy$$

$$= \int_2^3 [xy + y^2]_x^{2x} dx = \int_2^3 (2x^2 + 4x^2 - x^2 - x^2)\,dx$$

$$= 4\int_2^3 x^2\,dx = \frac{4}{3}x^3\Big|_2^3 = 25\frac{1}{3}.$$

6. Calculate $\iint\limits_D e^{x+\sin y}\cos y\,dx\,dy$, if the domain D is a rectangle $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \pi/2$.

7. Calculate $\iint\limits_D (x^2 + y^2)\,dx\,dy$, if the domain D is bounded by the lines $y = x$, $x = 0, y = 1, y = 2$.

8. Calculate $\iint\limits_D (3x^2 - 2xy + y)\,dx\,dy$, if the domain D is bounded by the lines $x = 0, x = y^2, y = 2$.

9. Change the order of integration in the integral

$$\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{1-x^2} f(x, y)\,dy.$$

Solution. The domain of integration D is bounded by the lines $x = -1, x = 1$, $y = -\sqrt{1 - x^2}$, $y = 1 - x^2$ (Fig. 4). Let us change the order of integration, for which purpose we shall represent the given domain as two subdomains (of the second kind): D_1, bounded on the left and on the right by the branches of the parabola $x = \pm\sqrt{1 - y}$ $(0 \leqslant y \leqslant 1)$ and D_2 bounded by the arcs of the circle $x = \pm\sqrt{1 - y^2}$ $(-1 \leqslant y \leqslant 0)$. Then we have

$$\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{1-x^2} f(x, y)\,dy = \int_0^1 dy \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x, y)\,dx + \int_{-1}^{0} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y)\,dx.$$

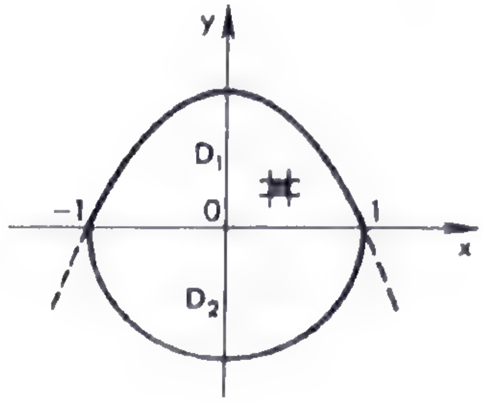


Fig. 4

10. Calculate $\int\limits_0^{2\pi} \cos^2 x\, dx \int\limits_0^{a} y\, dy$.

11. Calculate $\int\limits_1^{3} dx \int\limits_{x^3}^{x} (x - y)\, dy$.

12. Calculate $\iint\limits_D y \ln x\, dx\, dy$, if the domain D is bounded by the lines $xy = 1$, $y = \sqrt{x}$, $x = 2$.

13. Calculate $\iint\limits_D (\cos 2x + \sin y)\, dx\, dy$, if the domain D is bounded by the lines $x = 0, y = 0, 4x + 4y - \pi = 0$.

14. Calculate $\iint\limits_D (3x + y)\, dx\, dy$, if the domain D is specified by the inequalities $x^2 + y^2 \leqslant 9$, $y \geqslant (2/3)x + 3$.

15. Calculate $\iint\limits_D \sin (x + y)\, dx\, dy$, if the domain D is bounded by the lines $x = 0, y = \pi/2, y = x$.

16. Calculate $\iint\limits_D x\, dx\, dy$, if the domain D is a triangle with vertices $A(2; 3)$, $B(7; 2)$, $C(4; 5)$.

Change the order of integration:

17. $\int\limits_{-6}^{2} dx \int\limits_{x^2/4-1}^{2-x} f(x, y)\, dy$.

18. $\int\limits_1^{e} dx \int\limits_0^{\ln x} f(x, y)\, dy$.

19. $\int\limits_0^{1} dy \int\limits_{2-y}^{1+\sqrt{1-y^2}} f(x, y)\, dx$.

20. $\int\limits_0^{1} dx \int\limits_0^{x} f(x, y)\, dy$.

21. $\int\limits_0^1 dx \int\limits_{(1-x)^2/2}^{\sqrt{1-x^2}} f(x,y)\,dy.$

22. $\int\limits_0^{\pi} dx \int\limits_0^{\sin x} f(x,y)\,dy.$

1.2. Change of Variables in a Double Integral.

1.2.1. Double integral in polar coordinates. The transformation of a double integral from the rectangular coordinates x, y to the *polar coordinates* ρ, θ connected with the former by the relations $x = \rho\cos\theta$, $y = \rho\sin\theta$ can be carried out by means of the formula

$$\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_D f(\rho\cos\theta, \rho\sin\theta)\rho d\rho\, d\theta.$$

If the integration domain D is bounded by two rays $\theta = \alpha$, $\theta = \beta$ $(\alpha < \beta)$, emanating from the pole, and two curves $\rho = \rho_1(\theta)$ and $\rho = \rho_2(\theta)$, where $\rho_1(\theta)$ and $\rho_2(\theta)$ are single-valued functions for $\alpha \leqslant \theta \leqslant \beta$ and $\rho_1(\theta) \leqslant \rho_2(\theta)$, then the *double integral can be calculated by the formula*

$$\iint\limits_D F(\rho,\theta)\rho\, d\rho\, d\theta = \int\limits_\alpha^\beta d\theta \int\limits_{\rho_1(\theta)}^{\rho_2(\theta)} F(\rho,\theta)\rho\, d\rho,$$

where $F(\rho, \theta) = f(\rho\cos\theta, \rho\sin\theta)$, the integral $\int\limits_{\rho_1(\theta)}^{\rho_2(\theta)} F(\rho, \theta)\rho\, d\rho$ in which θ is taken to be constant being the first to be calculated.

1.2.2. Double integral in curvilinear coordinates. Suppose we transform the double integral from the rectangular coordinates x, y to the *curvilinear coordinates* u, v connected with the rectangular coordinates by the relations $x = x(u, v)$, $y = y(u, v)$, where the functions $x(u, v)$ and $y(u, v)$ possess continuous partial derivatives in the domain D' of the $uO'v$ plane, and the *transformation Jacobian* does not vanish in the domain D':

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \neq 0.$$

In this case, a mutual one-to-one correspondence, continuous in both directions, is established between the points of the domain D of the xOy plane and the points of the domain D' of the $uO'v$ plane (Fig. 5).

For this case, the *formula of the transformation of the double integrial* has the form

$$\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_{D'} f[x(u,v), y(u,v)]\,|J|\,du\,dv.$$

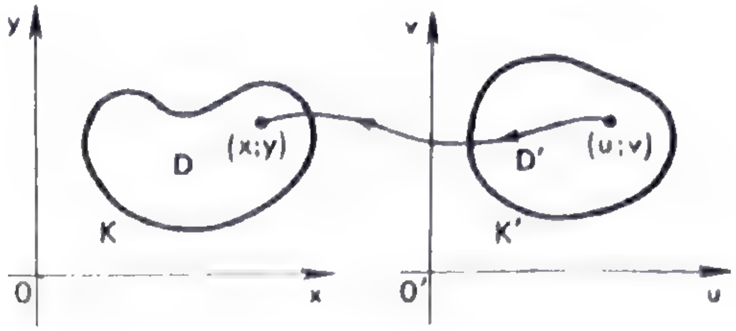


Fig. 5

And for the case of the polar coordinates it is

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho .$$

23. Having passed to polar coordinates, calculate $\iint\limits_D \sqrt{x^2+y^2}\,dx\,dy$, if D is the first quadrant of the circle $x^2+y^2 \leqslant a^2$.

Solution. Putting $x = \rho\cos\theta$, $y = \rho\sin\theta$, we have

$$\iint\limits_D \sqrt{x^2+y^2}\,dx\,dy = \iint\limits_D \sqrt{\rho^2\cos^2\theta + \rho^2\sin^2\theta}\;\rho\,d\rho\,d\theta$$

$$= \int\limits_0^{\pi/2} d\theta \int\limits_0^{a} \rho^2 d\rho = \frac{1}{3}\int\limits_0^{\pi/2} \rho^3\Big|_0^a d\theta = \frac{a^3}{3}\int\limits_0^{\pi/2} d\theta = \frac{\pi a^3}{6}.$$

24. Calculate $\iint\limits_D \ln(x^2+y^2)\,dx\,dy$, if the domain D is a disc between the circles $x^2+y^2 = e^2$ and $x^2+y^2 = e^4$.

Solution. We pass to polar coordinates:

$$\iint\limits_D \ln(x^2+y^2)dx\,dy = \iint\limits_D \ln\rho^2\cdot\rho\,d\rho\,d\theta$$

$$= 2\iint\limits_D \rho\ln\rho\,d\rho\,d\theta = 2\int\limits_0^{2\pi} d\theta \int\limits_e^{e^2} \rho\ln\rho\,d\rho.$$

Taking by parts the integral dependent on ρ we obtain

$$2\int\limits_0^{2\pi}\left[\frac{1}{2}\rho^2\ln\rho - \frac{1}{4}\rho^2\right]_e^{e^2} d\theta = \pi e^2(3e^2-1).$$

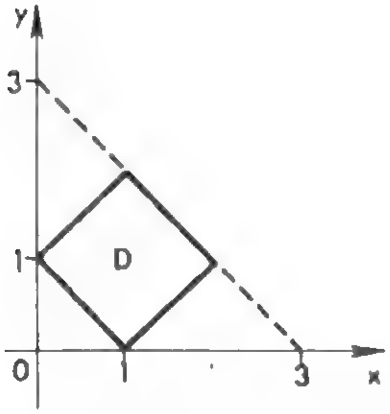


Fig. 6

25. Calculate $\iint_D (x+y)^3(x-y)^2\,dx\,dy$, if the domain D is a square bounded by the straight lines $x+y=1$, $x-y=1$, $x+y=3$, $x-y=-1$ (Fig. 6).

Solution. We set $x+y=u$, $x-y=v$, whence we have $x=(1/2)(u+v)$, $y=(1/2)(u-v)$. Then the Jacobian of transformation is

$$J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix}=-\frac{1}{2},\ \text{i.e. } |J|=\frac{1}{2}.$$

Consequently, $\iint_D (x+y)^3(x-y)^2\,dx\,dy=\frac{1}{2}\iint_{D'} u^3v^2\,du\,dv$.

Since the domain D' is also a square (Fig. 7), we have

$$\iint_D (x+y)^3(x-y)^2\,dx\,dy=\frac{1}{2}\int_1^3 u^3\,du\int_{-1}^1 v^2\,dv$$

$$=\frac{1}{2}\int_1^3 u^3\cdot\left[\frac{1}{3}v^3\right]_{-1}^1 du=\frac{1}{6}\int_1^3 u^3(1+1)\,du=\frac{1}{12}u^4\Big|_1^3=\frac{20}{3}.$$

Passing to polar coordinates, calculate the following double integrals:

26. $\iint_D \left(1-\frac{y^2}{x^2}\right)dx\,dy$, if the domain D is a circle $x^2+y^2\leqslant\pi^2$.

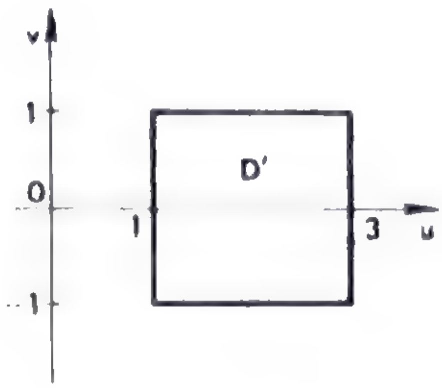


Fig. 7

27. $\iint\limits_D \frac{dx\,dy}{x^2+y^2+1}$, if the domain D is bounded by the semicircle $y = \sqrt{1-x^2}$ and the x-axis.

28. $\iint\limits_D (x^2+y^2)\,dx\,dy$, if the domain D is bounded by the circle $x^2+y^2 = 2ax$.

29. $\iint\limits_D \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,dx\,dy$, if the domain D is bounded by the lines $x^2+y^2 = \pi^2/9$, $x^2+y^2 = \pi^2$.

30. $\iint\limits_D \sqrt{x^2+y^2}\,dx\,dy$, if the domain D is bounded by the lines $x^2+y^2 = a^2$, $x^2+y^2 = 4a^2$.

31. Calculate $\int\limits_0^1 dx \int\limits_x^{2x} dy$, introducing new variables $x = u(1-v)$, $y = uv$.

32. Calculate $\iint\limits_D dx\,dy$, if the domain D is bounded by the lines $xy = 1$, $xy = 2$, $y = x$, $y = 3x$.

Hint. Perform the change of variables $x = (u/v)^{1/2}$, $y = (uv)^{1/2}$.

1.3. Calculating the Area of a Plane Figure

The *area of the plane figure* bounded by the domain D can be found from the formula

$$S = \iint\limits_D dx\,dy.$$

If the domain D is specified, say, by the inequalities $a \leqslant x \leqslant b$, $\varphi_1(x) \leqslant y \leqslant \leqslant \varphi_2(x)$, then

$$S = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dy.$$

If in polar coordinates the domain D is specified by the inequalities $\alpha \leqslant \theta \leqslant \beta$, $\varphi(\theta) \leqslant \beta \leqslant f(\theta)$, then

$$S = \iint_D \rho\, d\rho\, d\theta = \int_\alpha^\beta d\theta \int_{\varphi(\theta)}^{f(\theta)} \rho\, d\rho.$$

33. Calculate the area of the figure bounded by the lines $x = 4y - y^2$, $x + y = 6$.

Solution. Solving the system of equations $x = 4y - y^2$ and $x + y = 6$, we find the coordinates of the points of intersection of the given lines (the reader is recommended to make a drawing). As a result we shall have $A(4; 2)$, $B(3; 3)$, and, consequently,

$$S = \iint_D dx\, dy = \int_2^3 dy \int_{6-y}^{4y-y^2} dx = \int_2^3 x \,\Big|_{6-y}^{4y-y^2} dy$$

$$= \int_2^3 (-y^2 + 5y - 6)\, dy = \left[-\frac{1}{3} y^3 + \frac{5}{2} y^2 - 6y\right]_2^3 = \frac{1}{6} \text{ (sq. units).}$$

34. Calculate the area of the figure bounded by the circles $\rho = 1$, $\rho = (2/\sqrt{3}) \cos\theta$ (outside the circle $\rho = 1$; Fig. 8).

Solution. Let us find the coordinates of the point A; we have $1 = (2/\sqrt{3}) \cos\theta$, $\cos\theta = \sqrt{3}/2$, $\theta = \pi/6$, i.e. $A(1; \pi/6)$. Then

$$S = \iint_D \rho\, d\rho\, d\theta = 2 \int_0^{\pi/6} d\theta \int_1^{(2/\sqrt{3})\cos\theta} \rho\, d\rho = 2 \int_0^{\pi/2} \left[\frac{1}{2} \rho^2\right]_1^{(2/\sqrt{3})\cos\theta} d\theta$$

$$= \int_0^{\pi/6} \left(\frac{4}{3} \cos^2\theta - 1\right) d\theta = \int_0^{\pi/6} \left(\frac{2}{3} + \frac{2}{3} \cos 2\theta - 1\right) d\theta$$

$$= \frac{1}{3} \int_0^{\pi/6} (2 \cos 2\theta - 1)\, d\theta = \frac{1}{3} [\sin 2\theta - \theta]_0^{\pi/6}$$

$$= \frac{1}{3} \left(\sin\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{1}{18} (3\sqrt{3} - \pi) \text{ (sq. units).}$$

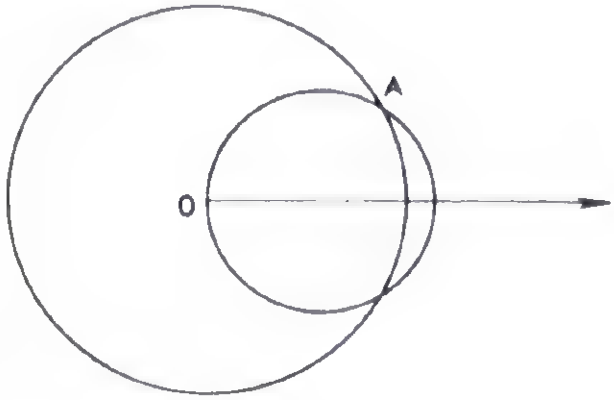


Fig. 8

35. Find the area bounded by the lemniscate $(x^2 + y^2)^2 = 2a^2 xy$.

Solution. Setting $x = \rho \cos \theta$, $y = \rho \sin \theta$, we transform the equation of the curve to polar coordinates. The result is $\rho^2 = 2a^2 \sin \theta \cos \theta = a^2 \sin 2\theta$.

It is evident that the change in the polar angle θ from 0 to $\pi/4$ is associated with the quarter of the desired area. It follows that

$$S = 4 \iint\limits_D \rho\, d\rho\, d\theta = 4 \int_0^{\pi/4} d\theta \int_0^{a\sqrt{\sin 2\theta}} \rho\, d\rho$$

$$= 2 \int_0^{\pi/4} \rho^2 \Big|_0^{a\sqrt{\sin 2\theta}} d\theta = 2a^2 \int_0^{\pi/4} \sin 2\theta\, d\theta = -a^2 \cos 2\theta \Big|_0^{\pi/4} = a^2.$$

36. Find the area of the figure bounded by the line $x^3 + y^3 = axy$ (the area of a loop; Fig. 9).

Solution. Let us transform the given equation to polar coordinates: $\rho^3 (\sin^3\theta + \cos^3\theta) = a\rho^2 \sin\theta \cos\theta$, i.e. $\rho = a \sin\theta \cos\theta/(\sin^3\theta + \cos^3\theta)$. The symmetry axis of the loop is the ray $\theta = \pi/4$, therefore,

$$S = 2 \iint\limits_D \rho\, d\rho\, d\theta = 2 \int_0^{\pi/4} d\theta \int_0^{a \sin\theta \cos\theta/(\sin^3\theta + \cos^3\theta)} \rho\, d\rho$$

$$= a^2 \int_0^{\pi/4} \frac{\sin^2\theta \cos^2\theta}{(\sin^3\theta + \cos^3\theta)^2}\, d\theta$$

$$= a^2 \int_0^{\pi/4} \frac{\tan^2\theta \cos^4\theta}{\cos^6\theta\, (1 + \tan^3\theta)^2}\, d\theta = \frac{a^2}{3} \int_0^{\pi/4} \frac{3 \tan^2\theta\, d(\tan\theta)}{(1 + \tan^3\theta)^2} =$$

$$= \frac{a^2}{3} \int_0^{\pi/4} \frac{d(1 + \tan^3\theta)}{(1 + \tan^3\theta)^2} = \left[-\frac{a^2}{3(1 + \tan^3\theta)} \right]_0^{\pi/4} = \frac{a^2}{6}.$$

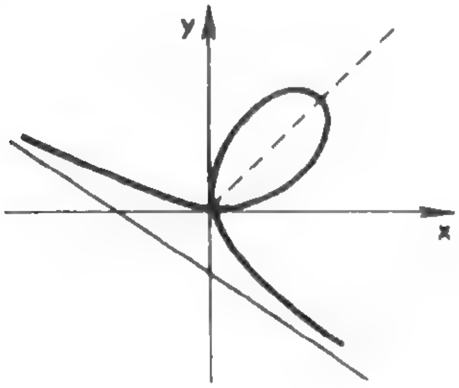


Fig. 9

37. Calculate the area bounded by the lines $x = y^2 - 2y$, $x + y = 0$.

38. Calculate the area bounded by the lines $y = 2 - x$, $y^2 = 4x + 4$.

39. Calculate the area bounded by the lines $y^2 = 4x - x^2$, $y^2 = 2x$ (outside the parabola).

40. Calculate the area bounded by the lines $3y^2 = 25x$, $5x^2 = 9y$.

41. Calculate the area bounded by the lines $y^2 + 2y - 3x + 1 = 0$, $3x - 3y - 7 = 0$.

42. Calculate the area of the figure which is closest to the origin and is bounded by the lines $y = \cos x$, $y = \cos 2x$, $y = 0$.

43. Calculate the area bounded by the lines $y = 4x - x^2$, $y = 2x^2 - 5x$.

44. Calculate the area bounded by the lines $x = 4 - y^2$, $x + 2y - 4 = 0$.

45. Calculate the area bounded by the lines $\rho = (2 - \cos\theta)$, $\rho = 2$ (outside the cardioid).

46. Calculate the area bounded by the lines $\rho = 2(1 + \cos\theta)$, $\rho = 2\cos\theta$.

47. Calculate the area bounded by the lines $y^2 = 4(1 - x)$, $x^2 + y^2 = 4$ (outside the parabola).

1.4. Calculating the Volume of a Body

The *volume of the cylindrical body* bounded above by the continuous surface $z = f(x, y)$, below by the plane $z = 0$ and on the side by a straight cylindrical surface cutting a domain D from the xOy plane, can be calculated by the formula

$$V = \iint_D f(x, y)\, dx\, dy.$$

48. Find the volume of the body bounded by the surfaces $y = 1 + x^2$, $z = 3x$, $y = 5$, $z = 0$ and located in the first octant.

Solution. The body is bounded above by the plane $z = 3x$, on the side by the parabolic cylynder $y = 1 + x^2$ and the plane $y = 5$. Hence it is a cylindrical body. The domain D is bounded by the parabola $y = 1 + x^2$ and the straight lines $y = 5$ and $x = 0$. Thus, we have

$$V = \iint_D 3x\, dx\, dy = 3\int_0^2 x\, dx \int_{1+x^2}^{5} dy$$

$$= 3\int_0^2 x\cdot [y]_{1+x^2}^{5}\,dx = 3\int_0^2 (4x - x^3)\,dx = 3\left[2x^2 - \frac{1}{4}x^4\right]_0^2 = 12 \text{ (cu. units)}.$$

49. Calculate the volume of the body bounded by the surfaces $z = 1 - x^2 - y^2$, $y = x$, $y = x\sqrt{3}$, $z = 0$ and lying in the first octant.

Solution. The given body is bounded above by the paraboloid $z = 1 - x^2 - y^2$. The integration domain D is a circular sector bounded by the arc of the circle $x^2 + y^2 = 1$, which is the line of intersection of the paraboloid with the $z = 0$ plane, and the straight lines $y = x$ and $y = x\sqrt{3}$. Consequently,

$$V = \iint_D (1 - x^2 - y^2)\,dx\,dy.$$

Since the domain of integration is a part of a circle and the integrand depends on $x^2 + y^2$, it is expedient to pass to the polar coordinates. In these coordinates, the equation of the circle $x^2 + y^2 = 1$ assumes the form $\rho = 1$, the integrand is equal to $1 - \rho^2$, and the limits of integration with respect to θ can be determined from the equations of the straight lines: $k_1 = \tan\theta_1 = 1$, i.e. $\theta_1 = \pi/4$; $k_2 = \tan\theta_2 = \sqrt{3}$, i.e. $\theta_2 = \pi/3$. Thus, we have

$$V = \iint_D (1 - \rho^2)\rho\,d\rho\,d\theta = \int_{\pi/4}^{\pi/3} d\theta \int_0^1 (\rho - \rho^3)\,d\rho$$

$$= \int_{\pi/4}^{\pi/3} \left[\frac{1}{2}\rho^2 - \frac{1}{4}\rho^4\right]_0^1 d\theta = \frac{1}{4}\int_{\pi/4}^{\pi/3} d\theta = \frac{\pi}{48} \text{ (cu. units)}.$$

50. Find the volume of the body bounded by the surfaces $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$.

Solution. We consider an eighth part of the given body (Fig. 10):

$$\frac{1}{8}V = \iint_D \sqrt{a^2 - x^2}\,dx\,dy = \int_0^a \sqrt{a^2 - x^2}\,dx \int_0^{\sqrt{a^2-x^2}} dy$$

$$= \int_0^a (a^2 - x^2)\,dx = \left[a^2x - \frac{1}{3}x^3\right]_0^a = \frac{2}{3}a^3.$$

Consequently, $V = 16a^3/3$.

51. Calculate the volume of the body bounded by the surfaces $x^2 + y^2 = 8$, $x = 0$, $y = 0$, $z = 0$, $x + y + z = 4$.

52. Calculate the volume of the body bounded by the surfaces $x = 2y^2$, $x + 2y + z = 4$, $y = 0$, $z = 0$.

53. Calculate the volume of the body bounded by the surfaces $x^2 + 4y^2 + z = 1$, $z = 0$.

54. Calculate the volume of the body bounded by the surfaces $z = x^2 + y^2$, $y = x^2$, $y = 1$, $z = 0$.

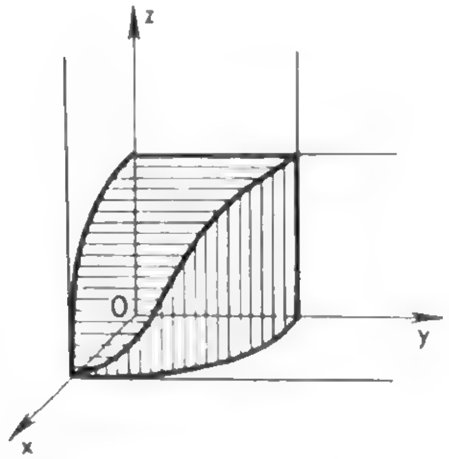


Fig. 10

55. Calculate the volume of the body bounded by the surfaces $z = 4 - x^2$, $2x + y = 4$, $x = 0$, $y = 0$, $z = 0$.

56. Calculate the volume of the body bounded by the surfaces $z^2 = xy$, $x = 0$, $y = 0$, $z = 0$, $y = 4$, $x = 1$.

57. Calculate the volume of the body bounded by the surfaces $z = 5x$, $x^2 + y^2 = 9$, $z = 0$.

58. Calculate the volume of the body bounded by the surfaces $x + y + z = 6$, $3x + 2y = 12$, $3x + y = 6$, $y = 0$, $z = 0$.

59. Calculate the volume of the body bounded by the surfaces $z = x + y + 1$, $y^2 = x$, $x = 1$, $y = 0$, $z = 0$.

60. Calculate the volume of the body bounded by the surfaces $z = 0$, $z = xy$, $x^2 + y^2 = 4$.

61. Calculate the volume of the body bounded by the surfaces $x^2/a^2 + y^2/b^2 = 1$, $y = 0$, $z = x/2$, $z = x$.

1.5. Calculating the Area of a Surface

If a smooth single-valued surface is specified by the equation $z = f(x, y)$, then the *area of the surface* is expressed by the formula

$$S = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

where D is the projection of the given surface on the xOy plane. Similarly, if the surface is specified by the equation $x = f(y, z)$, then

$$S = \iint_D \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial y}\right)^2}\, dy\, dz,$$

where D is the projection of the surface on the yOz plane; now if the equation of the

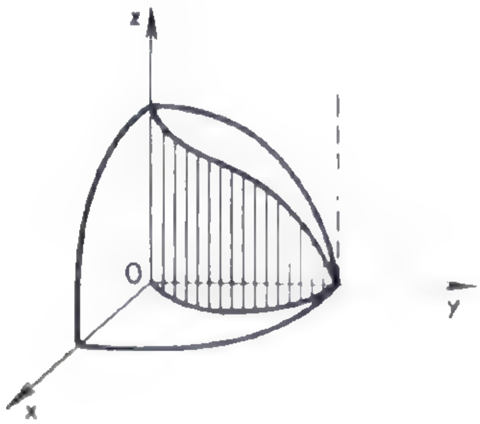


Fig. 11

surface has the form $y = f(x, z)$, then

$$S = \iint_D \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dx\, dz.$$

where D is the projection of the surface on the xOz plane.

62. Find the area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ contained within the cylinder $x^2 + y^2 = ay$ (Fig. 11).

Solution. It follows from the equation of the sphere (for the first octant) that

$$z = \sqrt{a^2 - x^2 - y^2}; \quad \frac{\partial z}{\partial x} = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}$$

$$\frac{\partial z}{\partial y} = -\frac{y}{\sqrt{a^2 - x^2 - y^2}};$$

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}.$$

The part of the sphere lying in the first octant is projected onto a half-disc bounded by the circle $x^2 + y^2 = ay$ and the Oy axis. The half disc is the domain of integration D.

The surface lies in four octants, and therefore, the desired area

$$S = 4a \iint_D \frac{dx\, dy}{\sqrt{a^2 - x^2 - y^2}}.$$

If we pass to polar coordinates, the equation of the surface assumes the form $\rho = a \sin \theta$ and

$$S = 4a \iint_D \frac{\rho \, d\rho \, d\theta}{\sqrt{a^2 - \rho^2}} = 4a \int_0^{\pi/2} d\theta \int_0^{a \sin \theta} \frac{\rho \, d\rho}{\sqrt{a^2 - \rho^2}}$$

$$= -4a \int_0^{\pi/2} \sqrt{a^2 - \rho^2} \, \Big|_0^{a \sin \theta} d\theta = -4a^2 \int_0^{\pi/2} (\cos \theta - 1) \, d\theta$$

$$= -4a^2 [\sin \theta - \theta]_0^{\pi/2} = 4a^2 \left(\frac{\pi}{2} - 1\right) \text{ (sq. units).}$$

63. Find the area of the part of the cone $z = \sqrt{x^2 + y^2}$ contained within the cylinder $x^2 + y^2 = 2x$ (Fig. 12).

Solution. The equation of the cone yields $\dfrac{\partial z}{\partial x} = \dfrac{x}{\sqrt{x^2 + y^2}}$, $\dfrac{\partial z}{\partial y} = \dfrac{y}{\sqrt{x^2 + y^2}}$. The domain of integration D is a disc bounded by the circle $x^2 + y^2 = 2x$, or $\rho = 2 \cos \theta$. So we have

$$S = \iint_D \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} \, dx \, dy$$

$$= \sqrt{2} \iint_D dx \, dy = \sqrt{2} \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} \rho \, d\rho$$

$$= 2\sqrt{2} \int_0^{\pi/2} \frac{1}{2} \rho^2 \, \Big|_0^{2\cos\theta} d\theta = 2\sqrt{2} \cdot \frac{1}{2} \int_0^{\pi/2} 4 \cos^2 \theta \, d\theta$$

$$= 2\sqrt{2} \int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta = 2\sqrt{2} \left[\theta + \frac{1}{2} \sin 2\theta\right]_0^{\pi/2} = \pi\sqrt{2} \text{(sq. units).}$$

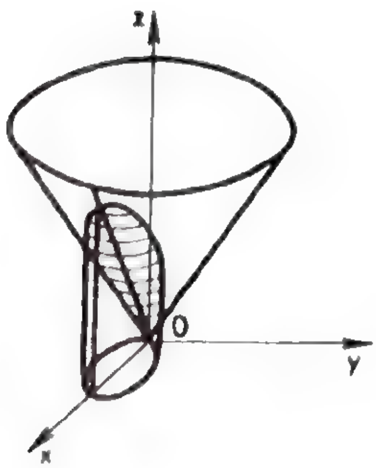


Fig. 12

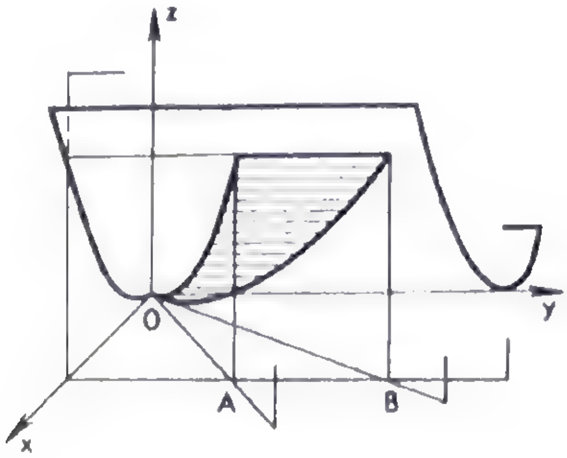


Fig. 13

64. Calculate the area of the surface of the cylinder $x^2 = 2z$ intercepted by the planes $x - 2y = 0$, $y = 2x$, $x = 2\sqrt{2}$ (Fig. 13).

Solution. The domain of integration is the triangle OAB. The equation of the cylinder yields $\frac{\partial z}{\partial x} = x$, $\frac{\partial z}{\partial y} = 0$. Then we have

$$S = \iint_D \sqrt{1 + x^2}\, dx\, dy = \int_0^{2\sqrt{2}} \sqrt{1 + x^2}\, dx \int_{x/2}^{2x} dy$$

$$= \int_0^{2\sqrt{2}} \frac{3}{2} x\sqrt{1 + x^2}\, dx = \frac{3}{4} \int_0^{2\sqrt{2}} (1 + x^2)^{1/2}\, d(1 + x^2)$$

$$= \frac{3}{4} \cdot \frac{2}{3} (1 + x^2)^{3/2} \Big|_0^{2\sqrt{2}} = 13 \text{ (sq. units)}.$$

65. Calculate the area of the part of the surface of the paraboloid $x = 1 - y^2 - z^2$ cut out by the cylinder $y^2 + z^2 = 1$.

Solution. The domain of integration is the circle $y^2 + z^2 = 1$ (located in the yOz plane). From the equation of the paraboloid we have $\frac{\partial x}{\partial y} = -2y$, $\frac{\partial x}{\partial z} = -2z$. It follows that

$$S = \iint_D \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dy\, dz = \iint_D \sqrt{1 + 4(y^2 + z^2)}\, dy\, dz.$$

Passing to polar coordinates, we obtain

$$S = \int_0^{2\pi} d\theta \int_0^1 \rho\sqrt{1 + 4\rho^2}\, d\rho\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{2}{3} \cdot \frac{1}{8} (1 + 4\rho^2)^{3/2} \right]_0^1 d\theta = \frac{5\sqrt{5} - 1}{12} \int_0^{2\pi} d\theta = \frac{5\sqrt{5} - 1}{6} \pi \text{ (sq. units).}$$

66. Find the area of the part of the surface $y = x^2 + z^2$ cut out by the cylinder $x^2 + z^2 = 1$ and lying in the first octant.
67. Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4$ cut out by the cylinder $x^2/4 + y^2 = 1$.
68. Find the area of the part of the plane $z = x$ which is contained within the cylinder $x^2 + y^2 = 4$ and lies above the plane $z = 0$.
69. Find the area of the part of the surface of the cylinder $z = x^2$ cut out by the planes $x + y = \sqrt{2}$, $x = 0$, $y = 0$.
70. Calculate the area of the surface of the cone $x^2 - y^2 - z^2 = 0$ lying within the cylinder $x^2 + y^2 = 1$.
71. Calculate the area of the surface of the cylinder $x^2 + z^2 = 4$ lying within the cylinder $x^2 + y^2 = 4$.
72. Find the area of the part of the surface $z^2 = 2xy$ cut out by the planes $x = 1$, $y = 4$, $z = 0$.

1.6. Applications of a Double Integral

If a plate occupies the domain D of the xOy plane and has the variable surface density $\gamma = \gamma(x, y)$, then the *mass* M of the plate is expressed by the double integral

$$M = \iint_D \gamma(x, y)\, dx\, dy.$$

The *static moments* of the plate about the Ox and Oy axes can be found by the formulas

$$M_x = \iint_D y\, \gamma(x, y)\, dx\, dy, \quad M_y = \iint_D x\gamma(x, y)\, dx\, dy.$$

In the case of a homogeneous plate $\gamma =$ const.
The *coordinates of the centre of gravity* of the plate can be found from the formulas

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M},$$

where M is the mass of the plate, and M_x, M_y are its static moments about the coordinate axes.
In the case of a homegeneous plate, the formulas assume the form

$$\bar{x} = \frac{\iint\limits_D x\,dx\,dy}{S}, \quad \bar{y} = \frac{\iint\limits_D y\,dx\,dy}{S},$$

where S is the area of the domain D.

The *moments of inertia* of the plate *about the Ox* and *Oy* axes are calculated by the formulas

$$I_x = \iint\limits_D y^2\gamma(x, y)dx\,dy, \quad I_y = \iint\limits_D x^2\gamma(x, y)\,dx\,dy,$$

and *the moment of inertia about the origin*, by the formula

$$I_0 = \iint\limits_D (x^2 + y^2)\gamma(x, y)\,dx\,dy = I_x + I_y.$$

Setting $\gamma(x, y) = 1$ in these formulas, we can obtain the formulas to calculate the geometric moments of inertia of a plane figure.

73. Find the coordinates of the centre of gravity of the figure bounded by the lines $y^2 = 4x + 4$, $y^2 = -2x + 4$ (Fig. 14).

Solution. The figure being symmetrical about the Ox axis, we have $\bar{y} = 0$. It remains to find $\bar{x}$.

Let us find the area of the given figure:

$$S = \iint\limits_D dx\,dy = 2\int_0^2 dy \int_{(y^2-4)/4}^{(4-y^2)/2} dx$$

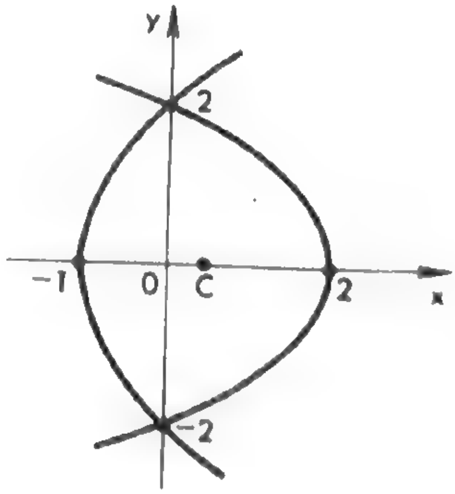


Fig. 14

$$= 2\int_0^2 \left(\frac{4-y^2}{2} - \frac{y^2-4}{4}\right) dy = 2\int_0^2 \left(3 - \frac{3y^2}{4}\right) dy = 6\left[y - \frac{1}{12}y^3\right]_0^2 = 8.$$

It follows that

$$\bar{x} = \frac{1}{8}\iint_D x\,dx\,dy = \frac{1}{8}\cdot 2\int_0^2 dy \int_{(y^2-4)/4}^{(4-y^2)/2} x\,dx$$

$$= \frac{1}{8}\int_0^2 \left[\frac{1}{4}(4-y^2)^2 - \frac{1}{16}(y^2-4)^2\right] dy$$

$$= \frac{1}{8}\int_0^2 \left(3 - \frac{3}{2}y^2 + \frac{3}{16}y^4\right) dy = \frac{1}{8}\left[3y - \frac{y^3}{2} + \frac{3y^5}{80}\right]_0^2 = \frac{2}{5}.$$

74. Find the coordinates of the centre of gravity of the figure bounded by the ellipse $x^2/25 + y^2/9 = 1$ and its chord $x/5 + y/3 = 1$.

Solution. We find the area of the segment:

$$S = \iint_D dx\,dy = \int_0^5 dx \int_{3(1-x/5)}^{(3/5)\sqrt{25-x^2}} dy$$

$$= \int_0^5 \left(\frac{3}{5}\sqrt{25-x^2} - 3 + \frac{3}{5}x\right) dx = \frac{15}{4}(\pi - 2).$$

Then we have

$$\bar{x} = \frac{1}{S}\iint_D x\,dx\,dy = \frac{4}{15(\pi-2)}\int_0^5 x\,dx \int_{3(1-x/5)}^{(3/5)\sqrt{25-x^2}} dy$$

$$= \frac{4}{15(\pi-2)}\int_0^5 \left[\frac{3}{5}x\sqrt{25-x^2} - 3x\left(1 - \frac{x}{5}\right)\right] dx$$

$$= \frac{4}{15(\pi-2)}\left[-\frac{3}{5}\cdot\frac{1}{2}\cdot\frac{2}{3}(25-x^2)^{3/2} - \frac{3x^2}{2} + \frac{x^3}{5}\right]_0^5$$

$$= \frac{4}{15(\pi-2)}\left(25 - \frac{75}{2} + 25\right) = \frac{10}{3(\pi-2)};$$

$$\bar{y} = \frac{1}{S}\iint_D y\,dx\,dy = \frac{4}{15(\pi-2)}\int_0^5 dx \int_{3(1-x/5)}^{(3/5)\sqrt{25-x^2}} y\,dy$$

$$= \frac{4}{15(\pi-2)}\cdot\frac{1}{2}\int_0^5\left[\frac{9}{25}(25-x^2) - 9\left(1-\frac{x}{5}\right)^2\right]dx$$

$$= \frac{2\cdot 9\cdot 2}{15(\pi-2)\cdot 25}\int_0^5 (5x - x^2)dx$$

$$= \frac{12}{125(\pi-2)}\left[\frac{5x^2}{2} - \frac{1}{3}x^3\right]_0^5 = \frac{12}{125(\pi-2)}\left(\frac{125}{2} - \frac{125}{3}\right) = \frac{2}{\pi-2}.$$

75. Calculate the polar moment of inertia of the figure bounded by the lines $x/a + y/b = 1$, $x = 0$, $y = 0$.

Solution. The moment of inertia about the origin is

$$I_0 = \iint_D (x^2+y^2)\,dx\,dy = \int_0^a dx \int_0^{(b/a)(a-x)} (x^2+y^2)\,dy$$

$$= \int_0^a \left[x^2y + \frac{1}{3}y^3\right]_0^{(b/a)(a-x)} dx = \int_0^a \left[\frac{b}{a}x^2(a-x) + \frac{1}{3}\frac{b^3}{a^3}(a-x)^3\right]dx$$

$$= \left[\frac{1}{3}bx^3 - \frac{b}{4a}x^4 - \frac{1}{3}\cdot\frac{b^3}{a^3}\cdot\frac{1}{4}(a-x)^4\right]_0^a = \frac{ab(a^2+b^2)}{12}.$$

76. Calculate the moment of inertia of the figure, bounded by the cordioid $\rho = a(1+\cos\theta)$, about the x-axis.

Solution. Passing to polar coordinates in the formula $I_x = \iint_D y^2\,dx\,dy$, we get

$$I_x = \iint_D \rho^2 \sin^2\theta\rho\,d\rho\,d\theta = \int_0^{2\pi}\sin^2\theta\,d\theta\int_0^{a(1+\cos\theta)}\rho^3 d\rho$$

$$= \int_0^{2\pi}\sin^2\theta\,\frac{1}{4}\rho^4\Big|_0^{a(1+\cos\theta)}d\theta = \frac{1}{4}a^4\int_0^{2\pi}\sin^2\theta(1+\cos\theta)^4 d\theta$$

$$= \frac{1}{4} a^4 \int_0^{2\pi} \sin^2\theta (1 + 4\cos\theta + 6\cos^2\theta + 4\cos^3\theta + \cos^4\theta) d\theta = \frac{21}{32} \pi a^4.$$

77. Determine the centre of gravity of the area bounded by the lines $y = x^2, y = 2x^2, x = 1, x = 2$.

78. Determine the centre of gravity of the area bounded by the cardioid $\rho = a(1 + \cos\theta)$.

79. Determine the centre of gravity of the half-segment of the parabola $y^2 = ax$ cut out by the straight lines $x = a, y = 0$ $(y > 0)$.

80. Find the centre of gravity of the area bounded by one loop of the curve $\rho = a \sin 2\theta$.

81. Find the centre of gravity of the area bounded by the parabolas $y^2 = x, x^2 = y$.

82. Find the centre of gravity of the area bounded by the parabola $y^2 = 2px$ and the straight line $x = 2p$.

83. Find the centre of gravity of the area bounded by the lines $y = \sqrt{2x - x^2}, y = 0$.

84. Calculate the moment of inertia of the area, bounded by the lines $y = 2\sqrt{x}$, $x + y = 3, y = 0$, about the x-axis.

85. Calculate the polar moment of inertia of the area bounded by the straight lines $x + y = 2, x = 0, y = 0$.

86. Calculate the moment of inertia of the area, bounded by the lines $y = 4 - x^2$, $y = 0$, about the x-axis.

87. Calculate the moment of inertia of the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ about its major axis.

88. Calculate the mass of a square plate with side a, whose density at any point is proportional to the square of the distance of that point from one of the vertices of the square.

89. Calculate the mass of a circular plate with radius r, if its density is inversely proportional to the distance of the point from the centre and is equal to δ at the edge of the plate.

90. Calculate the static moment of the plate shaped as a right triangle, with sides $|OA| = a$, $|OB| = b$, about the side OA, if its density at any point is equal to the distance of the point from the side OA.

1.7. Triple Integral

Suppose the function $f(x, y, z)$ is defined in a bounded closed spatial domain T. Let us partition the domain T arbitrarily into n subdomains $T_1, T_2, \ldots, T_n$ with diameters $d_1, d_2, \ldots, d_n$ and volumes $\Delta V_1, \Delta V_2, \ldots, \Delta V_n$. We take an arbitrary point $P_k(\xi_k; \eta_k; \zeta_k)$ in each subdomain and multiply the value of the function at the point P_k by the volume of the subdomain.

The *integral sum* for the function $f(x, y, z)$ over the domain T is the sum of the form $\sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k)\Delta V_k$.

The *triple integral* of the function $f(x, y, z)$ over the domain T is the limit of the integral sum under the condition that the greatest of the diameters of the subdomains tends to zero:

$$\iiint\limits_T f(x, y, z)dV = \lim_{\max d_k \to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k)\Delta V_k.$$

For a function continuous in the domain T this limit exists and does not depend on the method of partitioning the domain T into subdomains or on the choice of the points P_k (*theorem on the existence* of a triple integral).

If $f(x, y, z) > 0$ in the domain T, then the triple integral $\iiint\limits_T f(x, y, z)dV$ is the *mass* of the body occupying the domain T and possessing a variable density $\gamma = f(x, y, z)$ (physical interpretation of a triple integral).

The main properties of triple integrals are similar to those of double integrals.

In the Cartesian coordinates the triple integral is usually written as $\iiint\limits_T f(x, y, z)dx\,dy\,dz$.

Assume that the domain if integration T is specified by the inequalities $x_1 \leqslant x \leqslant x_2$, $y_1(x) \leqslant y \leqslant y_2(x)$, $z_1(x, y) \leqslant z \leqslant z_2(x, y)$, where $y_1(x)$, $y_2(x)$, $z_1(x, y)$ and $z_2(x, y)$ are continuous functions. Then the triple integral of the function $f(x, y, z)$ extended to the domain T can be calculated by the formula

$$\iiint\limits_T f(x, y, z)dxdydz = \int_{x_1}^{x_2} dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z)dz.$$

If, while calculating the triple integral, it is necessary to pass from the variables x, y, z to new variables u, v, w connected with x, y, z by the relations $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$, where the functions $x(u, v, w)$, $y(u, v, w)$, $z(u, v, w)$, continuous together with their first-order partial derivatives, establish a mutual one-to-one correspondence, continuous in both directions, between the points of the domain T of the $Oxyz$ space and the points of some domain T' of the $Ouvw$ space, and Jacobian J of the domain T does not vanish

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} \neq 0,$$

then use is made of the formula

$$\iiint\limits_T f(x, y, z)dxdydz = \iiint\limits_{T'} f[x(u, v, w), y(u, v, w), z(u, v, w)] \cdot |J|\,dudvdw.$$

In particular, in passing from the Cartesian coordinates x, y, z to the *cylindrical coordinates* ρ, φ, z (Fig. 15), connected with x, y, z by the relations

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi,$$

$$z = z (0 \leqslant \rho < +\infty, \quad 0 \leqslant \varphi < 2\pi, -\infty < z < +\infty),$$

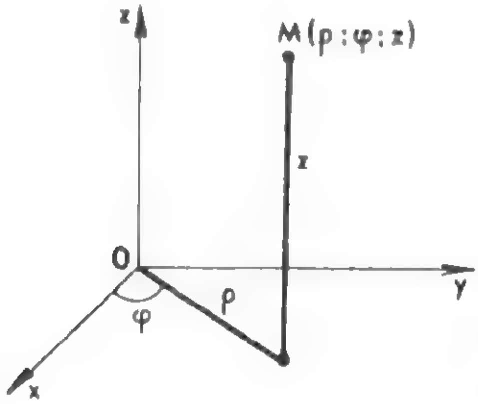


Fig. 15

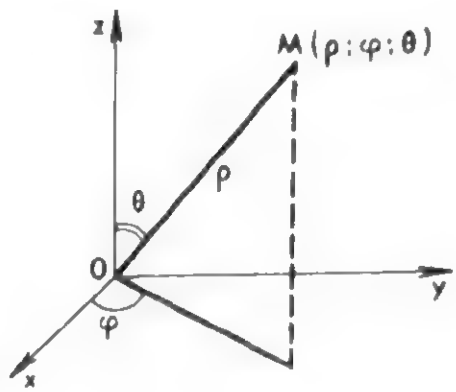


Fig. 16

the transformation Jacobian $J = B\rho$, and the *formula of transformation of the triple integral to the cylindrical coordinates* has the form

$$\iiint_T f(x, y, z)dx\,dy\,dz = \iiint_T f(\rho\cos\varphi, \rho\sin\varphi, z)\rho\,d\rho\,d\varphi\,dz.$$

In passing from the Cartesian coordinates x, y, z to the *spherical coordinates* ρ, φ, θ (Fig. 16), connected with x, y, z by the relations

$$x = \rho\sin\theta\cos\varphi, \quad y = \rho\sin\theta\sin\varphi,$$

$$z = \rho\cos\theta \quad (0 \leqslant \rho < +\infty, 0 \leqslant \varphi < 2\pi, 0 \leqslant \theta < \pi),$$

the transformation Jacobian $J = \rho^2\sin\theta$, and the *formula for transformation of the triple integral to the spherical coordinates* has the form

$$\iiint_T f(x, y, z)dx\,dy\,dz = \iiint_T f(\rho\sin\theta\cos\varphi, \rho\sin\theta\sin\varphi, \rho\cos\theta)\rho^2\sin\theta\,d\rho\,d\varphi\,d\theta.$$

91. Calculate $I = \iiint_T z\,dx\,dy\,dz$, where the domain T is specified by the inequalities $0 \leqslant x \leqslant 1/2, x \leqslant y \leqslant 2x, 0 \leqslant z \leqslant \sqrt{1 - x^2 - y^2}$.

Solution. We have

$$I = \int_0^{1/2} dx \int_x^{2x} dy \int_0^{\sqrt{1-x^2-y^2}} z\,dz = \frac{1}{2}\int_0^{1/2} dx \int_x^{2x} z^2 \Big|_0^{\sqrt{1-x^2-y^2}} dy$$

$$= \frac{1}{2}\int_0^{1/2} dx \int_x^{2x} (1 - x^2 - y^2)dy = \frac{1}{2}\int_0^{1/2} \left[y - yx^2 - \frac{1}{3}y^3\right]_x^{2x} dx$$

$$= \frac{1}{2}\int_0^{1/2} (2x - 2x^3 - \frac{8}{3}x^3 - x + x^3 + \frac{1}{3}x^3)dx$$

$$= \frac{1}{2}\int_0^{1/2} \left(x - \frac{10}{3}x^3\right) dx = \frac{1}{2}\left[\frac{1}{2}x^2 - \frac{5}{6}x^4\right]_0^{1/2} = \frac{1}{2}\left(\frac{1}{8} - \frac{5}{6}\cdot\frac{1}{16}\right) = \frac{7}{192}.$$

92. Calculate $I = \iiint\limits_T x^2 dxdydz$, if T is a sphere $x^2 + y^2 + z^2 \leqslant R^2$.

Solution. Let us pass to spherical coordinates. In the domain T the coordinates ρ, φ and θ vary as follows: $0 \leqslant \rho \leqslant R$, $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant \theta < \pi$. Consequently,

$$I = \iiint\limits_T \rho^4 \sin^3\theta \cos^2\varphi \, d\rho \, d\varphi \, d\theta = \int\limits_0^{\pi} \sin^3\theta \, d\theta \int\limits_0^{2\pi} \cos^2\varphi \, d\varphi \int\limits_0^{R} \rho^4 d\rho$$

$$= \frac{R^5}{5\cdot 2} \int\limits_0^{\pi} \sin^3\theta \, d\theta \left[\varphi + \frac{1}{2}\sin 2\varphi\right]_0^{2\pi} = \frac{\pi R^5}{5} \int\limits_0^{\pi} (\cos^2\theta - 1) d(\cos\theta) = \frac{4\pi R^5}{15}.$$

93. Calculate $\iiint\limits_T z\sqrt{x^2 + y^2}\, dxdydz$, if the domain T is bounded by the cylinder $x^2 + y^2 = 2x$ and by the planes $y = 0$, $z = 0$, $z = a$.

Solution. Let us pass to cylindrical coordinates. In these coordinates the equation of the cylinder assumes the form $\rho^2\cos^2\varphi + \rho^2\sin^2\varphi = 2\rho\cos\varphi$, or $\rho^2(\cos^2\varphi + \sin^2\varphi) = 2\rho\cos\varphi$, i.e. $\rho = 2\cos\varphi$. Consequently, in the domain T the coordinates ρ, φ and z vary as follows: $0 \leqslant \rho \leqslant 2\cos\varphi$, $0 \leqslant \varphi < \pi/2$, $0 \leqslant z \leqslant a$. Therefore,

$$\iiint\limits_T z\sqrt{x^2 + y^2}\, dxdydz = \iiint\limits_T z\rho \cdot \rho \, d\rho \, d\varphi \, dz$$

$$= \int\limits_0^{\pi/2} d\varphi \int\limits_0^{2\cos\varphi} \rho^2 d\rho \int\limits_0^{a} z\,dz = \frac{1}{2}a^2 \int\limits_0^{\pi/2} d\varphi \int\limits_0^{2\cos\varphi} \rho^2 d\rho$$

$$= \frac{4}{3}a^2 \int\limits_0^{\pi/2} \cos^3\varphi \, d\varphi = \frac{4}{3}a^2 \int\limits_0^{\pi/2} (1 - \sin^2\varphi) \cdot d(\sin\varphi)$$

$$= \frac{4}{3}a^2 \left[\sin\varphi - \frac{1}{3}\sin^3\varphi\right]_0^{\pi/2} = \frac{8}{9}a^2.$$

94. Calculate $\iiint\limits_T (x^2 + y^2)\, dxdydz$, if the domain T is the upper half of the sphere $x^2 + y^2 + z^2 \leqslant r^2$.

Solution. Let us introduce spherical coordinates; the new variables vary in the limits $0 \leqslant \rho \leqslant r$, or $0 \leqslant \varphi < 2\pi$, $0 \leqslant \theta < \pi/2$. Thus we have

$$\iiint\limits_T (x^2 + y^2)\, dxdydz = \iiint\limits_T \rho^4 \sin^3\theta \, d\rho \, d\varphi \, d\theta$$

$$= \int\limits_0^{r} \rho^4 d\rho \int\limits_0^{\pi/2} \sin^3\theta \, d\theta \int\limits_0^{2\pi} d\varphi = 2\pi \int\limits_0^{r} \rho^4 d\rho \int\limits_0^{\pi/2} (\cos^2\theta - 1) d(\cos\theta)$$

$$= 2\pi \int\limits_0^{r} \rho^4 d\rho \left[\frac{1}{3}\cos^3\theta - \cos\theta\right]_0^{\pi/2} = \frac{4}{15}\pi r^5.$$

95. Calculate $\iiint\limits_T (x^2 + y^2 + z^2)\, dxdydz$, if the domain T is a right parallelepiped specified by the inequalities $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, $0 \leqslant z \leqslant c$.

96. Calculate $\iiint\limits_T xyz\,dxdydz$, if the domain T is bounded by the sphere $x^2 + y^2 + z^2 = 1$ and by the planes $x = 0, y = 0, z = 0$.

97. Calculate $\iiint\limits_T xy^2z^3\,dxdydz$, if the domain T is bounded by the surfaces $z = xy, y = x, x = 1, z = 0$.

98. Calculate $\iiint\limits_T (2x + 3y - z)\,dxdydz$, if the domain T is a triangular prism bounded by the planes $z = 0$, $z = a, x = 0, y = 0, x + y = b$ $(a > 0, b > 0)$.

99. Calculate $\iiint\limits_T (x^2 + y + z^2)^3\,dxdydz$, if the domain T is bounded by the cylinder $x^2 + z^2 = 1$ and by the planes $y = 0, y = 1$.

100. Calculate $\iiint\limits_T (x + y + z)^2\,dxdydz$, where the domain T is the common part of the paraboloid $z \geqslant (x^2 + y^2)/(2a)$ and the sphere $x^2 + y^2 + z^2 \leqslant 3a^2$.

101. Calculate $\iiint\limits_T (x^2 + y^2)\,dxdydz$, where the domain T is bounded by the surfaces $z = (x^2 + y^2)/2, z = 2$.

102. Calculate $\iiint\limits_T dxdydz$, where the domain T is the sphere $x^2 + y^2 + z^2 \leqslant r^2$.

103. Calculate $\iiint\limits_T \sqrt{1 + (x^2 + y^2 + z^2)^{3/2}}\,dxdydz$, if T is the sphere $x^2 + y^2 + z^2 \leqslant 1$.

1.8. Applications of a Triple Integral

The *volume of the body* occupying the domain T is determined by the formula

$$V = \iiint\limits_T dxdydz.$$

If the density of the body is a variable quantity, i.e. $\gamma = \gamma(x, y, z)$, then the *mass* of the body, occupying the domain T, is determined by the formula

$$M = \iiint\limits_T \gamma(x, y, z)\,dxdydz.$$

The *coordinates of the centre of gravity* of the body are specified by the formulas

$$\bar{x} = \frac{1}{M}\iiint\limits_T \gamma x\,dxdydz, \quad \bar{y} = \frac{1}{M}\iiint\limits_T \gamma y\,dxdydz, \quad \bar{z} = \frac{1}{M}\iiint\limits_T \gamma z\,dxdydz.$$

For $\gamma = 1$, we have

$$\bar{x} = \frac{1}{V}\iiint\limits_T x\,dxdydz, \quad \bar{y} = \frac{1}{V}\iiint\limits_T y\,dxdydz, \quad \bar{z} = \frac{1}{V}\iiint\limits_T z\,dxdydz$$

($\bar{x}, \bar{y}, \bar{z}$ are the coordinates of the geometrical centre of gravity).

The *moments of inertia* (geometrical) about the coordinate axes are equal, respectively, to

$$I_x = \iiint_T (y^2 + z^2)dxdydz, \quad I_y = \iiint_T (z^2 + x^2)dxdydz,$$
$$I_z = \iiint_T (x^2 + y^2)dxdydz.$$

104. Compute the volume of the body bounded by the surfaces $hz = x^2 + y^2$, $z = h$ (Fig. 17).

Solution. The given body is bounded below by the paraboloid $z = (x^2 + y^2)/h$, above by the plane $z = h$, and is projected onto the circle $x^2 + y^2 \leqslant h^2$ of the xOy plane. We use the cylindrical coordinates in which the equation of the paraboloid assumes the form $z = \rho^2/h$. The volume of the body is equal to

$$V = \iiint_T dxdydz = \iiint_T \rho d\rho d\varphi dz = \int_0^{2\pi} d\varphi \int_0^h \rho d\rho \int_{\rho^2/h}^h dz$$
$$= \int_0^{2\pi} d\varphi \int_0^h \left(h - \frac{\rho^2}{h}\right) \rho d\rho = \int_0^{2\pi} \left[\frac{h\rho^2}{2} - \frac{\rho^4}{4h}\right]_0^h d\varphi$$
$$= \left(\frac{h^3}{2} - \frac{h^3}{4}\right) \int_0^{2\pi} d\varphi = \frac{\pi h^3}{2}.$$

105. Find the coordinates of the centre of gravity of a prismatic body bounded by the planes $x = 0$, $z = 0$, $y = 1$, $y = 3$, $x + 2z = 3$.

Solution. We find the volume of the body in question:

$$V = \iiint_T dxdydz = \int_0^3 dx \int_1^3 dy \int_0^{(3-x)/2} dz$$
$$= \int_0^3 dx \int_1^3 \frac{3-x}{2} dy = \int_0^3 (3 - x)dx = \left[3x - \frac{1}{2}x^2\right]_0^3 = \frac{9}{2}.$$

Then we have

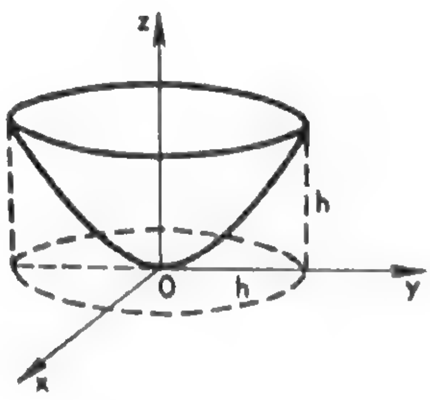


Fig. 17

$$\bar{x} = \frac{2}{9}\iiint\limits_T x\,dx\,dy\,dz = \frac{2}{9}\int\limits_0^3 x\,dx \int\limits_1^3 dy \int\limits_0^{(3-x)/2} dz$$

$$= \frac{2}{9}\int\limits_0^3 x\,dx \int\limits_1^3 \frac{3-x}{2}\,dy = \frac{2}{9}\int\limits_0^3 x(3-x)\,dx = \frac{2}{9}\left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3 = 1;$$

$$\bar{y} = \frac{2}{9}\iiint\limits_T y\,dx\,dy\,dz = \frac{2}{9}\int\limits_0^3 dx \int\limits_1^3 y\,dy \int\limits_0^{(3-x)/2} dz$$

$$= \frac{1}{9}\int\limits_0^3 dx \int\limits_1^3 y(3-x)\,dy = \frac{4}{9}\int\limits_0^3 (3-x)\,dx = \frac{4}{9}\left[3x - \frac{x^2}{2}\right]_0^3 = 2;$$

$$\bar{z} = \frac{2}{9}\iiint\limits_T z\,dx\,dy\,dz = \frac{2}{9}\int\limits_0^3 dx \int\limits_0^3 dy \int\limits_0^{(3-x)/2} z\,dz$$

$$= \frac{2}{9}\int\limits_0^3 \frac{(3-x)^2}{8}\,dx \int\limits_1^3 dy = \frac{1}{18}\left[\frac{-(3-x)^3}{3}\right]_0^3 = \frac{1}{2}.$$

106. Compute the volume of the body bounded by the surfaces $z = \sqrt{x^2 + y^2}$, $z = x^2 + y^2$.

107. Compute the volume of the body bounded by the plane $z = 0$, by the cylindrical surface $x = (x^2 + y^2)/2$ and by the sphere $x^2 + y^2 + z^2 = 4$ (in the interior of the cylinder).

108. Find the mass of the cube $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant a$, $0 \leqslant z \leqslant a$, if the density at the point (x, y, z) is $\gamma(x, y, z) = x + y + z$.

109. Find the coordinates of the centre of gravity of the body bounded by the surfaces $x + y = 1$, $z = x^2 + y^2$, $x = 0$, $y = 0$, $z = 0$.

110. Find the coordinates of the centre of gravity of the body bounded by the surfaces $z^2 = xy$, $x = 5$, $y = 5$, $z = 0$.

111. Find the coordinates of the centre of gravity of the body bounded by the planes $2x + 3y - 12 = 0$, $x = 0$, $y = 0$, $z = 0$ and by the cylindrical surface $z = y^2/2$.

112. Find the moment of inertia of the cube $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant a$, $0 \leqslant z \leqslant a$ about the edge of the cube.

1.9. Integrals Dependent on a Parameter. Differentiation and Integration under the Integral Sign

Let us consider the integral

$$I(\lambda) = \int\limits_a^b f(x, \lambda)\,dx, \tag{1}$$

in which λ is a variable parameter, and the function $f(x, \lambda)$ of two variables is defined for all the values of x in the interval $[a, b]$ and all the values of λ on the set $\{\lambda\}$. Under these conditions, integral (1) is a function of the parameter λ.

The question concerning the derivative of the function $I(\lambda)$ with respect to the parameter λ is of great significance. Assume that the function $f(x, \lambda)$ and the partial derivative $\dfrac{\partial f(x, \lambda)}{\partial \lambda}$ are continuous on the rectangle $a \leqslant x \leqslant b$, $\alpha \leqslant \lambda \leqslant \beta$. In this case, there exists a derivative

$$\frac{dI(\lambda)}{d\lambda} = \frac{\lambda}{d\lambda} \int_a^b f(x, \lambda)dx = \int_a^b \frac{\partial f(x, \lambda)}{\partial \lambda} dx \tag{2}$$

If it is admissible to interchange the signs of differentiation (with respect to λ) and integration (with respect to x), then it is said that function (1) can be *differentiated with respect to the parameter under the integral sign*. It is assumed that the limits of integration a and b in formula (2) do not depend on the parameter λ. Now if a and b depend on the parameter λ, then

$$\frac{dI(\lambda)}{d\lambda} = \int_{a(\lambda)}^{b(\lambda)} \frac{\partial f(x, \lambda)}{\partial \lambda} dx + b'(\lambda) f[b(\lambda), \lambda] - a'(\lambda) f[a(\lambda), \lambda]. \tag{3}$$

To differentiate, with an infinite limit of integration with respect to the parameter, the improper integral $\int_0^\infty f(x, \lambda)dx$, it is necessary that the integrals $\int_0^\infty f(x, \lambda)dx$ and $\int_0^\infty \dfrac{\partial f(x, \lambda)}{\partial \lambda} dx$, exist for $0 < \lambda < \infty$.

The formula for integration of the definite integral (1) under the integral sign with respect to the parameter λ in the interval $[\alpha, \beta]$ has the form

$$\int_\alpha^\beta I(\lambda)d\lambda = \int_\alpha^\beta d\lambda \int_a^b f(x, \lambda)dx = \int_a^b dx \int_\alpha^\beta f(x, \lambda)d\lambda. \tag{4}$$

The integrand $f(x, \lambda)$ must be a continuous function of two variables in the finite domain of integration. If we have an infinite domain of integration, then we deal with the improper multiple integral.

113. Find $\int_0^1 x^m (\ln x)^n dx$, where m and n are positive integers.

Solution. Let us consider the integral

$$\int_0^1 x^m dx = \frac{1}{m + 1};$$

here $f(x, m) = x^m$ is a continuous function in the interval $0 < x < 1$ for $m > 0$. Let

us find the derivative of this integral with respect to m:

$$\frac{d}{dm}\int_0^1 x^m dx = \int_0^1 x^m \ln x dx = -\frac{1}{(m+1)^2}.$$

Differentiating once again with respect to m, we get

$$\int_0^1 x^m (\ln x)^2 dx = \frac{2!}{(m+1)^3}.$$

After the n-fold differentiation with respect to m, we find

$$\int_0^1 x^m (\ln x)^n dx = (-1)^n \cdot \frac{n!}{(m+1)^{n+1}}.$$

114. Find $\int_0^\infty \frac{dx}{(x^2+\lambda)^{n+1}}$, where n is a positive integer and $\lambda > 0$.

Solution. Let us consider the integral

$$\int_0^\infty \frac{dx}{x^2+\lambda} = \frac{1}{\sqrt{\lambda}} \arctan \frac{x}{\sqrt{\lambda}} \Big|_0^\infty = \frac{\pi}{2} \cdot \frac{1}{\lambda^{1/2}}.$$

Differentiating with respect to the parameter λ, we have

$$\int_0^\infty \frac{dx}{(x^2+\lambda)^2} = \frac{1}{2} \cdot \frac{\pi}{2\lambda^{3/2}}.$$

Differentiating n times, we get

$$\int_0^\infty \frac{dx}{(x^2+\lambda)^{n+1}} = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)} \cdot \frac{\pi}{2\lambda^n \cdot \sqrt{\lambda}}.$$

115. Find $I(k,\lambda) = \int_0^\infty e^{-kx}\frac{\sin\lambda x}{x} dx$ and $I_1(\lambda) = \int_0^\infty \frac{\sin\lambda x}{x} dx$.

Solution. Differentiating the integral I with respect to λ, we find

$$\frac{dI}{d\lambda} = \int_0^\infty e^{-kx}\cos\lambda x dx$$

$$= \left[\frac{e^{-kx}}{k^2+\lambda^2}(\lambda\sin\lambda x - k\cos\lambda x)\right]_0^\infty = \frac{k}{k^2+\lambda^2}.$$

Now we can find I from the equation $\frac{dI}{d\lambda} = \frac{k}{k^2+\lambda^2}$; we have

$$I(k,\lambda) = \int_0^\infty e^{-kx}\frac{\sin\lambda x}{x} dx = \arctan\frac{\lambda}{k}.$$

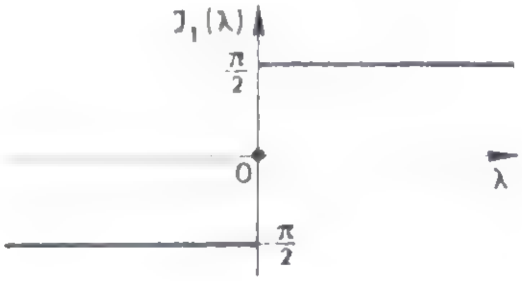


Fig. 18

We can find the integral $I_1(\lambda)$ by substituting the value $k = 0$ into the expression for $I(k, \lambda)$:

$$I_1(\lambda) = \int_0^\infty \frac{\sin \lambda x}{x}\,dx = \lim_{k \to +0} \arctan\frac{\lambda}{k} = \begin{cases} -\pi/2 & \text{for} \quad \lambda < 0, \\ 0 & \text{for} \quad \lambda = 0, \\ \pi/2 & \text{for} \quad \lambda > 0. \end{cases}$$

The graph of the function $I_1(\lambda) = \int_0^\infty \frac{\sin \lambda x}{x}\,dx$ consists of two half-lines and the point 0 (Fig. 18).

116. Find $I = \int_0^\infty \frac{e^{-x} - e^{-\lambda x}}{x}$.

Solution. Differentiating with respect to the parameter λ, we have

$$\frac{dI}{d\lambda} = \int_0^\infty e^{-\lambda x}\,dx = \frac{1}{\lambda}, \quad \text{i.e.} \quad \frac{dI}{d\lambda} = \frac{1}{\lambda}, \quad I = \ln \lambda.$$

117. Compute $I = \int_0^\infty e^{-x^2}dx$ (*Euler-Poisson integral*).

Solution. We get $x = \lambda t$, where $\lambda > 0$; then $dx = \lambda dt$ and $I = \lambda \int_0^\infty e^{-\lambda^2 t^2}dt$. We multiply both sides of the last equation by $e^{-\lambda^2}d\lambda$, and, using formula (4), integrate with respect to λ from 0 to ∞:

$$I \cdot \int_0^\infty e^{-\lambda^2}d\lambda = I^2 = \int_0^\infty e^{-\lambda^2} \cdot \lambda d\lambda \int_0^\infty e^{-\lambda^2 t^2}dt.$$

Changing the order of integration, we get

$$I^2 = \int_0^\infty dt \int_0^\infty e^{-(1+t^2)\lambda^2} \cdot \lambda d\lambda = \int_0^\infty \left[-\frac{1}{2(1+t^2)} e^{-(1+t^2)\lambda^2} \right]_0^\infty dt$$

$$= \frac{1}{2} \int_0^\infty \frac{dt}{1+t^2} = \frac{1}{2} \arctan t \,\Big|_0^\infty = \frac{\pi}{4}, \quad \text{i.e.} \quad I = \frac{\sqrt{\pi}}{2}.$$

118. Find $I(\lambda) = \int\limits_0^\infty e^{-x^2 - \lambda^2/x^2}dx.$

Solution. Differentiating with respect to the parameter λ, we have

$$\frac{dI}{d\lambda} = -2\int_0^\infty e^{-x^2 - \lambda^2/x^2}\lambda \cdot \frac{dx}{x^2}.$$

Then we change the variable of integration: $\lambda/x = z$, $(-\lambda/x^2)\,dx = dz$, $x^2 = \lambda^2/z^2$, with z varying from ∞ to 0. Thus we have

$$\frac{dI}{d\lambda} = 2\int_\infty^0 e^{-\lambda^2/z^2 - z^2}dz = -2\int_0^\infty e^{-\lambda^2/z^2 - z^2}dz, \quad \text{or} \quad \frac{dI}{d\lambda} = -2I.$$

Consequently, $\dfrac{dI}{I} = -2d\lambda$, $\ln I = -2\lambda + \ln C$, $I = Ce^{-2\lambda}$. To find C, we put $\lambda = 0$; then $I(0) = \int\limits_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$ (Euler-Poisson integral), i.e. $C = \sqrt{\pi}/2$.

Thus, the sought-for integral is $I = \dfrac{\sqrt{\pi}}{2}e^{-2\lambda}$.

119. Find $I = \displaystyle\int_0^\lambda \frac{\ln(1 + \lambda x)}{1 + x^2}\,dx.$

Solution. We find the total derivative $\dfrac{dI}{d\lambda}$ from formula (3):

$$\frac{dI}{d\lambda} = \int_0^\lambda \frac{x}{(1 + \lambda x)(1 + x^2)}\,dx + \frac{\ln(1 + \lambda\cdot\lambda)}{1 + \lambda^2}\cdot\frac{d\lambda}{d\lambda},$$

or

$$\frac{dI}{d\lambda} = \frac{\ln(1 + \lambda^2)}{1 + \lambda^2} + \int_0^\lambda \frac{x}{(1 + \lambda x)(1 + x^2)}\,dx.$$

Then we decompose the element of integration into partial fractions and integrate:

$$\int_0^\lambda \frac{x}{(1 + \lambda x)(1 + x^2)}\,dx = \int_0^\lambda \frac{-\lambda dx}{(1 + \lambda^2)(1 + \lambda x)} + \int_0^\lambda \frac{x + \lambda}{(1 + \lambda^2)(1 + x^2)}\,dx$$

$$= \left[-\frac{1}{1 + \lambda^2}\ln(1 + \lambda x) + \frac{1}{2(1 + \lambda^2)}\ln(1 + x^2) + \frac{\lambda}{1 + \lambda^2}\arctan x\right]_0^\lambda$$

$$= -\frac{\ln(1 + \lambda)^2}{1 + \lambda^2} + \frac{\ln(1 + \lambda^2)}{2(1 + \lambda^2)} + \frac{\lambda}{1 + \lambda^2}\arctan\lambda.$$

It follows that

$$\frac{dI}{d\lambda} = \frac{\ln(1 + \lambda^2)}{2(1 + \lambda^2)} + \frac{\lambda}{1 + \lambda^2}\arctan\lambda.$$

Hence

$$I = \int_0^{\lambda} \left[\frac{\ln(1+\lambda^2)}{2(1+\lambda^2)} + \frac{\lambda}{1+\lambda^2} \arctan\lambda \right] d\lambda.$$

Designating $\lambda = \tan\varphi$, we get

$$I = \int_0^{\varphi} \frac{\ln\sec^2\varphi}{2\sec^2\varphi} \sec^2\varphi\, d\varphi + \int_0^{\varphi} \frac{\tan\varphi}{\sec^2\varphi} \cdot \varphi\sec^2\varphi\, d\varphi$$

$$= -\int_0^{\varphi} \ln\cos\varphi\, d\varphi + \int_0^{\varphi} \varphi\tan\varphi\, d\varphi.$$

Taking the first integral by parts, we find

$$I = -\varphi\ln\cos\varphi \Big|_0^{\varphi} - \int_0^{\varphi} \varphi\tan\varphi\, d\varphi + \int_0^{\varphi} \varphi\tan\varphi\, d\varphi = -\varphi\ln\cos\varphi,$$

or, finally,

$$I = \frac{1}{2}\arctan\lambda \cdot \ln(1+\lambda^2).$$

Find the following integrals.

120. $\int_0^{\pi/2} \frac{\arctan(\lambda\sin x)}{\sin x} dx$. **121.** $\int_0^1 \frac{\arctan\lambda x}{x\sqrt{1-x^2}} dx$. **122.** $\int_0^{\pi} \frac{\arctan(\lambda\tan x)}{\tan x} dx$.

123. $\int_0^{\pi} \ln(1+\sin\alpha\cos x)\frac{dx}{\cos x}$. **124.** $\int_0^{\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx$. **125.** $\int_0^{\infty} \frac{\arctan\lambda x}{x(1+x^2)} dx$.

126. $\int_0^1 \frac{x^{\lambda} - x^{\mu}}{\ln x} dx;\ \lambda > 0, \mu > 0$. **127.** $\int_0^{\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x^2} dx;\ \alpha > 0, \beta > 0$.

128. $\int_0^1 \frac{\ln(1-\lambda^2x^2)}{x^2\sqrt{1-x^2}} dx;\ \lambda^2 < 1$.

1.10. Gamma-Function. Beta-Function

1.10.1. Gamma-function. The *gamma-function* (or *Euler's integral of the second kind*) is an integral of the form

$$\Gamma(p) = \int_0^{\infty} e^{-x}x^{p-1}dx. \tag{1}$$

Integral (1), which is a function of the parameter p, is improper since the upper limit is equal to infinity and, besides, the integrand increases indefinitely when $x \to 0$ and $p < 1$. Integral (1) converges for $p > 0$ and diverges for $p \leqslant 0$. The gamma-function is second only to elementary functions in its importance for analysis and its applications.

Main properties of the gamma-function

1°. The function $\Gamma(p)$ is continuous and possesses a continuous derivative $\Gamma'(p)$ for $p > 10$.

2°. There holds an equality

$$\Gamma(p + 1) = p\Gamma(p). \tag{2}$$

3°. The nth application of formula (2) results in a relation

$$\Gamma(p + n) = (p + n - 1)(p + n - 2) \dots (p + 1)p \cdot \Gamma(p). \tag{3}$$

4°. If we put $p = 1$ in formula (3) and take into account that $\Gamma(1) = \int\limits_0^\infty e^{-x}dx = 1$, we get an equality

$$\Gamma(n + 1) = n!. \tag{4}$$

If $n = 0$, then $0! = \Gamma(1) = 1$.

5°. The function $\Gamma(p)$ makes it possible to extend the notion of the factorial $n!$, defined only for the natural values of n, to the domain of any positive values of the argument. It follows from formula (2) that $\Gamma(p) = \dfrac{\Gamma(p + 1)}{p} \to +\infty$, as $p \to 0$, i.e. $\Gamma(0) = +\infty$.

6°. When $p = -n$, formula (2) yields

$$\Gamma(-n) = \frac{\Gamma(-n + 1)}{-n} = \frac{\Gamma(-n + 2)}{n(n - 1)} = -\frac{\Gamma(-n + 3)}{n(n - 1)(n - 2)}$$
$$= \dots = (-1)^n \frac{\Gamma(0)}{n!} = (-1)^n \cdot \infty,$$

i.e. $\Gamma(-n) = (-1)^n \cdot \infty$ $(n = 1, 2, 3, \dots)$.

7°. In a general case, the function $\Gamma(p)$ can be extended to the case of the negative values of the argument p. Since $\Gamma(p) = \dfrac{\Gamma(p + 1)}{p}$, it follows that $\Gamma(p + 1)$ is meaningful at $-1 < p < 0$.

If $-n < p < -(n - 1)$, then formula (3) yields

$$\Gamma(p) = \frac{\Gamma(p + n)}{p(p + 1)(p + 2) \dots (p + n - 1)}.$$

By means of the substitution $p + n = \alpha$, whence $p = -n + \alpha$, the last formula can be transformed to

$$\Gamma(\alpha - n) = \frac{(-1)^n \Gamma(\alpha)}{(1-\alpha)(2-\alpha)\dots(n-\alpha)} \tag{5}$$

and for $-n < p < -(n-1)$ the sign of $\Gamma(p)$ is defined by the factor $(-1)^n$.

8°. Using formula (2) it is possible to obtain the values of $\Gamma(p)$ for the half-argument:

$$\Gamma\left(m+\frac{1}{2}\right) = \Gamma\left[1+\left(m-\frac{1}{2}\right)\right]$$

$$= \left(m-\frac{1}{2}\right)\cdot\Gamma\left(m-\frac{1}{2}\right)$$

$$= \left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right)\cdot\Gamma\left(m-\frac{3}{2}\right)$$

$$= \dots = \left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right)\dots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{(2m-1)(2m-3)\dots5\cdot3\cdot1}{2^m}\Gamma\left(\frac{1}{2}\right),$$

or

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{(2m-1)!!}{2^m}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{(2m)!}{m!2^{2m}}\Gamma\left(\frac{1}{2}\right). \tag{6}$$

9°. There holds the *complement formula*

$$\Gamma(p)\cdot\Gamma(1-p) = \frac{\pi}{\sin p\pi} \quad (0 < p < 1). \tag{7}$$

If we set $p = 1/2$ in this formula, then $[\Gamma(1/2)]^2 = \pi/\sin(\pi/2) = \pi$, i.e. $\Gamma(1/2) = \sqrt{\pi}$.

Proceeding from the main properties, it is possible to compute $\Gamma(p)$ for any p. The values of the gamma-function are given in Table 1 on p. 478.

The graph of the function $\Gamma(p)$ is presented in Fig. 19.

129. Compute the Euler-Poisson integral $\int_0^\infty e^{-x^2}dx$.

Solution. We make the substitution $x^2 = t$, whence we have $x = \sqrt{t}$, $dx = dt/(2\sqrt{t})$ and, consequently,

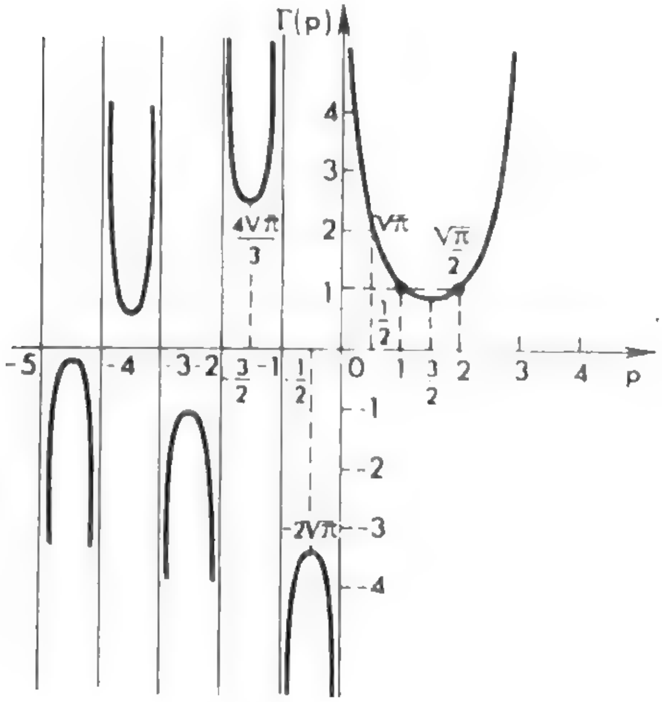


Fig. 19

$$\int_0^\infty e^{-x^2}dx = \frac{1}{2}\int_0^\infty e^{-t}t^{1/2}dt = \frac{1}{2}\int_0^\infty e^{-t}t^{1/2-1}dt$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

130. Compute $\Gamma(-1/2)$.

Solution. Using the formula $\Gamma(p) = \dfrac{\Gamma(n+1)}{p}$, we get

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma(-1/2+1)}{(-1/2)} = \frac{\Gamma(1/2)}{(-1/2)} = -2\sqrt{\pi}.$$

131. Compute $\Gamma(-9/2)$.

Solution. Using formula (5) for $\alpha = 1/2$ and $n = 5$, we get

$$\Gamma\left(\frac{1}{2}-5\right) = \Gamma\left(-\frac{9}{2}\right) = \frac{(-1)^5\cdot\Gamma(1/2)}{(1-1/2)(2-1/2)\ldots(5-1/2)}$$

$$= \frac{-\sqrt{\pi}}{(1/2)\cdot(3/2)\cdot(5/2)\cdot(7/2)\cdot(9/2)} = -\frac{32\sqrt{\pi}}{945}.$$

132. Compute $\Gamma(5/2)$.

Solution. Setting $m = 2$ in formula (6), we have

$$\Gamma\left(2+\frac{1}{2}\right) = \Gamma\left(\frac{5}{2}\right) = \frac{4!\Gamma(1/2)}{2!\cdot 2^4} = \frac{24\sqrt{\pi}}{2\cdot 16} = \frac{3}{4}\sqrt{\pi}.$$

133. Compute $\Gamma(-4/3)$.

Solution. Using the relation $\Gamma(p) = \dfrac{\Gamma(p+1)}{p}$, we have

$$\Gamma\left(-\frac{4}{3}\right) = \frac{\Gamma(-4/3+1)}{-4/3} = \frac{\Gamma(-1/3)}{-4/3} = \frac{\Gamma(-1/3+1)}{(-4/3)\cdot(-1/3)}$$

$$= \frac{9}{4}\Gamma\left(\frac{2}{3}\right) = \frac{9}{4}\cdot\frac{\Gamma(5/3)}{2/3} = \frac{27}{8}\Gamma\left(\frac{5}{3}\right).$$

With the aid of Table 1 on p. 478 we find that $\Gamma(5/3) = 0.9033$; consequently, $\Gamma(-4/3) = (27/8)\cdot 0.9033 = 3.0486$.

134. Compute: (1) $(-1/2)!$; (2) $(1/2)!$; (3) $(3/2)!$; (4) $(0.21)!$.

Solution. We find by formula (4):

(1) $(-1/2)! = \Gamma(-1/2+1) = \Gamma(1/2) = \sqrt{\pi} = 1.772$;

(2) $(1/2)! = \Gamma(1/2+1) = \Gamma(3/2) = (1/2)\Gamma(1/2) = \sqrt{\pi}/2 = 0.886$;

(3) $(3/2)! = \Gamma(3/2+1) = (3/2)\Gamma(3/2) = (3/2)\cdot(1/2)\Gamma(1/2) = 3\sqrt{\pi}/4 = 1.329$;

(4) $(0.21)! = \Gamma(0.21+1) = \Gamma(1.21) = 0.9156$ (from Table 1).

135. Compute $\Gamma(5/3)\cdot\Gamma(-5/3)$.

Solution. We find

$$\Gamma\left(\frac{5}{3}\right)\cdot\Gamma\left(-\frac{5}{3}\right) = \frac{2}{3}\Gamma\left(\frac{2}{3}\right)\cdot\frac{\Gamma(-2/3)}{-5/3}$$

$$= \frac{2}{3}\Gamma\left(\frac{2}{3}\right)\frac{\Gamma(1/3)}{(-5/3)(-2/3)} = \frac{3}{5}\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right).$$

Since the complement formula yields $\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right) = \dfrac{\pi}{\sin(\pi/3)} = \dfrac{2\pi}{\sqrt{3}}$, we have

$$\Gamma\left(\frac{5}{3}\right)\Gamma\left(-\frac{5}{3}\right) = \frac{3}{5}\cdot\frac{2\pi}{\sqrt{3}} = \frac{2\pi\sqrt{3}}{5}.$$

136. Show that $\Gamma\left(\frac{1}{2}+p\right)\cdot\Gamma\left(\frac{1}{2}-p\right) = \dfrac{\pi}{\cos p\pi}$.

Solution. Setting $p = \omega + 1/2$ in formula (7), we get

$$\Gamma\left(\omega+\frac{1}{2}\right)\cdot\Gamma\left[1-\left(\omega+\frac{1}{2}\right)\right] = \frac{\pi}{\sin(\pi/2+\omega\pi)},$$

or

$$\Gamma\left(\frac{1}{2}+\omega\right)\cdot\Gamma\left(\frac{1}{2}-\omega\right) = \frac{\pi}{\cos\omega\pi}.$$

Compute:

137. $\Gamma(0, 8)$. **138.** $\Gamma(-2.1)$. **139.** $\Gamma(3.2)$.
140. $\Gamma(7/2)$. **141.** $(-1/4)!$. **142.** $(1/3)!$. **143.** $(-2)!$.
144. $\Gamma(7/3) \cdot \Gamma(-7/3)$. **145.** $\Gamma(10/3) \cdot \Gamma(-10/3)$.
146. $\Gamma(1/4) \cdot \Gamma(-1/4)$. **147.** $\Gamma(5/4) \cdot \Gamma(-5/4)$.

148. Show that $\Gamma\left(-m + \frac{1}{2}\right) = \frac{(-2)^m}{(2m-1)!!}\sqrt{\pi}$.

149. Show that $\Gamma\left(m + \frac{1}{2}\right) \cdot \Gamma\left(-m + \frac{1}{2}\right) = (-1)^m \pi$ $(m = 1, 2, 3, \ldots)$.

1.10.2. Beta-function. The *beta-function* (or *Euler's integral of the first kind*) is the integral

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx. \tag{1}$$

Integral (1) is a function of two parameters, p and q, convergent for $p > 0$, $q > 0$.

The function B is symmetric with respect to the parameters, i.e. $B(p, q) = B(q, p)$.

If we perform a change of variable of integration, setting $x = \sin^2 t$, $dx = 2\sin t \times \cos t\, dt$, with t varying from 0 to $\pi/2$, then formula (1) assumes the form

$$B(p, q) = 2\int_0^{\pi/2} \sin^{2p-1}t \cos^{2q-1}t\, dt,$$

or

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right) \quad (m > 0, n > 0). \tag{2}$$

Many integrals encountered in applications can be reduced to integrals (1) and (2).

To compute the values of the beta-function, use is made of the following relationship between the beta-function and the gamma-function:

$$B(p, q) = \frac{\Gamma(p) \cdot \Gamma(q)}{\Gamma(p+q)}. \tag{3}$$

If $q = 1 - p$, then $B(p, 1-p) = \frac{\Gamma(p) \cdot \Gamma(1-p)}{\Gamma(1)} = \frac{\pi}{\sin p\pi}$ $(0 < p < 1)$.

Using a beta-function, it is easy to find the value of $\Gamma(1/2)$. Suppose $p = q = 1/2$; then $B(1/2, 1/2) = \frac{[\Gamma(1/2)]^2}{\Gamma(1)}$. Since $B(1/2, 1/2) = B(1/2, 1 - 1/2) = \pi/\sin(\pi/2) = \pi$, and $\Gamma(1) = 1$, it follows that $\Gamma(1/2) = \sqrt{\pi}$.

150. Compute $\int_0^{\pi/2} \sin^6 x \cos^8 x\, dx$.

Solution. Using formula (2) for $m = 6$, $n = 8$, we get

$$\int_0^{\pi/2} \sin^6 x \cos^8 x\,dx = \frac{1}{2}\,B\left(\frac{7}{2}, \frac{9}{2}\right) = \frac{1}{2}\,\frac{\Gamma(7/2)\Gamma(9/2)}{\Gamma(8)} = \frac{5\pi}{2^{12}}$$

(the values of $\Gamma(7/2)$ and $\Gamma(9/2)$ have been calculated by formula (6) in 1.10.1 for $m = 3$ and $m = 4$, and $\Gamma(8) = 7!$).

151. Compute $\displaystyle\int_0^{\pi} \frac{dt}{\sqrt{3 - \cos t}}$.

Solution. We set $\cos t = 1 - 2\sqrt{u}$; then $dt = \dfrac{du}{2\sqrt[4]{u^3}\sqrt{1 - \sqrt{u}}}$, $\sqrt{3 - \cos t} = \sqrt{2}\sqrt{1 + \sqrt{u}}$, with t varying from 0 to 1. Then we get

$$\int_0^{\pi} \frac{dt}{\sqrt{3 - \cos t}} = \frac{1}{2\sqrt{2}} \int_0^1 u^{-3/4}(1 - u)^{-1/2}du$$

$$= \frac{1}{2\sqrt{2}}\,B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{2\sqrt{2}}\,\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)}$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}} \cdot \frac{[\Gamma(1/4)]^2}{\Gamma(3/4)\Gamma(1/4)}.$$

Since $\Gamma\left(\frac{3}{4}\right) \cdot \Gamma\left(\frac{1}{4}\right) = \Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(1 - \frac{1}{4}\right) =$

$= \dfrac{\pi}{\sin(\pi/4)} = \pi\sqrt{2}$, and $\Gamma\left(\dfrac{1}{4}\right) = \dfrac{\Gamma(1.25)}{1/4} = 4 \cdot 0.9064 = 3.6256$, we have

$$\int_0^{\pi} \frac{dt}{\sqrt{3 - \cos t}} = \frac{\sqrt{\pi}}{2\sqrt{2}} \cdot \frac{(3.6256)^2}{\pi\sqrt{2}} = \frac{(3.6256)^2}{4\sqrt{\pi}} = 1.8545.$$

152. Compute $\displaystyle\int_0^1 \frac{dx}{\sqrt{1 - \sqrt[5]{x^2}}}$.

Solution. Let us rewrite the given integral in the form $\displaystyle\int_0^1 (1 - x^{2/5})^{-1/2}dx$. Next we make use of the substitution $x^{2/5} = t$; then we have $x = t^{5/2}$, $dx = (5/2)t^{3/2}dt$ and, consequently,

$$\int_0^1 \frac{dx}{\sqrt{1-\sqrt[5]{x^2}}} = \frac{5}{2}\int_0^1 t^{3/2}(1-t)^{-1/2}dt$$

$$= \frac{5}{2}\,\mathrm{B}\left(\frac{5}{2},\frac{1}{2}\right) = \frac{5}{2}\,\frac{\Gamma(5/2)\cdot\Gamma(1/2)}{\Gamma(3)} = \frac{15\pi}{16}.$$

153. Prove that if $I_1 = \int_0^1 \frac{dx}{\sqrt{1-x^4}}$ and $I_2 = \int_0^1 \frac{x^2dx}{\sqrt{1-x^4}}$, then $I_1 I_2 = \frac{\pi}{4}$.

Solution. We put $x^4 = t$, whence we have $dx = (1/4)t^{1/4-1}\,dt$. Then we obtain

$$I_1 = \frac{1}{4}\int_0^1 t^{1/4-1}(1-t)^{-1/2}\,dt = \frac{1}{4}\,\mathrm{B}\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{4}\cdot\frac{\Gamma(1/4)\cdot\Gamma(1/2)}{\Gamma(3/4)};$$

$$I_2 = \frac{1}{4}\int_0^1 t^{3/4-1}(1-t)^{-1/2}\,dt = \frac{1}{4}\,\mathrm{B}\left(\frac{3}{4},\frac{1}{2}\right)$$

$$= \frac{1}{4}\,\frac{\Gamma(3/4)\cdot\Gamma(1/2)}{\Gamma(5/4)} = \frac{\Gamma(3/4)\cdot\Gamma(1/2)}{\Gamma(1/4)},$$

because $\Gamma(5/4) = (1/4)\cdot\Gamma(1/4)$. Consequently,

$$I_1 I_2 = \frac{1}{4}\cdot\frac{\Gamma(1/4)\cdot\Gamma(3/4)\cdot[\Gamma(1/2)]^2}{\Gamma(3/4)\Gamma(1/4)} = \frac{1}{4}(\sqrt{\pi})^2 = \frac{\pi}{4}.$$

Compute:

154. $\int_0^{\pi/2} \sin^3 x \cos^5 x\,dx$. **155.** $\int_0^{\pi/2} \sin^4 x\,dx$.

156. $\int_0^{\pi/4} \sin^5 4x \cos^4 2x\,dx$.

157. $\int_0^{\overline{\pi/2}} \sin^{10} x \cos^4 x\,dx$.

158. $\int_0^1 \frac{dx}{\sqrt{1-x^a}}$, $a > 0$.

Hint. The substitution is $x^a = t$.

159. $\int_0^a x^{2n}\sqrt{a^2 - x^2}\, dx,\ a > 0.$

Hint. The substitution is $x^2/a^2 = t$.

160. $\int_0^1 x^6\sqrt{1 - x^2}dx.$

161. $\int_0^\infty \frac{x^{a-1}}{1 + x^b}\, dx.$

Hint. The substitution is $(1 + x^b)/x^b = 1/y$.

162. $\int_0^\infty \frac{dx}{x^a(1 + x)},\ 0 < a < 1.$

Hint. The substitution is $x = u/(1 - u)$.

163. $\int_0^\infty \frac{dx}{(1 + x)\sqrt[3]{x}}.$ 164. $\int_0^\pi \sin^6 x \cos^2 (x/2)\, dx.$

165. $\int_0^1 \frac{dx}{\sqrt{x - x^2}}.$ 166. $\int_0^1 x^3(1 - \sqrt[3]{x})^2 dx.$

Hint. The substitution is $x = t^3$.

167. $\int_0^1 x^{n-1}(1 - x^k)^{m-1}dx \quad (n > 0,\ m > 0).$

Hint. The substitution is $x^k = t$.

168. $\int_0^{\pi/2} \sqrt{\tan x}\, dx.$

Hint. The substitution is $\tan x = u^2$.

169. $\int_0^1 \frac{dx}{\sqrt[n]{1 - x^n}}.$ 170. $\int_0^1 \frac{dx}{\sqrt[4]{1 - x^4}}.$

171. $\int_0^{\pi/2} \tan^{2n-1}x\, dx \quad (0 < n < 1).$

172. Express $\int_0^{\pi/2} \sin^n x\, dx$ in terms of the gamma-function.

Chapter 2

Line Integrals and Surface Integrals

2.1. Line Integrals with Respect to Arc Length and to Coordinates

2.1.1. Line integrals with respect to arc length (line integrals of the first type). Suppose the function $f(x, y)$ is defined and continuous at the points of the arc AB of the smooth curve K, specified by the equation $y = \varphi(x)$ $(a \leqslant x \leqslant b)$.

Let us partition arbitrarily the arc AB into n elementary arcs by the points $A = A_0, A_1, A_2, \dots, A_n = B$; assume that Δs_k is the length of the arc $A_{k-1}A_k$. We choose an arbitrary point $M_k(\xi_k; \eta_k)$ on each elementary arc and multiply the value of the function $f(\xi_k, \eta_k)$ at that point by the length Δs_k of the corresponding arc.

The *integral sum* for the function $f(x, y)$ *over the length of the arc* AB is a sum of the form $\sum\limits_{k=1}^{n} f(\xi_k, \eta_k)\,\Delta s_k$.

The *line integral with respect to the length of the arc* AB of the function $f(x, y)$ (or a *line integral of the first type*) is the limit of the integral sum, under the condition that $\max \Delta s_k \to 0$:

$$\int\limits_{AB} f(x, y)\, ds = \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta s_k$$

(ds is the differential of the arc).

A line integral of the first type can be calculated by the formula

$$\int\limits_{AB} f(x, y)\, ds = \int\limits_{a}^{b} f[x, \varphi(x)]\sqrt{1 + [\varphi'(x)]^2}\, dx.$$

If the curve K is specified by the parametric equations $x = x(t)$, $y = y(t)$ $(t_1 \leqslant t \leqslant t_2)$, then

$$\int\limits_{K} f(x, y)\, ds = \int\limits_{t_1}^{t_2} f[x(t), y(t)]\sqrt{x'^2(t) + y'^2(t)}\, dt.$$

A line integral of the first type of the function of three variables $f(x, y, z)$ over a space curve can be determined in a similar way. If a space curve is specified by the equations $x = x(t)$, $y = y(t)$, $z = z(t)$ $(t_1 \leqslant t \leqslant t_2)$, then we have

$$\int\limits_{K} f(x, y, z)\, ds = \int\limits_{t_1}^{t_2} f[x(t), y(t), z(t)]\sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\, dt.$$

If $f(x, y) > 0$, then the line integral of the first type $\int_K f(x, y)\,ds$ is the *mass of the curve* K with varying linear density $\gamma = f(x, y)$ (physical interpretation).

Main properties of a line integral of the first type

1°. *A line integral of the first type is independent of the direction of integration:*

$$\int_{AB} f(x, y)\,ds = \int_{BA} f(x, y)\,ds.$$

2°. $\int_K [f_1(x, y) \pm f_2(x, y)]\,ds = \int_K f_1(x, y)\,ds \pm \int_K f_2(x, y)\,ds.$

3°. $\int_K cf(x, y)\,ds = c\int_K f(x, y)\,ds$, where $c = \text{const}$.

4°. *If the contour of integration K is divided into two parts K_1 and K_2, then*

$$\int_K f(x, y)\,ds = \int_{K_1} f(x, y)\,ds + \int_{K_2} f(x, y)\,ds.$$

2.1.2. Line integrals with respect to coordinates (line integrals of the second type). Assume the functions $P(x, y)$ and $Q(x, y)$ to be continuous at the points of the arc AB of the smooth curve K specified by the equation $y = \varphi(x)$ $(a \leqslant x \leqslant b)$.

The *integral sum* for the functions $P(x, y)$ and $Q(x, y)$ *with respect to the coordinates* is a sum of the form

$$\sum_{k=1}^{n} [P(\xi_k, \eta_k)\Delta x_k + Q(\xi_k, \eta_k)\Delta y_k],$$

where Δx_k and Δy_k are the projections of the elementary arc on the Ox and Oy axes.

The *line integral with respect to coordinates* (or *line integral of the second type*) of the expression $P(x, y)\,dx + Q(x, y)\,dy$ over the directed arc AB is the limit of the integral sum under the condition that $\max\Delta x_k \to 0$ and $\max\Delta y_k \to 0$:

$$\int_{AB} P(x, y)\,dx + Q(x, y)\,dy = \lim_{\substack{\max\Delta x_k \to 0 \\ \max\Delta y_k \to 0}} \sum_{k=1}^{n} [P(\xi_k, \eta_k)\Delta x_k + Q(\xi_k, \eta_k)\Delta y_k].$$

A line integral of the second type is the *work* performed by the variable force $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ over the path AB (mechanical interpretation).

Main properties of a line integral of the second type

1°. *A line integral of the second type changes sign to the opposite upon a change of the direction of integration:*

$$\int_{BA} P\,dx + Q\,dy = -\int_{AB} P\,dx + Q\,dy.$$

2°. $\int_{AB} P\,dx + Q\,dy = \int_{AB} P\,dx + \int_{AB} Q\,dy.$

The other properties are analogous to those of an integral of the first type. A line integral of the second type can be calculated by the formula

$$\int_K P(x, y)\,dx + Q(x, y)\,dy = \int_a^b \{P[x, \varphi(x)] + \varphi'(x) Q[x, \varphi(x)]\}\,dx.$$

If the curve K is specified by the parametric equations $x = x(t)$, $y = y(t)$, where $t_1 \leqslant t \leqslant t_2$, then we have

$$\int_K P(x, y)\,dx + Q(x, y)dy = \int_{t_1}^{t_2} \{P[x(t), y(t)]x'(t) + Q[x(t), y(t)]y'(t)\}\,dt.$$

An analogous formula is true for calculating a line integral of the second type over the space curve K: if the curve is specified by the equations $x = x(t)$, $y = y(t)$, $z = z(t)$, where $t_1 \leqslant t \leqslant t_2$, then

$$\int_K P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz$$

$$= \int_{t_1}^{t_2} \{P[x(t), y(t), z(t)]x'(t) + Q[x(t), y(t), z(t)]y'(t) + R[x(t), y(t), z(t)]z'(t)\}\,dt.$$

173. Compute $\int_K (x - y)\,ds$, where K is a line segment between $A(0; 0)$ and $B(4; 3)$.

Solution. The equation of the straight line AB has the form $y = (3/4)x$. We find $y'_x = 3/4$ and, consequently,

$$\int_K (x - y)\,ds = \int_0^4 \left(x - \frac{3}{4}x\right)\sqrt{1 + \frac{9}{16}}\,dx$$

$$= \frac{5}{16}\int_0^4 x\,dx = \frac{5}{32}x^2 \Big|_0^4 = \frac{5}{2}.$$

174. Compute $\int_K x^2y\,dy - y^2x\,dx$ if $x = \sqrt{\cos t}$, $y = \sqrt{\sin t}$, $0 \leqslant t \leqslant \pi/2$.

Solution. We find $dx = -\dfrac{\sin t}{2\sqrt{\cos t}}\,dt$, $dy = \dfrac{\cos t}{2\sqrt{\sin t}}\,dt$. Then we have

$$\int\limits_K x^2y\,dy - y^2x\,dx$$

$$= \int\limits_0^{\pi/2}\left(\cos t\cdot\sqrt{\sin t}\cdot\frac{\cos t}{2\sqrt{\sin t}} + \sin t\cdot\sqrt{\cos t}\cdot\frac{\sin t}{2\sqrt{\cos t}}\right)dt = \frac{\pi}{4}.$$

175. Find the mass M of the arc of the curve $x = t$, $y = t^2/2$, $z = t^3/3$ $(0 \leqslant t \leqslant 1)$, whose linear density varies by the law $\gamma = \sqrt{2y}$.

Solution. We have

$$M = \int\limits_K \sqrt{2y}\,ds = \int\limits_0^1 \sqrt{2\cdot\frac{1}{2}t^2}\sqrt{x'^2 + y'^2 + z'^2}\,dt$$

$$= \int\limits_0^1 t\cdot\sqrt{1 + t^2 + t^4}\,dt = \frac{1}{2}\int\limits_0^1\sqrt{\left(t^2 + \frac{1}{2}\right)^2 + \frac{3}{4}}\,d\left(t^2 + \frac{1}{2}\right)$$

$$= \frac{1}{2}\left[\frac{t^2 + \frac{1}{2}}{2}\cdot\sqrt{t^4 + t^2 + 1} + \frac{3}{8}\ln\left(t^2 + \frac{1}{2} + \sqrt{t^4 + t^2 + 1}\right)\right]_0^1$$

$$= \frac{1}{8}\left(3\sqrt{3} - 1 + \frac{3}{2}\ln\frac{3 + 2\sqrt{3}}{3}\right).$$

176. Find the coordinates of the centre of gravity of the arc of the cycloid $x = t - \sin t$, $y = 1 - \cos t$ $(0 \leqslant t \leqslant \pi)$.

Solution. The coordinates of the centre of gravity of the homogeneous arc of the curve K can be calculated by the formulas

$$\bar{x} = \frac{1}{s}\int\limits_K x\,ds,\quad \bar{y} = \frac{1}{s}\int\limits_K y\,ds,$$

where s is the arc length. We have

$$s = \int\limits_0^\pi \sqrt{x'^2 + y'^2}\,dt = \int\limits_0^\pi \sqrt{(1 - \cos t)^2 + \sin^2 t}\,dt$$

$$= 2\int\limits_0^\pi \sin\frac{t}{2}\,dt = -4\cos\frac{t}{2}\Bigg|_0^\pi = 4.$$

Then

$$\bar{x} = \frac{1}{4}\int_K x\,ds = \frac{1}{4}\int_0^\pi (t - \sin t)\,2\sin\frac{t}{2}\,dt$$

$$= \frac{1}{2}\int_0^\pi \left(t\sin\frac{t}{2} - \sin\frac{t}{2}\sin t\right)dt$$

$$= \frac{1}{2}\left[-2t\cos\frac{t}{2} + 4\sin\frac{t}{2} + \frac{4}{3}\sin^2\frac{t}{2}\right]_0^\pi = \frac{1}{2}\left(4 + \frac{4}{3}\right) = \frac{8}{3};$$

$$\bar{y} = \frac{1}{4}\int_K y\,ds = \frac{1}{4}\int_0^\pi (1 - \cos t)\,2\sin\frac{t}{2}\,dt$$

$$= \frac{1}{2}\int_0^\pi \left(\sin\frac{t}{2} - \sin\frac{t}{2}\cos t\right)dt$$

$$= \frac{1}{2}\left[-2\cos\frac{t}{2} + \frac{1}{3}\cos\frac{3t}{2} - \cos\frac{t}{2}\right]_0^\pi = \frac{4}{3}.$$

Compute the following line integrals:

177. $\int_{AB} (x^2 - y^2)\,dx + xy\,dy$, if the path form $A(1;\,1)$ to $B(3;\,4)$ is a line segment.

178. $\int_K (x - y)^2\,dx + (x + y)^2\,dy$, if K is the polygonal line OAB, where $O(0;\,0)$, $A(2;\,0)$, $B(4;\,2)$.

179. $\int_{AB} \frac{y\,ds}{\sqrt{x}}$, if AB is the arc of the semicubical parabola $y^2 + (4/9)x^3$ between $A(3;\,2\sqrt{3})$ and $B(8;\,32\sqrt{2}/3)$.

180. $\int_K y\,dx - (y + x^2)\,dy$, if K is the arc of the parabola $y = 2x - x^2$ which lies above the x-axis and is traversed clockwise.

181. $\int_K y\,dx + 2x\,dy$, if K is the contour of the rhomb, traversed counterclockwise, whose sides lie on the straight lines $x/3 + y/2 = \pm 1$, $x/3 - y/2 = \pm 1$.

182. $\int_K 2x\,dy - 3y\,dx$, if K is the contour of the triangle with vertices $A(1;\,2)$,

$B(3; 1)$, $C(2; 5)$ traversed counterclockwise.

183. $\int\limits_K \frac{dy}{x} - \frac{dx}{y}$, if $K - 1$ is a quarter of the circle $x = r\cos t$, $y = r\sin t$, traversed counterclockwise.

184. $\int\limits_K x^2y\,dx + x^3\,dy$, if K is the contour bounded by the parabolas $y^2 = x$, $x^2 = y$ and traversed counterclockwise.

185. Find the mass of the arc of the circle $x = \cos t$, $y = \sin t$ $(0 \leqslant t \leqslant \pi)$, if its linear density at the point $(x; y)$ is y.

186. Find the coordinates of the centre of gravity of a homogeneous arc of the curve $y = \cosh x$ $(0 \leqslant x \leqslant \ln 2)$.

187. Find the coordinates of the centre of gravity of a homogeneous arc of the curve $x = e^t\cos t$, $y = e^t\sin t$, $z = e^t$ $(-\infty \leqslant t \leqslant 0)$.

188. Compute $\int\limits_K \sqrt{x^2 + y^2}\,ds$, where K is the circle $x^2 + y^2 = ax$.

189. Compute $\int\limits_K \frac{ds}{x^2 + y^2 + z^2}$, where K is the first coil of the helical line $x = a\cos t$, $y = a\sin t$, $z = bt$.

190. Find the mass of the first coil of the helical line $x = \cos t$, $y = \sin t$, $z = t$, if the density at every point is equal to the radius vector of that point.

191. Compute $\int\limits_{OA} xy\,dx + yz\,dy + zx\,dz$, where OA is a quarter of the circle $x = \cos t$, $y = \sin t$, $z = 1$ traversed in the direction of the increases of the parameter t.

2.2. Independence of Line Integral of the Second Type on the Integration Contour. Finding the Function from Its Total Differential

Assume the functions $P(x, y)$ and $Q(x, y)$ to be continuous together with their partial derivatives of the first order in the simply connected domain D, and the contour K to lie entirely in that domain.

Then, *the necessary and sufficient condition for independence of the line integral* $\int\limits_K P(x, y)\,dx + Q(x, y)\,dy$ *of the integration contour is the satisfaction, in the domain* D, *of the identity*

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}.$$

If the indicated conditions are complied with, *the line integral around any closed*

contour C contained in the domain D is equal to zero:

$$\oint_C P(x, y)\, dx + Q(x, y)\, dy = 0.$$

To evaluate the integral

$$\int_{(x_0;\, y_0)}^{(x_1;\, y_1)} P(x, y)\, dx + Q(x, y)\, dy$$

independent of the integration contour (i.e. the condition $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ is satisfied), the polygonal line connecting the point $(x_0;\, y_0)$ and $(x_1;\, y_1)$, whose segments are parallel to the Ox and Oy axes, should be chosen as the most advantageous path of integration.

Under the indicated conditions, the element of integration $P(x, y)dx + Q(x, y)dy$ is the *total differential* of a certain single-valued function $U = U(x, y)$, i.e.

$$dU(x, y) = P(x, y)\, dx + Q(x, y)\, dy.$$

The function $U(x, y)$ (the antiderivative) can be found by taking the corresponding line integral along the polygon A_0A_1B, where $A_0(x_0;\, y_0)$ is an arbitrary fixed point, $B(x;\, y)$ is a variable point, and the point A_1 has the coordinates x and y_0. Then we have $y = y_0$ and $dy = 0$ along A_0A_1, and $x = \text{const}$, $dx = 0$ along A_1B. As a result we get the following formula:

$$U(x, y) = \int_{x_0}^{x} P(x, y_0)\, dx + \int_{y_0}^{y} Q(x, y)\, dy + C.$$

Similarly, integrating along the polygon A_0A_2B, where $A_2(x_0;\, y)$ we obtain

$$U(x, y) = \int_{y_0}^{y} Q(x_0, y)\, dy + \int_{x_0}^{x} P(x, y)\, dx + C.$$

192. Compute $I = \int_{(1;\, 1)}^{(2;\, 3)} (x + 3y)\, dx + (y + 3x)\, dy.$

Solution. The given integral is independent of the contour of integration, because

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x + 3y) = 3; \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(y + 3x) = 3,$$

i.e. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (on the entire xOy plane).

As the path of integration we choose the polygonal line whose segments are parallel to the coordinate axes. On the first segment we have $y = 1$, $dy = 0$, $1 \leqslant x \leqslant 2$, on the second $x = 2$, $dx = 0$, $1 \leqslant y \leqslant 3$. Consequently,

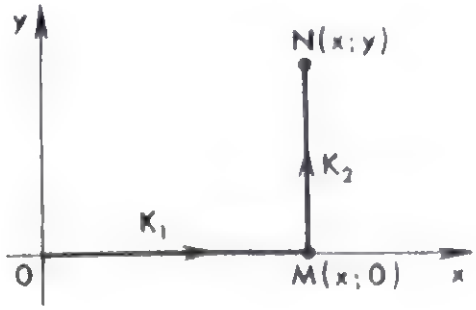


Fig. 20

$$I = \int_1^2 (x + 3)\,dx + \int_1^3 (y + 6)\,dy = \left[\frac{x^2}{2} + 3x\right]_1^2 + \left[\frac{y^2}{2} + 6y\right]_1^3$$

$$= 2 + 6 - \frac{1}{2} - 3 + \frac{9}{2} + 18 - \frac{1}{2} - 6 = 20\frac{1}{2}.$$

193. Find the antiderivative U, if

$$dU = [y + \ln(x + 1)]\,dx + (x + 1 - e^y)\,dy.$$

Solution. We have $P = y + \ln(x + 1)$; $Q = x + 1 - e^y$; $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 1$. Assume that $x_0 = 0$, $y_0 = 0$ and the contour K is the polygonal line OMN (Fig. 20). Then

$$U(x, y) = \int_0^x \ln(x + 1)\,dx + \int_0^y (x + 1 - e^y)\,dy$$

$$= [x \ln(x + 1) - x + \ln(x + 1)]_0^x + [xy + y - e^y]_0^y$$

$$= (x + 1)\ln(x + 1) - x + xy + y - e^y + 1 + C.$$

194. Find $U(x, y)$ if

$$dU = \left(\frac{1}{x} + \frac{1}{y}\right)dx + \left(\frac{2}{y} - \frac{x}{y^2}\right)dy.$$

Solution. We have

$$P = \frac{1}{x} + \frac{1}{y}, \quad Q = \frac{2}{y} - \frac{x}{y^2}; \quad \frac{\partial P}{\partial y} = -\frac{1}{y^2} = \frac{\partial Q}{\partial x}.$$

Here we cannot take the origin as the point $(x_0; y_0)$ since at $x = 0$ and $y = 0$ the functions $P(x, y)$ and $Q(x, y)$ are not defined. So we shall take the point $A_0(1, 1)$, for instance, as the point $(x_0; y_0)$. Then we have

$$U(x, y) = \int_1^x \left(\frac{1}{x} + 1\right)dx + \int_1^y \left(\frac{2}{y} - \frac{x}{y^2}\right)dy = \ln x + 2\ln y + \frac{x}{y} - 1 + C.$$

Find the antiderivative $U(x, y)$ from its total differential:

195. $dU = [e^{x+y} + \cos(x - y)]\, dx + [e^{x+y} - \cos(x - y) + 2]\, dy$.

196. $dU = (1 - e^{x-y} + \cos x)\, dx + (e^{x-y} + \cos y)\, dy$.

197. $dU = (x^2 - 2xy^2 + 3)\, dx + (y^2 - 2x^2y + 3)\, dy$.

198. $dU = (2x - 3xy^2 + 2y)\, dx + (2x - 3x^2y + 2y)\, dy$.

199. $dU = (\sinh x + \cosh y)\, dx + (x \sinh y + 1)\, dy$.

200. $dU = (\arcsin x - x \ln y)\, dx - \left(\arcsin y + \dfrac{x^2}{2y}\right) dy$.

201. Compute $\int\limits_{(0;\,0)}^{(\pi;\,\pi)} (x + y)\, dx + (x - y)\, dy$ over various contours connecting the points $O(0; 0)$ and $M(\pi; \pi)$: (1) over the straight line OM; (2) over the curve $y = x + \sin x$; (3) over the polygonal line OPM, where $P(\pi; 0)$; (4) over the parabola $y = x^2/\pi$.

202. Compute $\oint\limits_K x\, dy + y\, dx$ around various closed contours: (1) over the circle $x = \cos t$, $y = \sin t$; (2) over the contour bounded by an arc of the parabola $y = x^2$ and the line segment $y = 1$.

2.3. Green's Formula

If C is the boundary of the domain D and the functions $P(x, y)$ and $Q(x, y)$ are continuous together with their partial derivatives $\dfrac{\partial Q}{\partial x}$ and $\dfrac{\partial P}{\partial y}$ in the closed domain D (the boundary C inclusive), then the following *Green's formula* holds true:

$$\oint\limits_C P\, dx + Q\, dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy.$$

In this case the direction in which the contour C is traversed is chosen so that the domain D remains on the left.

203. Applying Green's formula, evaluate $I = \oint\limits_C 2(x^2 + y^2)\, dx + (x + y)^2\, dy$, if C is the contour of the triangle with vertices $L(1; 1)$, $M(2; 2)$, $N(1; 3)$ traversed counterclockwise. Check the result by direct integration.

Solution. Here $P(x, y) = 2(x^2 + y^2)$, $Q(x, y) = (x + y)^2$. We find $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 2(x + y) - 4y = 2(x - y)$. Thus we have

$$I = \oint\limits_C 2(x^2 + y^2)\, dx + (x + y)^2\, dy = \iint\limits_D 2(x - y)\, dx\, dy,$$

where the domain D is the triangle LMN. The equation of the straight line LM is $y = x$, the equation of MN is $y = -x + 4$. Let us calculate the double integral over the given domain:

$$I = 2\int_1^2 dx \int_x^{4-x} (x - y)\, dy = 2\int_1^2 \left[xy - \frac{1}{2}y^2\right]_x^{4-x} dx$$

$$= 2\int_1^2 \left[x(4 - x) - \frac{1}{2}(4 - x)^2 - x^2 + \frac{1}{2}x^2\right] dx$$

$$= 4\int_1^2 (4x - x^2 - 4)\, dx = 4\left[2x^2 - \frac{1}{3}x^3 - 4x\right]_1^2 = -\frac{4}{3}.$$

Let us now evaluate directly the line integral around the contour C consisting of the segments LM, MN, NL:

$$I = \int_{LM} 2(x^2 + y^2)\, dx + (x + y)^2\, dy + \int_{MN} 2(x^2 + y^2)\, dx$$

$$+ (x + y)^2\, dy + \int_{NL} 2(x^2 + y^2)\, dx + (x + y)^2\, dy.$$

The equation of LM is $y = x$ and, consequently, $dy = dx$, $1 \leqslant x \leqslant 2$.

The equation of MN is $y = -x + 4$ and, consequently, $dy = -dx$, $2 \geqslant y \geqslant 1$.

The equation of ML is $x = 1$ and, hence, $dx = 0$, $3 \geqslant y \geqslant 1$.

Thus we have

$$I = \int_1^2 [2(x^2 + x^2)\, dx + (x + x)^2\, dx] + \int_2^1 [2[x^2 + (4 - x)^2]\, dx$$

$$+ (x - x + 4)^2(-dx)] + \int_3^1 (1 + y)^2\, dy = 8\int_1^2 x^2\, dx$$

$$+ \int_2^1 (4x^2 - 16x + 16)\, dx + \int_3^1 (1 + y)^2\, dy$$

$$= \left[\frac{8}{3}x^3 - \frac{4}{3}x^3 + 8x^2 - 16x\right]_1^2 + \frac{1}{3}(1 + y)^3\Big|_3^1 = -\frac{4}{3}.$$

204. Applying Green's formula, evaluate $\oint_C - x^2y\, dx + xy^2\, dy$, where C is the circle $x^2 + y^2 = R^2$ traversed counterclockwise.

Solution. Here $P(x, y) = -x^2y$, $Q(x, y) = xy^2$. Then $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = x^2 + y^2$. Consequently,

$$I = \oint_C - x^2y\,dx + x^2y\,dy = \iint_D (x^2 + y^2)\,dx\,dy.$$

Let us introduce polar coordinates: $x = \rho\cos\theta$, $y = \rho\sin\theta$, $0 \leqslant \theta < 2\pi$, hence

$$I = \iint_D \rho^2 \cdot \rho\,d\rho\,d\theta = \int_0^{2\pi} d\theta \int_0^R \rho^3\,d\rho = \frac{1}{4}R^4 \cdot \int_0^{2\pi} d\theta = \frac{\pi R^4}{2}.$$

205. By means of Green's formula transform the line integral $I = \oint_C [x + \ln(x^2 + y^2)]\,dx + y\ln(x^2 + y^2)\,dy$, where the contour C is the boundary of the domain D.

206. Applying Green's formula, calculate $\oint_C \sqrt{x^2 + y^2}\,dx + y[xy + \ln(x + \sqrt{x^2 + y^2})]\,dy$, where C is the contour of the rectangle $1 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 2$.

2.4. Computing the Area

The area S of the figure bounded by the simple closed contour C can be found from the formula

$$S = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

The integration contour is traversed so that the bounded domain remains on the left (positive direction).

207. Calculate the area of the figure bounded by the curves $y = x^2$, $x = y^2$, $8xy = 1$ (we mean the area adjoining the origin; Fig. 21).

Solution. Solving simultaneously the equations of the curves, we find $A(1/2; 1/4)$, $B(1/4; 1/2)$. Consequently,

$$S = \frac{1}{2}\int_{OA} x\,dy - y\,dx = \frac{1}{2}\int_{AB} x\,dy - y\,dx$$

$$+ \frac{1}{2}\int_{BO} x\,dy - y\,dx = \frac{1}{2}\int_0^{1/2} x^2\,dx - \frac{1}{8}\int_{1/2}^{1/4} \frac{dx}{x}$$

$$- \frac{1}{4}\int_{1/4}^{0} \sqrt{x}\,dx = \frac{1 + 3\ln 2}{24} \approx 0.13 \text{ (sq. units)}.$$

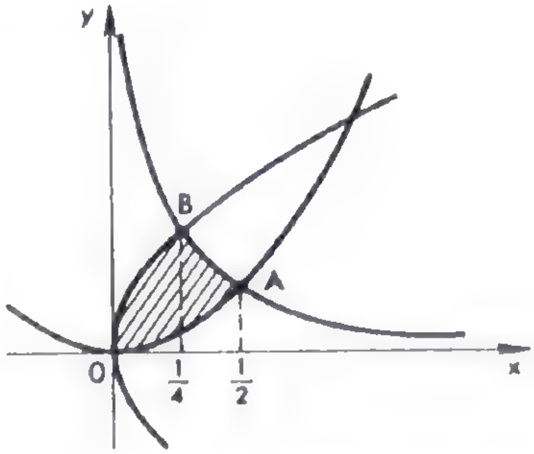


Fig. 21

208. Calculate the area, bounded by the astroid $x = a\cos^3 t$, $y = a\sin^3 t$, first constructing the curve.

Solution. The area is calculated by the formula $S = \frac{1}{2}\oint\limits_C x\,dy - y\,dx$, where $dy = 3a\sin^2 t\cos t\,dt$, $dx = -3a\cos^2 t\sin t\,dt$, $0 \leqslant t \leqslant 2\pi$. Consequently,

$$S = \frac{1}{2}\int_0^{2\pi}(3a^2\cos^4 t\sin^2 t + 3a^2\sin^4 t\cos^2 t)\,dt$$

$$= \frac{3}{2}a^2\int_0^{2\pi}\sin^2 t\cos^2 t\,dt = \frac{3a^2}{8}\int_0^{2\pi}\sin^2 2t\,dt$$

$$= \frac{3a^2}{16}\int_0^{2\pi}(1 - \cos 4t)\,dt = \frac{3}{16}a^2\left[t - \frac{1}{4}\sin 4t\right]_0^{2\pi} = \frac{3\pi a^2}{8}.$$

209. Calculate the area bounded by the parabolas $y^2 = x$, $x^2 = y$.

210. Calculate the area bounded by the ellipse $x = a\cos t$, $y = b\sin t$.

211. Calculate the area of the quadrangle with vertices $A(6; 1)$, $B(4; 5)$, $C(1; 6)$, $D(-1; 1)$.

212. Calculate the area of the figure bounded by the contour $OABCO$, if $A(1; 3)$, $B(0; 4)$, $C(-1; 2)$, $O(0; 0)$; OA, BC, CO being the line segments and AB, the arc of the parabola $y = 4 - x^2$.

213. Calculate the area bounded by the cardioid $x = 2r\cos t - r\cos 2t$, $y = 2r\sin t - r\sin 2t$.

2.5. Surface Integrals

Suppose $F(x, y, z)$ is a continuous function and $z = f(x, y)$ is a smooth surface S, where $f(x, y)$ is given in a certain domain D of the xOy plane. The *surface in-*

tegral of the first type is the limit of the integral sum under the condition that $\max d_k \to 0$:

$$\lim_{\max d_k \to 0} \sum_{k=1}^{n} F(\xi_k, \eta_k, \zeta_k)\Delta S_k = \iint_S F(x, y, z)\, dS,$$

where ΔS_k is the area of the kth element of the surface S, the point $(\xi_k; \eta_k; \zeta_k)$ belongs to that element, d_k is the diameter of the element, $F(x, y, z)$ is defined at every point of the surface S.

The value of the integral does not depend on the choice of the side of the surface S over which the integration is performed.

If the projection D of the surface S on the xOy plane is single-valued, the corresponding *surface integral of the first type can be computed by the formula*

$$\iint_S F(x, y, z)\, dS = \iint_D F[x, y, f(x, y)] \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy.$$

If $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$ are continuous functions and S^+ is a side of the smooth surface S, characterized by the direction of the normal $\mathbf{n}$ $(\cos\alpha; \cos\beta; \cos\gamma)$, the corresponding *surface integral of the second type* is expressed as follows:

$$\iint_{S^+} P\, dy\, dz + Q\, dz\, dx + R\, dx\, dy = \iint_S (P \cos\alpha + Q \cos\beta + R \cos\gamma)\, dS.$$

Passing to the other side S^- of the surface, the integral changes sign to the opposite.

If the surface S is specified by an equation in an implicit form $\Phi(x, y, z) = 0$, then the direction cosines of the normal to that surface are found from the formulas:

$$\cos\alpha = \frac{\dfrac{\partial\Phi}{\partial x}}{\pm\sqrt{\left(\dfrac{\partial\Phi}{\partial x}\right)^2 + \left(\dfrac{\partial\Phi}{\partial y}\right)^2 + \left(\dfrac{\partial\Phi}{\partial z}\right)^2}},$$

$$\cos\beta = \frac{\dfrac{\partial\Phi}{\partial y}}{\pm\sqrt{\left(\dfrac{\partial\Phi}{\partial x}\right)^2 + \left(\dfrac{\partial\Phi}{\partial y}\right)^2 + \left(\dfrac{\partial\Phi}{\partial z}\right)^2}},$$

$$\cos\gamma = \frac{\dfrac{\partial\Phi}{\partial z}}{\pm\sqrt{\left(\dfrac{\partial\Phi}{\partial x}\right)^2 + \left(\dfrac{\partial\Phi}{\partial y}\right)^2 + \left(\dfrac{\partial\Phi}{\partial z}\right)^2}},$$

where the sign before the radical must be coherent with the side of the surface.

The *moments of inertia of a part of the surface* about the coordinate axes are expressed in term of the surface integrals:

$$I_{Ox} = \iint_S (y^2 + z^2)\, dS, \quad I_{Oy} = \iint_S (x^2 + z^2)\, dS, \quad I_{Oz} = \iint_S (x^2 + y^2)\, dS,$$

The *coordinates of the centre of gravity of a part of the surface* can be found by the formulas

$$\bar{x} = \frac{1}{S}\iint_S x\, dS, \quad \bar{y} = \frac{1}{S}\iint_S y\, dS, \quad \bar{z} = \frac{1}{S}\iint_S z\, dS,$$

where S is the area of the given part of the surface.

214. Compute $I = \iint_S (x^2 + y^2)\, dS$, where S is a part of the conic surface $z^2 = x^2 + y^2$ contained between the planes $z = 0$ and $z = 1$.

Solution. We have

$$z = \sqrt{x^2 + y^2}, \quad \frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}},$$

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

$$= \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}}\, dx\, dy = \sqrt{2} \cdot dx\, dy.$$

Then the sought-for integral is transformed to the double integral

$$I = \iint_D (x^2 + y^2) \cdot \sqrt{2}\, dx\, dy.$$

The domain of integration D is the circle $x^2 + y^2 \leqslant 1$; therefore

$$I = \sqrt{2} \iint_D (x^2 + y^2)\, dx\, dy = 4\sqrt{2} \cdot \int_0^{\pi/2} d\theta \int_0^1 \rho^3 d\rho = \sqrt{2} \int_0^{\pi/2} d\theta = \frac{\pi}{2}\sqrt{2}.$$

215. Find the moment of inertia of the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ about the z-axis.

Solution. We have

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}; \quad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}.$$

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

$$= \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}}\, dx\, dy = \frac{a\, dx\, dy}{\sqrt{a^2 - x^2 - y^2}},$$

$$I_{Oz} = \iint\limits_S (x^2 + y^2)\, dS = \iint\limits_D (x^2 + y^2)\cdot \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy.$$

The integration domain is the projection of the hemisphere on the xOy plane, that is, the circle $x^2 + y^2 \leqslant a^2$, and, therefore, passing to polar coordinates, we obtain

$$I_{Oz} = \iint\limits_D \rho^2 \frac{a}{\sqrt{a^2 - \rho^2}}\, \rho\, d\rho\, d\theta = 4a \int_0^{\pi/2} d\theta \int_0^a \frac{\rho^3\, d\rho}{\sqrt{a^2 - \rho^2}} = \frac{4}{3} \pi a^4$$

(the inner integral can be computed by means of the substitution $\rho = a \sin t$).

216. Calculate the coordinates of the centre of gravity of the plane $z = x$, bounded by the planes $x + y = 1$, $y = 0$, $x = 0$ (Fig. 22).

Solution. Let us find the area of the indicated part of the plane $z = x$. We have $\frac{\partial z}{\partial x} = 1$, $\frac{\partial z}{\partial y} = 0$; therefore,

$$S = \iint\limits_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

$$= \sqrt{2} \int_0^1 dx \int_0^{1-x} dy = \sqrt{2} \int_0^1 (1 - x)\, dx = -\frac{\sqrt{2}}{2}(1 - x)^2 \Big|_0^1 = \frac{\sqrt{2}}{2}.$$

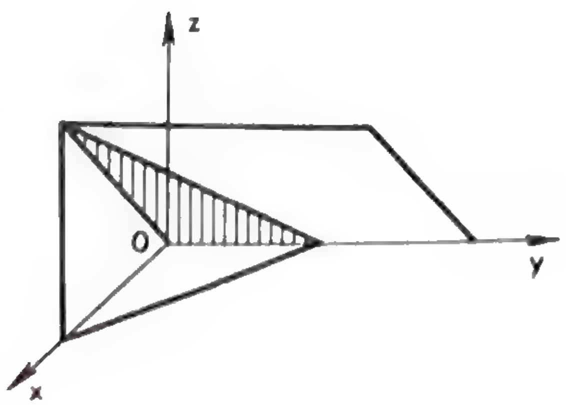


Fig. 22

Then we have

$$\bar{x} = \frac{1}{S}\iint\limits_S x\,dS = \frac{2}{\sqrt{2}}\int\limits_0^1 x\,dx\int\limits_0^{1-x}\sqrt{2}\,dy$$

$$= 2\int\limits_0^1 x(1-x)\,dx = 2\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{3};$$

$$\bar{y} = \frac{1}{S}\iint\limits_S y\,dS = \frac{2}{\sqrt{2}}\int\limits_0^1 dx\int\limits_0^{1-x}\sqrt{2}\,y\,dy = \int\limits_0^1 (1-x)^2\,dx = -\frac{1}{3}(1-x)^3\Big|_0^1 = \frac{1}{3};$$

$$\bar{z} = \frac{1}{S}\iint\limits_S z\,dS = \frac{1}{S}\iint\limits_S x\,dS = \frac{1}{3}$$

(we have used the equation of the plane $z = x$).

217. Find the coordinates of the centre of gravity of the part of the surface $z = 2 - (x^2 + y^2)/2$ lying above the plane xOy.

218. Find the moment of inertia of the paraboloid $z = (x^2 + y^2)/2$ about the z-axis for $0 \leqslant z \leqslant 1$.

219. Compute $\iint\limits_S xyz\,dS$, where S is the part of the surface $z = x^2 + y^2$ located between the planes $z = 0$ and $z = 1$.

2.6. Stokes' Formula and Gauss-Ostrogradsky's Formula. Elements of Field Theory

If the functions $P = P(x, y, z)$, $Q = Q(x, y, z)$, $R = R(x, y, z)$ are continuous together with their first-order partial derivatives on the surface S, and C is a closed contour bounding the surface S, then the following *Stokes' formula* holds true:

$$\oint\limits_C P dx + Q dy + R dz$$

$$= \iint\limits_S \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\gamma\right] dS.$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the normal to the surface S; the direction of the normal is defined so that the contour C seems to be traversed counterclockwise, when observed from the tip of the normal.

If the functions $P = P(x, y, z)$, $Q = Q(x, y, z)$, $R = R(x, y, z)$ are continuous together with their first-order partial derivatives in the closed domain T of the space bounded by the closed smooth surface S, then the *Gauss-Ostrogradsky formula* holds true:

$$\oiint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,dS = \iiint_T \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz,$$

where $\cos\alpha$, $\cos\gamma$ are the direction cosines of the outer normal to the surface S.

If the variable vector $\mathbf{F}$ is the vector function of a point of the space, $\mathbf{F} = \mathbf{F}(M) = \mathbf{F}(r)$, where $M(x; y; z)$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then the vector defines the *vector field*. In the coordinate form

$$\mathbf{F} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k},$$

where $P = P(x, y, z)$, $Q = Q(x, y, z)$, $R = R(x, y, z)$ are the projections of the vector $\mathbf{F}$ on the coordinate axes.

The *divergence* of the vector field $\mathbf{F}(M) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is the scalar

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

The *rotation* (*curl*) of the vector field $\mathbf{F}(M) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is the vector

$$\operatorname{rot}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

The *flux* of the vector field $\mathbf{F}(M)$ across the surface S to the side defined by the unit vector of the normal $\mathbf{n} = \cos\alpha \cdot \mathbf{i} + \cos\beta \cdot \mathbf{j} + \cos\gamma \cdot \mathbf{k}$ to the surface S is the surface integral

$$\iint_S \mathbf{F}\mathbf{n}\,dS = \iint_S F_n\,dS = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,dS.$$

The *Gauss-Ostrogradsky formula in vector form* is

$$\oiint_S F_n\,dS = \iiint_T \operatorname{div}\mathbf{F}\,dV,$$

that is, ***the integral of the divergence of the vector field*** $\mathbf{F}$ ***extending over some volume*** T ***is equal to the flux of the vector across the surface*** S ***bounding the given volume.***

The *line integral* of the vector $\mathbf{F}$ over the curve K is the line integral

$$\int_K \mathbf{F}d\mathbf{r} = \int_K P\,dx + Q\,dy + R\,dz.$$

which is the *work* performed by the vector field along the curve K. If the contour C is closed, the line integral

$$\oint_C \mathbf{F}d\mathbf{r} = \oint_C P\,dx + Q\,dy + R\,dz$$

is known as the *circulation* of the vector field $\mathbf{F}(M)$ around the contour C.

Stokes' formula in vector form is

$$\oint_C \mathbf{F}d\mathbf{r} = \iint_S \mathbf{n} \cdot \operatorname{rot}\mathbf{F}dS,$$

that is, *the circulation of the vector around the contour of a certain surface is equal to the flux of the rotation across the surface.*

220. Find the integral $\oiint (x\cos\alpha + y\cos\beta + z\cos\gamma)dS$, extending over the surface of some body.

Solution. We have from the Gauss-Ostrogradsky formula

$$\oiint_S (x\cos\alpha + y\cos\beta + z\cos\gamma)dS$$

$$= \iiint_T \left(\frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{(\partial z)}{\partial z}\right)dx\,dy\,dz$$

$$= 3\iiint_V \partial x\partial y\partial z = 3V,$$

where V is the volume of the body.

221. Applying the Gauss-Ostrogradsky formula, transform the surface integral

$$I = \oiint_S \frac{\partial u}{\partial x}\partial y\partial z + \frac{\partial u}{\partial y}\partial x\partial z + \frac{\partial u}{\partial z}\partial x\partial y$$

into the integral over the volume.

Solution. The given integral can be written as follows:

$$I = \oiint_S \left(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma\right)dS,$$

and, by the Gauss-Ostrogradsky formula, the last integral is equal to

$$\iiint_T \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)\right]\partial x\partial y\partial z$$

$$= \iiint_T \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\partial x\partial y\partial z.$$

Consequently,

$$I = \iiint_T \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\partial x\partial y\partial z.$$

222. Applying Stokes' formula, find $I = \oint_C x^2y^3\,dx + dy + z\,dz$, if C is the circle $x^2 + y^2 = r^2, z = 0$.

Solution. We have by Stokes' formula

$$I = \oint_C x^2y^3dx + dy + zdz$$

$$= \iint_S \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\gamma\right] dS,$$

where $P = x^2y^3$, $Q = 1$, $R = z$. Consequently, on the right-hand side we get $-\iint_S 3x^2y^2 \cos\gamma\, dS$, where $dS\cos\gamma = dx\,dy$. Thus we have

$$I = -3\iint_D x^2y^2\,dx\,dy.$$

Setting $x = \rho\cos\theta$, $y = \rho\sin\theta$, we get

$$I = -3\iint_D \rho^5 \sin^2\theta\cos^2\theta\, d\rho\, d\theta$$

$$= -12 \iint_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta \int_0^r \rho^5 d\rho$$

$$= -2r^6 \int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta = -\frac{r^6}{2}\int_0^{\pi/2} \sin^2 2\theta\, d\theta$$

$$= -\frac{r^6}{4}\int_0^{\pi/2} (1 - \cos 4\theta)\, d\theta = -\frac{r^2}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = -\frac{\pi r^6}{8}.$$

223. Find the flux of the radius vector **r** across the outer side of the surface of the right circular cylinder if the origin coincides with the centre of the lower base of the cylinder, R is the radius of the base of the cylinder, and h is its altitude (Fig. 23).

Solution. To calculate the flux of the vector **r** across the outer side of the surface of the cylinder, it is necessary to compute the flux of the vector across the lower base, lateral surface and the upper base of the cylinder.

We have $Q_{\text{low.base}} = \iint_S r_n dS$, but the projection of the radius vector **r** on the outer normal to the base of the cylinder is equal to zero, and therefore $Q_{\text{low.base}} = 0$.

The projection of the radius vector on the normal to the lateral surface is equal to the radius of the base of the cylinder, i.e. $r_n = R$; then $Q_{\text{lat.sur}} = \iint_S R\,dS = R \cdot S_{\text{lat.sur}} = 2\pi R^2 h$.

The projection of the radius vector on the normal to the upper base is h; consequently, $Q_{\text{up.base}} = h \iint_S dS = h \cdot S_{\text{base}} = \pi R^2 h$.

Thus, the flux of the vector **r** across the outer side of the surface of the cylinder is equal to

$$Q = 2\pi R^2 h + \pi R^2 h = 3\pi R^2 h.$$

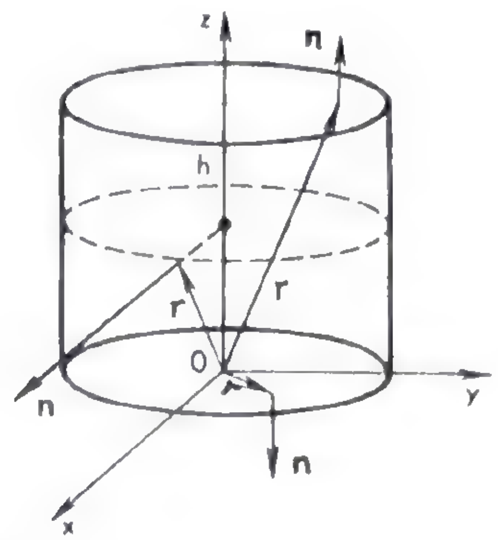


Fig. 23

224. Find the flux of the vector field $\mathbf{F} = (2z - x)\mathbf{i} + (x + 2z)\mathbf{j} + 3z\mathbf{k}$ across the side of the triangle S cut from the plane $x + 4y + z - 4 = 0$ by the coordinate planes in the direction of the normal to the plane which forms an acute angle with the Oz axis.

Solution. We have

$$\iint_S (2z - x)\,dy\,dz + (x + 2z)\,dz\,dx + 3z\,dx\,dy$$

$$= \iint_S (2z + 4y + z - 4)\,dy\,dz + (x + 2z)\,dz\,dx + 3(4 - x - 4y)\,dx\,dy$$

$$= \int_0^1 dy \int_0^{4-4y} (3z + 4y - 4)\,dz + \int_0^4 dz \int_0^{4-z} (x + 2z)\,dx$$

$$+ 3\int_0^4 dx \int_0^{1-x/4} (4 - x - 4y)\,dy$$

$$= \int_0^1 \left[\frac{3}{2} \cdot 16(1 - y)^2 - 16(1 - y)^2\right] dy$$

$$+ \int_0^4 \left[\frac{1}{2}(4 - z)^2 + 2z(4 - z)\right] dx + 3\int_0^4 \left[\frac{1}{4}(4 - x)^2 - \frac{(4 - x)^2}{8}\right] dx = 42\frac{2}{3}.$$

225. Find the circulation of the vector field $\mathbf{F} = (x + 3y + 2z)\mathbf{i} + (2x + z)\mathbf{j} + (x + y)\mathbf{k}$ around the contour of the triangle MNP, where $M(2; 0; 0)$, $N(0; 3; 0)$, $P(0; 0; 1)$.

Solution. In accordance with Stokes' formula, $\oint_C \mathbf{F}\,d\mathbf{r} = \iint_S \mathbf{n} \cdot \operatorname{rot} \mathbf{F}\,dS$,

where

$$\operatorname{rot}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + 3y + 2z & 2x + z & x - y \end{vmatrix}$$

$$= \left[\frac{\partial(x-y)}{\partial y} - \frac{\partial(2x+z)}{\partial z}\right]\mathbf{i} + \left[\frac{\partial(x-y)}{\partial x} - \frac{\partial(x+3y+2z)}{\partial z}\right]\mathbf{j}$$

$$+ \left[\frac{\partial(2x+z)}{\partial x} + \frac{\partial(x+3y+2z)}{\partial y}\right]\mathbf{k} = -2\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

Here C is the contour of the triangle MNP with positive orientation.

The triangle lies in the plane $x/2 + y/3 + z/1 = 1$, or $3x + 2y + 6z - 6 = 0$. Consequently,

$$\iint_S \mathbf{n}\cdot\operatorname{rot}\mathbf{F}\,dS = \iint_S (\operatorname{rot}\mathbf{F})_x dydz + (\operatorname{rot}\mathbf{F})_y dzdx + (\operatorname{rot}\mathbf{F})_z dxdy$$

$$= -2\iint_{D_{yz}} dydz \iint_{D_{zx}} dzdx - \iint_{D_{xy}} dxdy = -2\int_0^3 dy \int_0^{1-y/3} dz + \int_0^1 dz \int_0^{2-2z} dx$$

$$- \int_0^2 dx \int_0^{3-3x/2} dy = -2\left[y - \frac{y^2}{6}\right]_0^3 + \left(2z - z^2\right)_0^1 - \left(3x - \frac{3}{4}x^2\right)_0^2 = -5.$$

226. Find the divergence of the vector field $\mathbf{A} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$.

Solution. By definition, $\operatorname{div}\mathbf{A} = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$, where $A_x = x^2$, $A_y = y^2$, $A_z = z^2$. It follows that $\operatorname{div}\mathbf{A} = 2(x + y + z)$.

227. Find the circulation of the vector $\mathbf{A} = -\omega y\mathbf{i} + \omega x\mathbf{j}$ around the circle $x = a\cos t$, $y = a\sin t$ in the positive direction.

Solution. We have

$$\oint_C \mathbf{A}d\mathbf{r} = \oint_C -\omega y dx + \omega x dy = \omega \int_0^{2\pi} (a^2\sin^2 t + a^2\cos^2 t)dt = 2\pi a^2\omega.$$

228. Find the flux of the vector field $\mathbf{F} = (y - x)\mathbf{i} + (x + y)\mathbf{j} + y\mathbf{k}$ across the side of the triangle S cut from the plane $x + y + z - 1 = 0$ by the coordinate planes.

229. Find the circulation of the vector field $\mathbf{F} = (x + y)\mathbf{i} + (x - z)\mathbf{j} + (y + z)\mathbf{k}$ around the contour of the triangle ABC, where $A(0; 0; 0)$, $B(0; 1; 0)$, $C(0; 0; 1)$.

230. Show that $\operatorname{rot}(\operatorname{grad} u) = 0$, that is, the rotation of the gradient of any scalar is zero.

Solution. Since the projections of the vector of the gradient are the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$, we have

$$\operatorname{rot}(\operatorname{grad} u) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 u}{\partial y \partial z}\right) - \mathbf{j}\left(\frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 u}{\partial x \partial z}\right) + \mathbf{k}\left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y}\right) = 0.$$

231. Applying Stokes' formula, find the line integral $\oint_C (y + z)dx + (z + x)dy + (x + y)dz$, where C is the circle $x^2 + y^2 + z^2 = a^2, x + y + z = 0$.

232. Find the integral $\iint_S (x^3 \cos\alpha + y^3 \cos\beta + z^3 \cos\gamma)dS$ taken over the surface of the sphere $x^2 + y^2 + z^2 = a^2$, where α, β, γ are the angles made by the outer normal with the coordinate axes.

233. Find $\iint_S [(z^2 - y^2)\cos\alpha + (x^2 - z^2)\cos\beta + (y^2 - x^2)\cos\gamma]dS$, where S is the outer side of the surface of the hemisphere $x^2 + y^2 + z^2 = a^2$ $(z \geqslant 0)$.

234. Compute $\iint_S (x\cos\alpha + y\cos\beta + z\cos\gamma)dS$, where S is the outer side of the surface of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

235. Evaluate $\iint_S x dy dz + y dx dz + z dx dy$, where S is the outer side of the surface of the cylinder $x^2 + y^2 = a^2$ $(-h \leqslant x \leqslant h)$.

236. Find the flux of the vector $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ across the lateral surface of the cone $x^2 + y^2 \leqslant (R^2/h^2)z^2, 0 \leqslant z \leqslant h$.

237. Find the divergence of the gradient of the function $u = e^{x+y+z}$.

238. Find div $(\mathbf{u} \times \mathbf{v})$, where $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{v} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$.

239. Find the flux of the radius vector $\mathbf{r}$ across the outer side of the surface of a right circular cone, if h is the height of the cone and R is the radius of its base.

240. Find the circulation of the vector $\mathbf{A} = -y\mathbf{i} + x\mathbf{j}$ around the circle $x^2 + (y - 1)^2 = 1$.

241. Find the circulation of the vector $\mathbf{u} = (x + z)\mathbf{i} + (x - y)\mathbf{j} + x\mathbf{k}$ around the ellipse $x^2/a^2 + y^2/b^2 = 1$.

242. Find rot$(\mathbf{r} \cdot \mathbf{a})\mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

243. Find rot$(\mathbf{r} \cdot \mathbf{a})\mathbf{b}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} - \mathbf{k}$.

244. Show that div$(\mathbf{a} \times \mathbf{b}) = \mathbf{b}$ rot $\mathbf{a} - \mathbf{a}$ rot $\mathbf{b}$.

245. Show that div $(f\mathbf{A}) = f$ div $\mathbf{A} + \mathbf{A}$ grand f.

Chapter 3

Series

3.1. Number Series

Suppose $u_1, u_2, u_3, \ldots, u_n, \ldots$, where $u_n = f(n)$, is an infinite number sequence. The expression

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

is called an infinite *number series*, and the numbers $u_1, u_2, u_3, \ldots, u_n$, the *terms* of the series; $u_n = f(n)$ is known as the *general term*. The series is often written as ten as $\sum_{n=1}^{\infty} u_n$.

The sum of the first n terms of a number series is denoted by S_n and is called the nth *partial sum of the series*:

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n .$$

A series is *convergent* if its nth partial sum S_n tends to a finite limit as n increases indefinitely, i.e. if $\lim_{n \to \infty} S_n = S$. The number S is called the *sum* of the series. If the nth partial sum does not tend to a finite limit as $n \to \infty$, then the series is *divergent*.

The series

$$a + aq + aq^2 + \ldots + aq^{n-1} + \ldots \quad (|q| < 1)$$

set up from the terms of any decreasing geometric progression is convergent and possesses the sum $a/(1 - q)$.

The series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots,$$

called *harmonic* series, diverges.

Here are the main theorems concerning convergent number series.

1. *If the series*

$$u_1 + u_2 + u_3 + \ldots$$

converges, then so does the series

$$u_{m+1} + u_{m+2} + u_{m+3} + \ldots,$$

obtained from the given series by discarding the first m terms (the last series is called the mth *remainder* of the original series); conversely, *the convergence of the mth remainder of a series implies the convergence of the given series.*

2. *If the series*

$$u_1 + u_2 + u_3 + \dots$$

converges, and the number S is its sum, then so does the series

$$au_1 + au_2 + au_3 + \dots,$$

and the sum of the last series is equal to aS.

3. *If the series*

$$u_1 + u_2 + u_3 + \dots, \quad v_1 + v_2 + v_3 + \dots,$$

possessing the sums S and σ respectively, converges, then so does the series

$$(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) + \dots,$$

the sum of the last series being equal to $S + \sigma$.

4. *If the series*

$$u_1 + u_2 + u_3 + \dots$$

converges, then $\lim\limits_{n\to\infty} u_n = 0$, *that is, when* $n \to \infty$ *the limit of the general term of the convergent series is equal to zero* (**necessary condition** for convergence of a series).

Thus, if $\lim\limits_{n\to\infty} u_n \neq 0$, then series diverges.

Below we enumerate the most important conditions for convergence and divergence of series with positive terms.

The 1st comparison test. *Suppose we are given two series with positive terms:*

$$u_1 + u_2 + u_3 + \dots + u_n + \dots \tag{1}$$

and

$$v_1 + v_2 + v_3 + \dots + v_n + \dots, \tag{2}$$

every term of series (1) *not exceeding the respective term of series* (2), *i.e.* $u_n \leqslant v_n$ $(n = 1, 2, 3, \dots)$. *Then, in case series* (2) *converges, series* (1) *converges too, and if series* (1) *diverges, then series* (2) *diverges too.*

This test remains valid even if the inequalities $u_n < v_n$ are satisfied not for all n but only beginning with some number $n = N$.

The 2nd comparison test. *If there exists a finite nonzero limit* $\lim\limits_{n\to\infty} \times (u_n/v_n) = k$, *then both series* $\sum\limits_{n=1}^{\infty} u_n$ and $\sum\limits_{n=1}^{\infty} v_n$ *are simultaneously convergent or simultaneously divergent.*

Cauchy's test. *If there exists a limit* $\lim\limits_{n\to\infty} \sqrt[n]{u_n} = C$ *for the series*

$$u_1 + u_2 + u_3 + \dots + u_n + \dots,$$

then this series converges for $C < 1$, *and diverges for* $C > 1$.

D'Alembert's test. *If there exists* $\lim_{n\to\infty}(u_{n+1}/u_n) = D$ *for the series*

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots,$$

then this series converges for $D < 1$ *and diverges for* $D > 1$.

Integral test. *If, for* $x \geqslant 1$, $f(x)$ *is a continuous, positive and monotonic decreasing function, then the series* $\sum_{n=1}^{\infty} u_n$, *where* $u_n = f(n)$, *converges or diverges according as the integral* $\int_1^{\infty} f(x)\,dx$ *converges or diverges.*

Let us now consider series whose terms are of different signs.

An *alternating series* is a series of the form

$$u_1 - u_2 + u_3 - u_4 + \ldots,$$

where $u_n > 0$ $(n = 1, 2, 3, \ldots)$.

Test for convergence of an alternating series (Leibniz' test). *An alternating series converges if the absolute values of its terms decrease and the general term tends to zero,* that is, if the following two conditions are fulfilled: (1) $u_1 > u_2 > u_3 > \ldots$ and (2) $\lim_{n\to\infty} u_n = 0$.

Let us take the nth partial sum of an alternating series for which Leibniz' test is complied with:

$$S_n = u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1}u_n.$$

Assume that R_n is the nth remainder of the series. It can be written as the difference between the sum S of the series and the nth partial sum S_n, i.e. $R_n = S - S_n$. It is easy to see that

$$R_n = (-1)^n(u_{n+1} - u_{n+2} + u_{n+3} - u_{n+4} + \ldots).$$

The absolute value $|R_n|$ is found by means of the inequality $|R_n| < u_{n+1}$.

Let us now discuss some properties of sign-changing series (that is, alternating series and series arbitrarily changing their signs).

A sign-changing series

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

converges if the series

$$|u_1| + |u_2| + |u_3| + \ldots + |u_n| + \ldots$$

converges.

In that case the original series $\sum_{n=1}^{\infty} u_n$ is called *absolutely convergent.*

The convergent series $\sum_{n=1}^{\infty} u_n$ is called *conditionally convergent* if the series $\sum_{n=1}^{\infty} |u_n|$ diverges.

If the series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent, then the series obtained after any commutation of the infinite number of its terms is absolutely convergent and possesses the same sum as the original series.

If the series $\sum_{n=1}^{\infty} u_n$ is conditionally convergent, then upon a commutation of the infinite number of its terms the sum of the series may change. In particular, making a requisite commutation of the terms of a conditionally convergent series, we can turn it into a divergent series.

If the series $u_1 + u_2 + u_3 + \dots$ and $v_1 + v_2 + v_3 + \dots$ are absolutely convergent and possess the numbers S_1 and S_2 as their sums, then the series

$$u_1v_1+(u_1v_2+v_1u_2)+(u_1v_3+u_2v_2+u_3v_1)+\dots+(u_1v_n+u_2v_{n-1}+\dots+u_nv_1)+\dots,$$

is also absolutely convergent.

This series is called the *product* of the series

$$u_1 + u_2 + u_3 + \dots \quad \text{and} \quad v_1 + v_2 + v_3 + \dots.$$

Its sum is S_1S_2.

246. Given the general term of the series $u_n = \dfrac{n}{10^n + 1}$. Write the first four terms of the series.

Solution. If $n = 1$, then $u_1 = 1/11$; if $n = 2$, then $u_2 = 2/101$; if $n = 3$, then $u_3 = 3/1001$; if $n = 4$, then $u_4 = 4/10001$; ... . The series can be written as

$$\frac{1}{11} + \frac{2}{101} + \frac{3}{1001} + \frac{4}{10001} + \dots .$$

247. Find the general term of the series

$$\frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \frac{7}{2^4} + \dots.$$

Solution. The successive numerators form the arithmetic progression 1, 3, 5, 7, ... ; we find the nth term of the progression by the formula $a_n = a_1 + d(n - 1)$. Here $a_1 = 1$, $d = 2$, therefore, $a_n = 2n - 1$. The successive denominators form the geometric progression 2, 2^2, 2^3, 2^4, ... ; the nth term of this progression is $b_n = 2^n$. Consequently, the general term of the series $u_n = (2n - 1)/2^n$.

248. Find the general term of the series

$$\frac{2}{3} + \left(\frac{3}{7}\right)^2 + \left(\frac{4}{11}\right)^3 + \left(\frac{5}{15}\right)^4 + \dots .$$

Solution. The exponent of each term coincides with the number of that term in the series; therefore, the exponent of the nth term is n. The numerators of the fractions 2/3, 3/7, 4/11, 5/15, ... form an arithmetic progression with the first term 2 and the difference 1. Therefore, the nth numerator is $n + 1$. The denominators form an arithmetic progression with the first term 3 and the difference 4. Consequently, the nth denominator is $4n - 1$. Hence the general term of the series is

$$u_n = \left(\frac{n+1}{4n-1}\right)^n.$$

249. Find the sum of the series $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} + \ldots$.

Solution. We have $u_n = \frac{1}{(2n-1)(2n+1)}$. Since

$$u_n = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right),$$

we have

$$u_1 = \frac{1}{2}\left(1 - \frac{1}{3}\right), u_2 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right),$$

$$u_3 = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right),$$

$$u_4 = \frac{1}{2}\left(\frac{1}{7} - \frac{1}{9}\right), \ldots .$$

Consequently,

$$S_n = \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$+ \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \ldots\right.$$

$$\left.\ldots + \frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right).$$

Since $\lim\limits_{n\to\infty} S_n = \frac{1}{2} \lim\limits_{n\to\infty}\left(1 - \frac{1}{2n+1}\right) = \frac{1}{2}$, the series converges and its sum is equal to 1/2.

250. Find the sum of the series $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \ldots$.

Solution. We represent the general term of the series $u_n = \frac{1}{n(n+1)(n+2)}$ as the sum of the partial fractions

$$\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}.$$

Multiplication by the denominator of the left-hand side yields the identity

$$1 \equiv A(n+1)(n+2) + Bn(n+2) + Cn(n+1).$$

Setting consecutively $n = 0, -1, -2$, we find: $1 = 2A$; $A = 1/2$ at $n = 0$; $1 = -B$; $B = -1$ at $n = -1$; $1 = 2C$; $C = 1/2$ at $n = -2$. Thus we have

$$u_n = \frac{1}{2} \cdot \frac{1}{n} - \frac{1}{n+1} + \frac{1}{2} \cdot \frac{1}{n+2}, \quad \text{i.e.}$$

$$u_n = \frac{1}{2}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right).$$

Hence

$$u_1 = \frac{1}{2}\left(1 - \frac{2}{2} + \frac{1}{3}\right), \quad u_2 = \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right),$$
$$u_3 = \frac{1}{2}\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right), \quad u_4 = \frac{1}{2}\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right), \ldots$$

and

$$S_n = \frac{1}{2}\left(1 - \frac{2}{2} + \frac{1}{3} + \frac{1}{2} - \frac{2}{3}\right.$$
$$+ \frac{1}{4} + \frac{1}{3} - \frac{2}{4} + \frac{1}{5} + \frac{1}{4} - \frac{2}{5} + \frac{1}{6} + \ldots$$
$$\left.\ldots + \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1} + \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) =$$
$$\frac{1}{2}\left(\frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}\right).$$

Thus $\lim\limits_{n\to\infty} S_n = 1/4$; consequently, the series converges and its sum is equal to $1/4$.

251. Test the series

$$\frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \ldots$$

for convergence.

Solution. The series consists of the terms of an infinitely decreasing geometric progression and is, therefore, convergent. Let us find the sum of the series. Here $a = 2/3$, $q = 1/2$ (the denominator of the progression). Consequently,

$$S = \frac{a}{1-q} = \frac{2/3}{1-1/2} = \frac{4}{3}.$$

252. Test the series

$$\frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \ldots$$

for convergence.

Solution. The given series has been obtained from the harmonic series by discarding the first ten terms. This means that it is divergent.

253. Test the series

$$\frac{1}{2} + \frac{2}{5} + \frac{3}{8} + \frac{4}{11} + \dots$$

for convergence.

Solution. We find the general term of the series $u_n = n/(3n - 1)$. Since

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{3n-1} = \lim_{n\to\infty} \frac{1}{3 - 1/n} = \frac{1}{3},$$

i.e. $\lim\limits_{n\to\infty} u_n \neq 0$, the series is divergent (the necessary condition is not fulfilled).

254. Test the series

$$0.6 + 0.51 + 0.501 + 0.5001 + \dots$$

for convergence.

Solution. Here $u_n = 0.5 + (0.1)^n$, $\lim\limits_{n\to\infty} u_n = 0.5 \neq 0$ and the series diverges.

255. Test the series $\sum\limits_{n=1}^{\infty} \frac{1}{2^n + 1}$ for convergence.

Solution. The terms of the given series are smaller than the respective terms of the series $\sum\limits_{n=1}^{\infty} \frac{1}{2^n}$, that is, the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$. But the last series converges as an infinitely decreasing geometric progression. Consequently, the given series converges too.

256. Test the series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

for convergence if $p < 1$.

Solution. The terms of the series, beginning with the second, are larger than the respective terms of the geometric series. Consequently, the series diverges.

257. Test for convergence the series with the general term $u_n = \dfrac{1}{4 \cdot 2^n - 3}$.

Solution. Let us compare this series with the series possessing the general term $v_n = 1/2^n$ (that is, an infinitely decreasing geometric progression):

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2^n}{4 \cdot 2^n - 3} = \lim_{n\to\infty} \frac{1}{4 - 3/2^n} = \frac{1}{4}.$$

Since the limit is finite and different from zero and the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, so does the given series.

258. Test the series

$$\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \ldots$$

for convergence.

Solution. Here $u_n = 1/(3n - 1)$. Let us compare this series with a harmonic series in which $v_n = 1/n$:

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{n}{3n-1} = \frac{1}{3}.$$

We see that the given series diverges.

259. Test the series

$$\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \left(\frac{4}{9}\right)^4 + \ldots$$

for convergence.

Solution. We have $u_n = \left(\frac{n}{2n+1}\right)^n$. It is convenient here to apply Cauchy's test since $\sqrt[n]{u_n} = \frac{n}{2n+1}$, and the limit for the last fraction is easy to find:

$$C = \lim_{n \to \infty} \sqrt[n]{u_n} = \lim_{n \to \infty} \frac{n}{2n+1} = \lim_{n \to \infty} \frac{1}{2 + 1/n} = \frac{1}{2}.$$

Since $C = 1/2 < 1$, the series converges.

260. Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n}\left(1 + \frac{1}{2}\right)^n$ for convergence.

Solution. We apply Cauchy's test:

$$u_n = \frac{1}{2^n}\left(1 + \frac{1}{n}\right)^{n^2}, \sqrt[n]{u_n} = \frac{1}{2}\left(1 + \frac{1}{n}\right)^n,$$

$$C = \lim_{n \to \infty} \sqrt[n]{u_n} = \frac{1}{2} \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \frac{1}{2} e;$$

since $C > 1$, the series deverges.

261. Test the series

$$\frac{2}{1} + \frac{2^2}{2^{10}} + \frac{2^3}{3^{10}} + \ldots + \frac{2^n}{n^{10}} + \ldots$$

for convergence.

Solution. We apply D'Alembert's test. We have $u_n = 2^n/n^{10}$, $u_{n+1} =$

$= 2^{n+1}/(n+1)^{10}$, $u_{n+1}/u_n = 2n^{10}/(n+1)^{10}$. Hence

$$D = \lim_{n\to\infty} \frac{2n^{10}}{(n+1)^{10}} = \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^{10}} = 2.$$

Since $D > 1$, the series diverges.

262. Test the series

$$\frac{1}{\sqrt{3}} + \frac{2}{3} + \frac{3}{3\sqrt{3}} + \frac{4}{9} + \frac{5}{9\sqrt{3}} + \ldots$$

for convergence.

Solution. Here $u_n = n/3^{n/2}$, $u_{n+1} = (n+1)/3^{(n+1)/2}$, $u_{n+1}/u_n = (n+1)/(n\sqrt{3})$; therefore,

$$D = \lim_{n\to\infty} \frac{n+1}{n\sqrt{3}} = \lim_{n\to\infty} \frac{1+1/n}{\sqrt{3}} = \frac{1}{\sqrt{3}}; \quad D < 1.$$

Consequently, the series converges.

263. Test the series

$$\frac{10}{1!} + \frac{10^2}{2!} + \frac{10^3}{3!} + \ldots$$

for convergence.

Solution. We have $u_n = 10^n/n!$, $u_{n+1} = 10^{n+1}/(n+1)!$, $u_{n+1}/u_n = 10/(n+1)$, $D = \lim\limits_{n\to\infty} 10/(n+1) = 0$, $D < 1$. The series converges.

264. Test the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

for convergence.

Solution. We have $u_n = 1/n^2$, $u_{n+1} = 1/(n+1)^2$, $u_{n+1}/u_n = n^2/(1+n^2) = 1/(1+1/n)^2$, $D = \lim\limits_{n\to\infty} (u_{n+1}/u_n) = 1$. Since $D = 1$, D'Alembert's test does not give the solution of the problem.

Let us apply the integral test: $u_n = \dfrac{1}{n^2}$, consequently, $f(x) = \dfrac{1}{x^2}$, $\displaystyle\int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^\infty = 1$. The integral converges (is a finite quantity), therefore, the given series converges too.

265. Test the series

$$\frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \frac{1}{4\ln 4} + \ldots$$

for convergence.

Solution. We apply the integral test:

$$u_n = \frac{1}{(n+1)\ln(n+1)}, \quad f(x) = \frac{1}{(x+1)\ln(x+1)},$$

$$\int_1^\infty \frac{dx}{(x+1)\ln(x+1)} = \int_1^\infty \frac{\frac{dx}{x+1}}{\ln(x+1)} = \ln\ln(x+1)\Big|_1^\infty = \infty.$$

The integral diverges and, consequently, the given series diverges too.

266. Test the series

$$\frac{1}{2} - \frac{2}{2^2+1} + \frac{3}{3^2+1} - \frac{4}{4^2+1} + \ldots$$

for convergence.

Solution. We apply Leibniz' test. Since

$$\frac{2}{2^2+1} = \frac{1}{2+1/2}, \quad \frac{3}{3^2+1} = \frac{1}{3+1/3}, \quad \frac{4}{4^2+1} = \frac{1}{4+1/4}, \quad \ldots,$$

it follows that

$$\frac{1}{2} > \frac{2}{2^2+1} > \frac{3}{3^2+1} > \frac{4}{4^2+1} > \ldots .$$

Consequently, the first condition of Leibniz' test is fulfilled. Since $u_n = n/(n^2+1)$, we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+1/n} = 0,$$

that is, the second condition is fulfilled. The series converges.

267. Test the series

$$1.1 - 1.01 + 1.001 - 1.0001 + \ldots$$

for convergence.

Solution. The first condition of Leibniz' test is fulfilled: $1.1 > 1.01 > 1.001 > 1.0001 > \ldots$; on the other hand, $u_n = 1 + \frac{1}{10^n}$, $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1 + \frac{1}{10^n}\right) = 1$. Since $\lim_{n\to\infty} u_n \neq 0$, the necessary condition for convergence of the series is not fulfilled. The series diverges.

268. Test the series

$$1 - 1 + 1 - 1 + \ldots$$

for convergence.

Solution. The general term of the series does not tend to zero, therefore, the series diverges.

269. Test the series

$$1-\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\frac{1}{2^4}-\frac{1}{2^5}+\ldots$$

for convergence.

Solution. Let us set up a series of absolute values:

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots.$$

The series is an infinitely decreasing geometric progression and, consequently, it is convergent. Hence, the given series converges as well, and its convergence is absolute.

270. Find the product of the absolutely convergent series

$$1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\ldots+\frac{2^n}{n!}+\ldots$$

and

$$1+\frac{3}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\ldots+\frac{3^n}{n!}+\ldots.$$

Solution. In accordance with the definition presented on p. 75, the product of the series is the series

$$1+\left(\frac{2}{1!}+\frac{3}{1!}\right)+\left(\frac{2^2}{2!}+\frac{2}{1!}\cdot\frac{3}{1!}+\frac{3^2}{2!}\right)+\left(\frac{2^3}{3!}+\frac{2^2}{2!}\cdot\frac{3}{1!}+\frac{2}{1!}\cdot\frac{3^2}{2!}+\frac{3^3}{3!}\right)+\ldots$$
$$+\left(\frac{2^n}{n!}+\frac{2^{n-1}}{(n-1)!}\cdot\frac{3}{1!}+\frac{2^{n-2}}{(n-2)!}\cdot\frac{3^2}{2!}+\ldots+\frac{3^n}{n!}\right)+\ldots,$$

or

$$1+\frac{1}{1!}(2+3)+\frac{1}{2!}(2^2+2\cdot 2\cdot 3+3^2)$$
$$+\frac{1}{3!}\cdot(2^3+3\cdot 2^2\cdot 3+3\cdot 2\cdot 3^2+3^3)+\ldots+\frac{1}{n!}\Bigg(2^n$$
$$+\frac{n!}{(n-1)!\cdot 1!}\cdot 2^{n-1}\cdot 3+\frac{n!}{(n-2)!\cdot 2!}\cdot 2^{n-2}\cdot 3^2+\ldots+3^n\Bigg)+\ldots.$$

Since $\dfrac{n!}{(n-k)!k!}=C_n^k\,(k=1,2,\ldots)$, the series can be rewritten in the form

$$1+\frac{2+3}{1!}+\frac{(2+3)^2}{2!}+\frac{(2+3)^3}{3!}+\ldots+\frac{(2+3)^n}{3!}+\ldots,$$

or

$$1 + \frac{5}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \ldots + \frac{5^n}{n!} + \ldots.$$

271. Write the first four terms of the series $\sum\limits_{n=1}^{\infty} \frac{n}{10^n + n}$.

272. Write the first four terms of the series $\sum\limits_{n=1}^{\infty} \frac{9}{100^n - 1}$.

Derive the formulas for the general terms of the following series:

273. $\frac{10}{7} + \frac{100}{9} + \frac{1000}{11} + \frac{10\,000}{13} + \ldots.$ **274.** $\frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \frac{7}{8} + \ldots.$

275. $\frac{2}{1} + \frac{2^2}{1 \cdot 2} + \frac{2^3}{1 \cdot 2 \cdot 3} + \frac{2^4}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots.$ **276.** $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots.$

Find the sums of the following series:

277. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots.$ **278.** $\sum\limits_{n=1}^{\infty} \frac{2n + 1}{n^2(n + 1)^2}$. **279.** $\frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7 \cdot 9} + \ldots.$ **280.** $1 + \frac{m - 1}{m} + \left(\frac{m - 1}{m}\right)^2 + \left(\frac{m - 1}{m}\right)^3 + \ldots$ $(m > 1)$.

281. Use the necessary condition to show that the series $\frac{1}{9} + \frac{9}{12} + \frac{3}{29} + \frac{4}{39} + \ldots$ diverges.

282. Show that the series $\sum\limits_{n=1}^{\infty} \frac{4n}{(2^n + 1)^2}$ diverges.

Test the following series for convergence with the aid of the first comparison test:

283. $\frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \frac{1}{\ln 5} + \ldots.$ **284.** $\sum\limits_{n=1}^{\infty} \frac{2n}{5^n + 1}$.

Test the following series for convergence, using the second comparison test:

285. $\frac{2 + 1}{5 + 1} + \frac{2^2 + 1}{5^2 + 1} + \frac{2^3 + 1}{5^3 + 1} + \ldots.$

Hint. Compare with the series $\frac{2}{5} + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \ldots.$

286. $\frac{1}{2\cdot 1-1}+\frac{\sqrt{2}}{2\cdot 2-1}+\frac{\sqrt{3}}{2\cdot 3-1}+\frac{\sqrt{4}}{2\cdot 4-1}+\dots.$

Using Cauchy's test, investigate the convergence of the following series:

287. $\sum_{n=1}^{\infty}\left(\frac{2n^2+2n+1}{5n^2+2n+1}\right)^n$. **288.** $3+(2.1)^2+(2.01)^3+(2.001)^4+\dots.$

Using D'Alembert's test, investigate the convergence of the following series:

289. $\frac{10}{11}+\left(\frac{10}{11}\right)^2\cdot 2^5+\left(\frac{10}{11}\right)^3\cdot 3^5+\left(\frac{10}{11}\right)^4\cdot 4^5+\dots.$

290. $\frac{11}{10}+\left(\frac{11}{10}\right)^2\cdot\frac{1}{2^5}+\left(\frac{11}{10}\right)^3\cdot\frac{1}{3^5}+\left(\frac{11}{10}\right)^4\cdot\frac{1}{4^5}+\dots.$

Use the integral test to investigate the convergence of the following series:

291. $\sum_{n=1}^{\infty}\frac{1}{n^p}$, if $p>1$. **292.** $\frac{1}{9\ln 9}+\frac{1}{19\ln 19}+\frac{1}{29\ln 29}+\dots.$

Test the following sign-changing series for convergence and establish the nature of the convergence:

293. $\frac{1}{2}-\frac{4}{5}+\frac{7}{8}-\frac{10}{11}+\dots.$

294. $1.1-1.02+1.003-1.0004+\dots.$

295. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(n+1)}{n^2+n+1}$.

296. $\frac{1}{10}+\frac{7}{10^2}-\frac{13}{10^3}+\frac{19}{10^4}+\frac{25}{10^5}-\frac{31}{10^6}+\dots.$

297. $3\frac{1}{2}+3\frac{1}{4}-3\frac{1}{8}-3\frac{1}{16}+3\frac{1}{32}+3\frac{1}{64}-3\frac{1}{128}-3\frac{1}{256}+\dots.$

Test the following series for convergence:

298. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots.$ **299.** $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\dots.$

300. $\frac{1}{10}+\frac{1}{20}+\frac{1}{30}+\frac{1}{40}+\dots.$ **301.** $\frac{1}{8}+\frac{1}{18}+\frac{1}{28}+\frac{1}{38}+\dots.$

302. $1 + \frac{2}{5} + \frac{3}{5^2} + \frac{4}{5^3} + \ldots$. **303.** $\frac{1!}{5} + \frac{2!}{5^2} + \frac{3!}{5^3} + \ldots$.

304. $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \ldots$. **305.** $1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$.

306. $\frac{1}{2\ln 2 \cdot \ln\ln 2} + \frac{1}{3\ln 3 \cdot \ln\ln 3} + \frac{1}{4\ln 4 \cdot \ln\ln 4} + \ldots$.

307. $\frac{1}{2} + \frac{1}{2^3+1} + \frac{1}{3^3+1} + \frac{1}{4^3+1} + \ldots$.

308. $\frac{2}{2^3+1} - \frac{3}{3^3+2} + \frac{4}{4^3+3} - \frac{5}{5^3+4} + \ldots$.

309. $1 - 2 + 3 - 4 + 5 - 6 + \ldots$. **310.** $1 - \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} - \frac{1}{6^4} + \ldots$.

311. $\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} + \frac{5}{6} - \frac{6}{7} + \ldots$.

312. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$. **313.** $\frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + \ldots$.

314. Find the product of the absolutely convergent series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ and $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots$.

315. Show that the series $1 - \frac{2}{1!} + \frac{2^2}{2!} - \frac{2^3}{3!} + \ldots$ is absolutely convergent and raise it to the second power (multiply by itself).

3.2. Functional Series

The series

$$u_1(x) + u_2(x) + u_3(x) + \ldots + u_n(x) + \ldots,$$

whose terms are the functions of x, is called *functional*. The collection of the values of x at which the functions $u_1(x), u_2(x), \ldots, u_n(x) \ldots$ are defined and the series $\sum_{n=1}^{\infty} u_n(x)$ converges, is called the *domain of the convergence* of the functional series. Most often, such a domain is some interval of the x-axis. Each value of the domain of convergence X is associated with a certain value of the quantity $\lim_{n\to\infty} \sum_{n=1}^{n} u_n(x)$. This quantity, which is a function of x, is called the *sum* of the functional series and is denoted $S(x)$.

Let us represent the sum of the series as $S(x) = S_n(x) + R_n(x)$, where

$$S_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x), \quad R_n(x) = u_{n+1}(x) + u_{n+2}(x) + \ldots$$

($R_n(x)$ is the remainder of the functional series).

The convergent functional series $\sum\limits_{n=1}^{\infty} u_n(x)$ is said to be *uniformly convergent* in a certain domain X if for every arbitrarily small number $\varepsilon > 0$ there is a positive integer N such that at $n \geqslant N$ there holds an inequality $|R_n(x)| < \varepsilon$ for every x of the domain X. In this case, the sum $S(x)$ of the uniformly convergent series $\sum\limits_{n=1}^{\infty} u_n(x)$ in the domain X, where $u_n(x)$ ($n = 1, 2, 3, \ldots$) are continuous functions, is a continuous function.

Let us formulate the **Weierstrass sufficient condition** for uniform convergence of a functional series.

If the absolute values of the functions $u_1(x), u_2(x), \ldots, u_n(x), \ldots$ do not exceed the positive numbers $a_1, a_2, \ldots a_n, \ldots$ in a certain domain X, and the number series

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

converges, then the functional series

$$u_1(x) + u_2(x) + u_3(x) + \ldots + u_n(x) + \ldots$$

uniformly converges in that domain.

In conclusion, we shall formulate two theorems relating to integration and differentiation of functional series.

1. *If the series $u_1(x) + u_2(x) + u_3(x) + \ldots$, where $u_1(x), u_2(x), u_3(x) \ldots$ are continuous functions, uniformly converges in a certain domain X and possesses a sum $S(x)$, then the series*

$$\int\limits_a^b u_1(x)\,dx + \int\limits_a^b u_2(x)\,dx + \int\limits_a^b u_3(x)\,dx + \ldots$$

converges and possesses a sum $\int\limits_a^b S(x)\,dx$ (the interval $[a, b]$ belongs to the domain X).

2. Assume that the functions $u_1(x), u_2(x), u_3(x), \ldots$ are defined in a certain domain X and possess derivatives $u'_1(x), u'_2(x), u'_3(x), \ldots$ in that domain.

If the series $\sum\limits_{n=1}^{\infty} u'_n(x)$ uniformly converges in that domain, then its sum is equal to the derivative of the sum of the original series:

$$\sum_{n=1}^{\infty} u'_n(x) = \left\{\sum_{n=1}^{\infty} u_n(x)\right\}'_x .$$

316. Test the functional series

$$\frac{4-x}{7x+2} + \frac{1}{3}\left(\frac{4-x}{7x+2}\right)^2 + \frac{1}{5}\left(\frac{4-x}{7x+2}\right)^3 + \ldots$$

for convergence at the points $x = 0$, $x = 1$.

Solution. At $x = 0$, the series is

$$2 + \frac{1}{3} \cdot 2^2 + \frac{1}{5} \cdot 2^3 + \frac{1}{7} \cdot 2^4 + \ldots .$$

Here $u_n = 2^n/(2n - 1)$, $u_{n+1} = 2^{n+1}/(2n + 1)$. We apply the D'Alembert test:

$$D = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2^{n+1}(2n - 1)}{2^n(2n + 1)} = 2 \lim_{n\to\infty} \frac{2n - 1}{2n + 1} = 2 \lim_{n\to\infty} \frac{2 - 1/n}{2 + 1/n} = 2,$$

i.e. $D > 1$. Consequently, the series diverges.

At the point $x = 1$ we get a series

$$\frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3^2} + \frac{1}{5} \cdot \frac{1}{3^3} + \ldots .$$

Here $u_n = 1/(3^n(2n - 1))$, $u_{n+1} = 1/(3^{n+1} \cdot (2n + 1))$; we find

$$D = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3^n \cdot (2n - 1)}{3^{n+1} \cdot (2n + 1)} = \frac{1}{3} \lim_{n\to\infty} \frac{2n - 1}{2n + 1} = \frac{1}{3},$$

that is, the series converges.

317. Find the domain of convergence of the series

$$\frac{1}{1 + x^2} + \frac{1}{1 + x^4} + \frac{1}{1 + x^6} + \ldots .$$

Solution. We find the general term of the series, $u_n = 1/(1 + x^{2n})$. If $|x| < 1$, then $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{1 + x^{2n}} = 1$; since $\lim_{n\to\infty} u_n \neq 0$, the series diverges. If $|x| = 1$, we again obtain a divergent series $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$.

If $|x| > 1$, the terms of the given series are smaller than the terms of the infinitely decreasing geometric progression $\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} + \ldots$, that is, the series converges.

Thus we see that the domain of convergence of the series is specified by the inequality $|x| > 1$. Hence the series converges if $1 < x < +\infty$ or $-\infty < x < -1$.

318. Show that the series

$$\frac{1}{x^2 + 1} - \frac{1}{x^4 + 2} + \frac{1}{x^6 + 3} - \frac{1}{x^8 + 4} + \ldots$$

uniformly converges for all the values of x $(-\infty < x < +\infty)$.

Solution. For any value of x the given series converges in accordance with Leibniz' test, therefore its remainder can be evaluated with the aid of the inequality $|R_n(x)| < |u_{n+1}(x)|$, i.e.

$$|R_n(x)| < \frac{1}{x^{2n+2} + n + 1} < \frac{1}{n + 1}.$$

Since the inequalitites $\frac{1}{n+1} \leqslant \varepsilon$ and $n \geqslant \frac{1}{\varepsilon} - 1$ are equivalent, we can take $n \geqslant N$, where N is some positive integer satisfying the condition $N \geqslant \frac{1}{\varepsilon} - 1$. Then we shall arrive at the inequality $|R_n(x)| < \varepsilon$. Thus we see that the given series uniformly converges in the interval $(-\infty, +\infty)$.

319. Show that the series $\sum\limits_{n=1}^{\infty} x^n$ is nonuniformly convergent in the interval $(-1, 1)$.

Solution. In the indicated interval the series converges as an infinitely decreasing geometric progression. We have $R_n(x) = x^{n+1} + x^{n+2} + x^{n+3} + \ldots$, i.e. $R_n(x) = x^{n+1}/(1 - x)$. But $\lim\limits_{n \to -1+0} |R_n(x)| = 1/2$, $\lim\limits_{n \to 1-0} R_n(x) = \infty$. Consequently, assuming $\varepsilon > 1/2$, we cannot obtain a satisfied inequality for any value of x. Thus the series $\sum\limits_{n=1}^{\infty} x^n$ is nonuniformly convergent.

320. Show with the aid of the Weierstrass test that the series

$$\sin x + \frac{1}{2^2} \cdot \sin^2 2x + \frac{1}{3^2} \cdot \sin^3 3x + \ldots$$

is uniformly convergent in the interval $(-\infty, +\infty)$.

Solution. Since $\left| \frac{1}{n^2} \sin^n nx \right| \leqslant \frac{1}{n^2}$ and the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ converges, it follows that the given series uniformly converges at any values of x.

321. Can the theorem on the term-by-term differentiation of series be applied to the series

$$\arctan x + \arctan \frac{x}{2\sqrt{2}} + \arctan \frac{x}{3\sqrt{3}} + \ldots + \arctan \frac{x}{n\sqrt{n}}\ ?$$

Solution. Let us compare the given series with the convergent series

$$x + \frac{x}{2^{3/2}} + \frac{x}{3^{3/2}} + \ldots + \frac{x}{n^{3/2}} + \ldots$$

(at any fixed x). Then $u_n(x) = \arctan(x/n^{3/2})$, $v_n(x) = x/n^{3/2}$. Since $\arctan \alpha$ and α are equivalent infinitesimals, it follows that $\lim\limits_{n \to \infty} \frac{u_n(x)}{v_n(x)} = 1$, and we infer by the second comparison test that the given series is convergent.

Let us find the derivative of the general term of the given series:

$$u_n'(x) = \frac{1/n^{3/2}}{1 + x^2/n^3} = \frac{n^{3/2}}{x^2 + n^3}.$$

The series set up from the derivatives has the form

$$\frac{1}{x^2+1} + \frac{2\sqrt{2}}{x^2+2^3} + \frac{3\sqrt{3}}{x^2+3^3} + \frac{4\sqrt{4}}{x^2+4^3} + \ldots .$$

Note that the terms of the last series are smaller than the respective terms of the convergent series $1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \ldots$. Consequently, in accordance with the Weierstrass test, a series set up from the derivatives converges uniformly in the interval $(-\infty, +\infty)$ and so we can apply the theorem on differentiation of series to the given series.

322. Can we apply the theorem on integration of functional series in the interval $[\pi/4, \pi/3]$ to the series

$$\cos x + \frac{1}{2} \cdot \cos 2x + \frac{1}{2^2} \cdot \cos 3x + \frac{1}{2^3} \cdot \cos 4x + \ldots ?$$

Solution. At any value of x, the terms of the given series are smaller in absolute value than the respective terms of the infinitely decreasing geometric progression

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots .$$

Therefore, in accordance with the Weierstrass test, the given series converges uniformly in the interval $(-\infty, +\infty)$ and, consequently, we can apply to it the theorem on integration of series for any finite interval $[a, b]$ and, in particular, for the interval $[\pi/4, \pi/3]$.

323. Test the functional series

$$\frac{3x+1}{x^2+x+1} + \left(\frac{3x+1}{x^2+x+1}\right)^2 + \left(\frac{3x+1}{x^2+x+1}\right)^3 + \ldots$$

for convergence at the points $x = 1$, $x = 2$, $x = 3$.

324. Test the functional series

$$\frac{1!}{1}(x^2 - 4x + 6) + \frac{2!}{2^2}(x^2 - 4x + 6)^2 + \frac{3!}{3^3}(x^2 - 4x + 6)^3 + \ldots$$

for convergence at the points $x = 1$ and $x = 2$.

325. Find the domain of convergence of the series

$$1 + e^{-x} + e^{-2x} + e^{-3x} + \ldots .$$

326. Find the domain of convergence of the series

$$1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \ldots .$$

327. Find the domain of convergence of the series

$$\frac{1}{x^2+1}+\frac{1}{2^2(x^2+1)^2}-\frac{1}{3^2(x^2+1)^3}+\frac{1}{4^2(x^2+1)^4}+\ldots.$$

328. Show that the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^n\cdot n}{x^4+n^2}$ uniformly converges in the interval $(-\infty, +\infty)$.

329. Show that the series

$$\frac{2x+1}{x+2}+\frac{1}{2}\left(\frac{2x+1}{x+2}\right)^2+\frac{1}{4}\left(\frac{2x+1}{x+2}\right)^3+\frac{1}{8}\left(\frac{2x+1}{x+2}\right)^4+\ldots$$

uniformly converges in the interval $[-1, 1]$.

330. Show that the series

$$x+\frac{x^2}{2}+\frac{x^3}{4}+\frac{x^4}{8}+\ldots$$

converges nonuniformly in the interval $(-2, 2)$.

331. Show that the series

$$\frac{\sin x+\sqrt{3}\cos x}{3}+\frac{(\sin x+\sqrt{3}\cos x)^2}{3^2}+\frac{(\sin x+\sqrt{3}\cos x)^3}{3^3}+\ldots$$

converges in the interval $(-\infty, +\infty)$ and establish the nature of the convergence.

332. Can the theorem on differentiation of functional series be applied to the series

$$\sin x+\frac{1}{2^2}\cdot\sin\frac{x}{2}+\frac{1}{3^2}\cdot\sin\frac{x}{3}+\frac{1}{4^2}\cdot\sin\frac{x}{4}+\ldots?$$

333. Can the theorem on integration of functional series in any finite interval $[a, b]$ be applied to the series

$$1+\frac{\cos x}{1!}+\frac{\cos^2 x}{2!}+\frac{\cos^3 x}{3!}+\ldots?$$

334. Can the theorem on differentiation of functional series be applied to the series

$$(x^2+1)+2(x^2+1)^2+3(x^2+1)^3+4(x^2+1)^4+\ldots?$$

3.3. Power Series

A functional series of the form

$$a_0+a_1(x-a)+a_2(x-a)^2+\ldots+a_n(x-a)^n+\ldots,$$

where $a, a_0, a_1, \ldots, a_n$ are real numbers, is called a *power series*.

Here is the main property of power series: *if a power series converges at $x = x_0$, it is also convergent (absolutely) at any value of x satisfying the inequality $|x-a| < |x_0-a|$* ± **Abel's theorem**).

One of the corollaries of Abel's theorem is the existence, for any power series, of the *interval of convergence* $|x - a| < R$, or $a - R < x < a + R$ with centre at the point a, in whose interior the power series is absolutely convergent and outside of which it is divergent. At the end points of the interval of convergence (at the points $x = a \pm R$) different power series behave differently: some of them converge absolutely at both end points, others either converge conditionally at both end points or converge conditionally at one end point and diverge at the other, still others diverge at both end points.

The number R, half the length of the interval of convergence, is called the *radius of convergence* of a power series. In special cases, the radius of convergence R of a series may be equal to zero or to infinity. If $R = 0$, then the power series converges only at $x = a$; now if $R = \infty$, the series converges throughout the number axis.

One of the following methods may be used in the search for the interval and radius of convergence of a power series.

1. If neither of the coefficients $a_1, a_2, \ldots, a_n, \ldots$ of the series is equal to zero, that is, the series contains all integral positive degrees of the difference $x - a$, then

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|, \tag{1}$$

under the condition that the limit (finite or infinite) exists.

2. If the original series has the form

$$a_0 + a_1(x - a)^p + a_2(x - a)^{2p} + \ldots + a_n(x - a)^{np} + \ldots$$

(where p is some definite positive integer: 2, 3, . . .), then

$$R = \sqrt[p]{\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|}. \tag{2}$$

3. If among the coefficients of a series there are some equal to zero and the sequence of the degrees of the difference $x - a$ remaining in the series is arbitrary (that is, it does not form an arithmetic progression as in the previous case), then the radius of convergence can be found from the formula

$$R = \frac{1}{\lim\limits_{n \to \infty} \sqrt[n]{|a_n|}}, \tag{3}$$

which mades use only of the nonzero values of a_n. (This formula also holds for cases 1 and 2.)

4. In all cases we can find the interval of convergence by means of a direct usage of D'Alembert's test or Cauchy's test, applying them to the series set up from the absolute values of the terms of the original series.

Having written the series as

$$u_0(x) + u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$$

(here $u_0 = a_0$, $u_n(x) = a_n(x - a)^N$, where there may be any dependence of N on n, with a_n designating not the coefficient in $(x - a)^n$ but the coefficient in the nth term

of the series), the interval of convergence can be found from the inequality

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} < 1 \quad \text{or} \quad \lim_{n\to\infty} \sqrt[n]{|u_n|} < 1.$$

Note the following property of power series: *the series obtained by means of termwise differentiation and integration of a power series have the same interval of convergence and their sum within the interval of convergence is equal, respectively, to the derivative and the integral of the sum of the original series.*

Thus, if $S(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$, then

$$S'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}, \qquad \int_a^x S(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n(x-a)^{n+1}}{n+1},$$

where $-R < x - a < R$.

Operations of termwise differentiation and integration of a power series can be performed any number of times. Consequently, the sum of a power series within its interval of convergence is an infinitely differentiable function.

335. Test the power series

$$x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$$

for convergence.

Solution. Here $a_n = 1/n$, $a_{n+1} = 1/(n+1)$. Let us find the radius of convergence of the series:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$$

Consequently, the series converges for the values of x satisfying the double inequality $-1 < x < 1$.

Let us now test the series for convergence at the end points of the interval. If $x = 1$, we get the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$, which, as is known, is divergent. If $x = -1$, we get the series $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots$, which converges since it satisfies the conditions of Leibniz' test.

Thus, the domain of convergence of a power series is specified by the double inequality $-1 \leqslant x < 1$.

336. Test the series

$$(x-2) + \frac{1}{2^2}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \ldots$$

for convergence.

Solution. Here $a_n = 1/n^2$, $a_{n+1} = a/(n+1)^2$;

$$R = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 = 1.$$

Consequently, the series converges if $-1 < x - 2 < 1$, i.e. $1 < x < 3$.

Now we shall test the series for convergence at the end points of the interval. If $x = 3$, we get the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$, which is convergent since the series $1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$ converges at $p > 1$. If $x = 1$, we get the series $-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots$. The series converges (absolutely) since the series set up from the absolute values of its terms is convergent.

Thus, the power series converges for the values of x satisfying the double inequality $1 \leqslant x \leqslant 3$.

337. Test the series

$$1!(x-5) + 2!(x-5)^2 + 3!(x-5)^3 + \ldots$$

for convergence.

Solution. Here $a_n = n!$, $a_{n+1} = (n+1)!$;

$$R = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1 \cdot 2 \cdot 3 \ldots n}{1 \cdot 2 \cdot 3 \ldots n(n+1)} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

The series converges only for $x - 5 = 0$, that is at the point $x = 5$.

338. Test the series

$$\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

for convergence.

Solution. We have $a_n = 1/n!$, $a_{n+1} = 1/(n+1)!$, $a_0 = 0$;

$$R = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty.$$

Consequently, the series converges for any value of x. Incidentally, we infer from this that the limit of the general term of the series is zero for any value of x, i.e. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

339. Test the series

$$1 + \frac{x^3}{10} + \frac{x^6}{10^2} + \frac{x^9}{10^3} + \ldots$$

for convergence.

Solution. The series is a geometric progression with the denominator $q = x^3/10$. It converges if $|x^3/10| < 1$ and diverges if $|x^3/10| \geqslant 1$. Consequently, the interval

of convergence of the series is defined by the double inequality $-\sqrt[3]{10} < x < \sqrt[3]{10}$. The same result can be obtained by using formulas (2) and (3).

340. Test the series

$$2x^5 + \frac{4x^{10}}{3} + \frac{8x^5}{5} + \frac{16x^{20}}{7} + \ldots$$

for convergence.

Solution. Setting $x^5 = t$, we get the series

$$2t + \frac{4t^2}{3} + \frac{8t^3}{5} + \frac{16t^4}{7} + \ldots. \qquad (*)$$

Here $a_n = 2^n/(2n-1)$, $a_{n+1} = 2^{n+1}/(2n+1)$. We find the radius of convergence of series $(*)$:

$$R = \lim_{n\to\infty} \frac{2^n(2n+1)}{2^{n+1}(2n-1)} = \frac{1}{2}\lim_{n\to\infty} \frac{2n+1}{2n-1} = \frac{1}{2}\lim_{n\to\infty} \frac{2+1/n}{2-1/n} = \frac{1}{2}.$$

Thus the series converges if $|t| < 1/2$.

We test the series for convergence at the end points of the interval. If $t = 1/2$, we get $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$. The series diverges (it can be compared with the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots$, whose terms are the terms of the harmonic series multiplied by $1/2$). At $t = -1/2$ we get the series $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \ldots$, which converges conditionally. Consequently, series $(*)$ converges if $-1/2 \leqslant t < 1/2$. Thus the given series converges if $-1/2 \leqslant x^5 < 1/2$, i.e. $-1/\sqrt[5]{2} \leqslant x \leqslant 1/\sqrt[5]{2}$. We can obtain the same result by using formula (2).

341. Test the series

$$\sum_{k=1}^{\infty} \left(\frac{k+1}{2k+1}\right)^k (x-2)^{2k}$$

for convergence.

Solution. In the given case we have $a_n = 0$ for $n = 2k-1$ and $a_n = \left(\frac{k+1}{2k+1}\right)^k$ for $n = 2k$. To find the radius of convergence, it is most convenient to use formula (3). We find

$$R = \frac{1}{\lim\limits_{k\to\infty} \sqrt[2k]{\left(\frac{k+1}{2k+1}\right)^2}} = \lim_{k\to\infty} \sqrt{\frac{2k+1}{k+1}} = \sqrt{2}.$$

Now we test the series at the end points of the interval of convergence. Setting $x - 2 = \sqrt{2}$, we get the number series

$$\sum_{k=1}^{\infty}\left(\frac{k+1}{2k+1}\right)^k \cdot 2^k = \sum_{k=1}^{\infty}\left(\frac{k+1}{k+\frac{1}{2}}\right)^k = \sum_{k=1}^{\infty}\left(1+\frac{1}{2k+1}\right)^k.$$

But $\lim\limits_{k\to\infty}\left(1+\dfrac{1}{2k+1}\right)^k = \sqrt{e} \neq 0$. Thus, for $x - 2 = \sqrt{2}$, the series diverges. The same is true when $x - 2 = -\sqrt{2}$. Thus we see that the domain of convergence of the given series is $2 - \sqrt{2} < x < 2 + \sqrt{2}$.

342. Test the series

$$\sum_{n=1}^{\infty} \frac{(x-1)^{n(n+1)}}{n^n}$$

for convergence.

Solution. We apply Cauchy's test, setting $u_n = \dfrac{(x-1)^{n(n+1)}}{n^n}$. Then we have

$$\sqrt[n]{|u_n|} = \frac{|x-1|^{n+1}}{n}; \quad \lim_{n\to\infty}\sqrt[n]{|u_n|} = \begin{cases} 0 & \text{for} \quad |x-1| \leqslant 1, \\ \infty & \text{for} \quad |x-1| > 1. \end{cases}$$

Thus we see that the series converges if $|x - 1| \leqslant 1$, that is, in the interval $0 \leqslant x \leqslant 2$.

343. Test the series

$$\sum_{n=1}^{\infty} \frac{x^{n(n-1)/2}}{n!}$$

for convergence.

Solution. We apply D'Alembert's test, setting $u_n = x^{n(n-1)/2}/n!$; $u_{n+1} = x^{n(n+1)/2}/(n+1)!$. Then we have

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{|x|^n}{n+1}; \quad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \begin{cases} 0 & \text{for} \quad |x| \leqslant 1, \\ \infty & \text{for} \quad |x| > 1. \end{cases}$$

The series converges if $|x| \leqslant 1$, that is, in the interval $-1 \leqslant x \leqslant 1$.

344. Find the sum of the series $1 + 2x + 3x^2 + 4x^3 + \ldots$ ($|x| < 1$), differentiating the series $1 + x + x^2 + x^3 + \ldots$ ($|x| < 1$) termwise.

Solution. Using the formula for the sum of the terms of an infinitely decreasing geometric progression $\left(S = \dfrac{a}{1-q}\right)$, we obtain

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}.$$

It remains to differentiate the equality obtained:

$$1 + 2x + 3x^2 + 4x^3 + \ldots = \frac{1}{(1-x)^2}.$$

345. Find the sum of the series

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots \quad (|x| < 1).$$

Solution. Integrating the equation

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

in the limits from 0 to x, we get

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots = -\ln(1-x).$$

The series converges in the interval $[-1, 1)$.

Test the following power series for convergence:

346. $\frac{x+1}{1!} + \frac{(x+1)^2}{3!} + \frac{(x+1)^3}{5!} + \ldots.$

347. $(x-4) + \frac{1}{\sqrt{2}}(x-4)^2 + \frac{1}{\sqrt{3}}(x-4)^3 + \ldots.$

348. $\frac{x-1}{2} + \frac{(x-1)^2}{2^2} + \frac{(x-1)^3}{2^3} + \ldots.$ **349.** $x + (2x)^2 + (3x)^3 + (4x)^4 + \ldots.$

350. $5x + \frac{5^2 \cdot x^2}{2!} + \frac{5^3 \cdot x^3}{3!} + \frac{5^4 \cdot x^4}{4!} + \ldots.$ **351.** $x^2 + \frac{x^4}{2} + \frac{x^6}{3} + \frac{x^8}{4} + \ldots.$

Hint. Put $x^2 = t$.

352. $\frac{x^3}{8} + \frac{x^6}{8^2 \cdot 5} + \frac{x^9}{8^3 \cdot 9} + \frac{x^{12}}{8^4 \cdot 13} + \ldots.$ **353.** $\frac{x}{2+3} + \frac{x^2}{2^2+3^2} + \frac{x^3}{2^3+3^3} + \ldots.$

354. $\frac{1}{2} \cdot \frac{x-1}{2} + \frac{2}{3}\left(\frac{x-1}{2}\right)^2 + \frac{3}{4}\left(\frac{x-1}{2}\right)^3 + \frac{4}{5}\left(\frac{x-1}{2}\right)^4 + \ldots.$

355. $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \frac{x^4}{4 \cdot 5} + \ldots.$

Find the sums of the following series:

356. $\frac{1}{a} + \frac{2x}{a^2} + \frac{3x^2}{a^3} + \frac{4x^3}{a^4} + \ldots$, if $|x| < a$.

357. $\frac{x^2}{2a} + \frac{x^3}{3a^2} + \frac{x^4}{4a^3} + \ldots$, if $-a \leqslant x < a$.

358. $\frac{1 \cdot 2}{a^2} + \frac{2 \cdot 3}{a^3} \cdot x + \frac{3 \cdot 4}{a^4} \cdot x^2 + \ldots$, if $|x| < a$.

359. $-2x + 4x^3 - 6x^5 + 8x^7 - \ldots$, if $|x| < 1$.

3.4. Expansion of Functions into Power Series

Any function infinitely differentiable in the interval $|x - x_0| < r$, i.e. $x_0 - r < x < x_0 + r$, can be expanded in that interval into the infinite *Taylor's power series*

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots$$

converging to it if the condition

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} = 0,$$

is fulfilled in that interval, $R_n(x)$ being the remainder of Taylor's formula, $c = x_0 + \theta(x - x_0)$, $0 < \theta < 1$.

For $x_0 = 0$ we get the so-called *Maclaurin's series*:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots.$$

If the inequality $|f^{(n)}(x)| < M$, with M being a positive constant, is satisfied, for any n, in a certain interval containing a point x_0, then $\lim\limits_{n\to\infty} R_n = 0$ and the function $f(x)$ can be expanded into Taylor's series.

Given below are the expansions of the following functions into Taylor's series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \quad -\infty < x < +\infty;$$

$$\sinh x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^4}{5!} + \ldots, \quad -\infty < x < +\infty;$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots, \quad -\infty < x < +\infty;$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, \quad -\infty < x < +\infty;$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, \quad -\infty < x < +\infty;$$

$$(1 + x)^m = 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots.$$

This last expansion is valid

$$\begin{aligned} &\text{for} \quad m \geqslant 0, && \text{if} \quad -1 \leqslant x \leqslant 1; \\ &\text{for} -1 < m < 0, && \text{if} \quad -1 < x \leqslant 1; \\ &\text{for} \quad m \leqslant -1, && \text{if} \quad -1 < x < -1; \end{aligned}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \quad -1 < x \leqslant 1;$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots, \quad -1 \leqslant x \leqslant 1.$$

360. Expand the function $f(x) = 2^x$ into a power series.

Solution. We find the values of the function and of its derivatives for $x = 0$:

$$f(x) = 2^x, \qquad f(0) = 2^0 = 1,$$
$$f'(x) = 2^x \ln 2, \qquad f'(0) = \ln 2,$$
$$f''(x) = 2^x \ln^2 2, \qquad f''(0) = \ln^2 2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f^{(n)}(x) = 2^x \cdot \ln^n 2; \qquad f^{(n)}(0) = \ln^n 2.$$

Since $0 < \ln 2 < 1$, it follows that for a fixed x there holds an inequality $|f^{(n)}(x)| < 2^x$ for any n. Consequently, the function can be represented as the sum of Taylor's series

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots.$$

In the given case

$$2^x = 1 + x \cdot \ln 2 + \frac{x^2 \cdot \ln^2 2}{2!} + \frac{x^3 \cdot \ln^3 2}{3!} + \ldots, -\infty < x < +\infty.$$

This expansion can be obtained otherwise: it is sufficient to replace x by $x \ln 2$ in the expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

361. Expand the function $f(x) = \sin^2 x$ into a power series.

Solution. Let us differentiate the function $n + 1$ times:

$$f(x) = \sin^2 x,$$
$$f'(x) = 2 \sin x \cos x = \sin 2x,$$
$$f''(x) = 2 \cos 2x = 2 \sin\left(2x + \frac{\pi}{2}\right),$$
$$f'''(x) = -2^2 \cdot \sin 2x = 2^2 \sin\left(2x + 2 \cdot \frac{\pi}{2}\right),$$
$$f^{\mathrm{IV}}(x) = -2^3 \cdot \cos 2x = 2^3 \cdot \sin\left(2x + 3 \cdot \frac{\pi}{2}\right),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f^{(n)}(x) = 2^{n-1} \sin \left[2x + \frac{\pi}{2}(n-1)\right],$$

$$f^{(n+1)}(x) = 2^{n} \cdot \sin\left(2x + \frac{\pi}{2} \cdot n\right).$$

We find the values of the functions $f(x), f'(x), f''(x), \ldots, f^{(n)}(x)$ at the point $x = 0$ and the value of $f^{(n+1)}(x)$ at the point $x = c$ (see the equality specifying R_n). We obtain $f(0) = 0, f'(0) = 0, f''(0) = 2, f'''(0) = 0, f^{IV}(0) = -2^3, f^{V}(0) = 0, f^{IV}(0) = 2^5, \ldots, f^{(n+1)}(c) = 2^n \cdot \sin(2c + \pi n/2)$.

Now we find the remainder:

$$R_n = \frac{2^n \cdot \sin(2c + \pi n/2)}{(n+1)!} \cdot x^{n+1}, \text{i.e. } R_n = \frac{1}{2}\frac{(2x)^{n+1}}{(n+1)!} \sin(2c + \pi n/2).$$

Since $\lim\limits_{n\to\infty} \dfrac{(2x)^{n+1}}{(n+1)!} = 0$ for any x and $\sin(2c + \pi n/2)$ is a limited quantity, it follows that $\lim\limits_{n\to\infty} R_n = 0$. Consequently, the function $f(x) = \sin^2 x$ can be represented as the sum of Taylor's series:

$$\sin^2 x = \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + \ldots.$$

There is another solution to the problem. We replace $\cos 2x$ by its expansion into a power series in the equation $\sin^2 x = \dfrac{1}{2}(1 - \cos 2x)$:

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots.$$

Having performed simple transformations, we get the expansion $\sin^2 x$ obtained above.

362. Expand e^{-x^2} into a power series.

Solution. In the expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad (-\infty < x < +\infty)$$

we substitute $-x^2$ for x and obtain

$$e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \ldots \quad (-\infty < x < +\infty).$$

363. Expand $\ln x$ into a series by the degrees of $x - 1$.

Solution. In the expansion

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad (-1 < x \leqslant 1)$$

we substitute $x - 1$ for x and obtain

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots \quad (0 < x \leqslant 2).$$

364. Expand $1/x$ into a series by the degrees of $x - 2$.

Solution. We make use of the equality $\frac{1}{x} = \frac{1/2}{1 + (x-2)/2}$. We can consider the right-hand side of this equality as the sum of an infinitely decreasing geometric progression with the first term $a = 1/2$ and the denominator $q = -(x-2)/2$. Hence we get

$$\frac{1}{x} = \frac{1}{2} - \frac{1}{2} \cdot \frac{x-2}{2} + \frac{1}{2}\left(\frac{x-2}{2}\right)^2 - \frac{1}{2}\left(\frac{x-2}{2}\right)^3 + \ldots,$$

i.e.

$$\frac{1}{x} = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \ldots .$$

Since $|(x-2)/2| < 1$, we have $0 < x < 4$.

Expand the following functions into power series:

365. $f(x) = 3^x$. **366.** $f(x) = e^{-2x}$. **367.** $f(x) = \cos^2 x$. **368.** $f(x) = \sinh^2 x$. **369.** $f(x) = \ln(x + a), a > 0$. **370.** $f(x) = \sqrt{x + a}, a > 0$. **371.** $f(x) = \cosh^2 (x^2)$.

3.5. Approximate Calculations of the Values of Functions with the Aid of Power Series

It may be of use to bear in mind the expansions into power series of the functions e^x, $\sinh x$, $\cosh x$, $\sin x$, $\cos x$, $(1 + x)^m$, $\ln(1 + x)$, $\arctan x$ presented in the previous section.

The following formula is efficient for taking the logarithms:

$$\ln(t+1) = \ln t + 2\left[\frac{1}{2t+1} + \frac{1}{3(2t+1)^3} + \frac{1}{5(2t+1)^5} + \ldots\right].$$

The series on the right-hand side of the equality converges the faster the larger is t.

To obtain an approximate value of the function $f(x)$ in its expansion into a power series, the first n terms are retained (n being a finite quantity) and the other terms are discarded. To evaluate the error in the approximation obtained, it is necessary to calculate the sum of the discarded terms. If the given series is of constant sign, then the series set up from the discarded terms is compared with an infinitely decreasing geometric progression. In the case of an alternating series whose terms satisfy Leibniz' test, use is made of the evaluation $|R_n| < u_{n+1}$, where u_{n+1} is the first of the discarded terms of the series.

372. Evaluate the error of the approximate equality

$$e^x \approx 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}, \quad 0 < x < n + 1.$$

Solution. The error of this approximation is determined by the sum of the terms following $x^n/n!$ in the expansion of e^x

$$R_n = \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \frac{x^{n+3}}{(n+3)!} + \ldots,$$

or

$$R_n = \frac{x^n}{n!}\left[\frac{x}{n+1} + \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} + \ldots\right].$$

Replacing each of the factors $n + 2$, $n + 3$, $n + 4$, ... by the smaller quantity $n + 1$, we get the inequality

$$R_n < \frac{x^n}{n!}\left[\frac{x}{n+1} + \left(\frac{x}{n+1}\right)^2 + \left(\frac{x}{n+1}\right)^3 + \ldots\right].$$

Summing up the infinitely decreasing geometric progression in the square brackets, we get

$$R_n < \frac{x^n}{n!} \cdot \frac{x/(n+1)}{1 - x/(n+1)}, \quad \text{i.e.} \quad R_n < \frac{x^n}{n!} \cdot \frac{x}{n+1-x}.$$

373. Calculate $\sqrt{e}$ with an accuracy to within 0.00001.

Solution. Using the expansion of e^x into a series, we get

$$\sqrt{e} = e^{1/2} = 1 + \frac{1}{1! \cdot 2} + \frac{1}{2! \cdot 2^2} + \frac{1}{3! \cdot 2^3} + \ldots.$$

Let us determine the number n so that the error of the approximate equality

$$\sqrt{e} \approx 1 + \frac{1}{1! \cdot 2} + \frac{2}{2! \cdot 2^2} + \ldots + \frac{1}{n! \cdot 2^n}$$

should not exceed 0.00001. We make use of the evaluation of the error given in the previous example. Setting $x = 1/2$, we get

$$R_n < \frac{1}{n!2^n} \frac{1/2}{n + 1/2}, \quad \text{i.e.} \quad R_n < \frac{1}{n!2^n} \cdot \frac{1}{2n+1}.$$

We try different values of n and find at which value the inequality $R_n < 0.00001$ is satisfied. Putting $n = 3$, for instance, we get $R_3 < 1/(8 \cdot 6 \cdot 7)$, i.e. $R_3 < 1/336$. Next we try $n = 5$. Now we have $R_5 < 1/(32 \cdot 120 \cdot 11)$, i.e. $R_5 < 1/42240$. Suppose, finally, $n = 6$; it follows that $R_6 < 1/(64 \cdot 720 \cdot 13)$, i.e. $R_6 < 1/100000$. Thus we assume $n = 6$:

$$\sqrt{e} = 1 + \frac{1}{1! \cdot 2} + \frac{1}{2! \cdot 2^2} + \frac{1}{3! \cdot 2^3} + \frac{1}{4! \cdot 2^4} + \frac{1}{5! \cdot 2^5} + \frac{1}{6! \cdot 2^6}.$$

We sum up the summands:

$$\begin{array}{rl}
1.000000 & \\
0.500000 & \\
0.125000 & \\
0.020833 & \text{(one-sixth as great as the preceding summand)} \\
+0.002604 & \text{(one-eighth as great as the preceding summand)} \\
0.000260 & \text{(one-tenth as great as the preceding summand)} \\
0.000022 & \text{(one-twelfth as great as the preceding summand)} \\
\hline
1.648719. &
\end{array}$$

Hence $\sqrt{e} = 1.64872$. We have calculated each summand with an accuracy to within 0.000001, in order that the error in summation should not exceed 0.00001.

374. Calculate $1/\sqrt[5]{e}$ with an accuracy to within 0.00001.

Solution. We have

$$1/\sqrt[5]{e} = e^{-1/5} = 1 - \frac{1}{1! \cdot 5} + \frac{1}{2! \cdot 5^2} - \frac{1}{3! \cdot 5^3} + \ldots .$$

We make use of the approximate equality

$$1/\sqrt[5]{e} \approx 1 - \frac{1}{1! \cdot 5} + \frac{1}{2! \cdot 5^2} - \frac{1}{3! \cdot 5^3} + \frac{1}{4! \cdot 5^4}.$$

We have taken 5 summands since the sign-changing series satisfies the conditions of Leibniz' test and, therefore, in its absolute value, the error in approximation must be smaller than the first discarded term of the series. The first discarded term is equal to $1/(5! \cdot 5^5)$. It is easy to see that $1/(5! \cdot 5^5) < 0.00001$.

Let us sum up the summands occupying the odd and the even places:

$$\begin{array}{r}
1.000000 \\
+0.020000 \\
0.000067 \\
\hline
1.020067;
\end{array}
\qquad
\begin{array}{r}
0.200000 \\
+0.001333 \\
\hline
0.201333.
\end{array}$$

Subtracting the second sum from the first, we get $1.020067 - 0.201333 = 0.818734$. Thus we have $1/\sqrt[5]{e} \approx 0.81873$.

375. Using the expansion of $\cos x$ into a series, calculate $\cos 18°$ with an accuracy to within 0.0001.

Solution. We have

$$\cos 18° = \cos \frac{\pi}{10} = 1 - \frac{1}{2!}\left(\frac{\pi}{10}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{10}\right)^4 - \ldots;$$

$$\pi/10 = 0.31416, (\pi/10)^2 = 0.09870, (\pi/10)^4 = 0.00974.$$

It is sufficient to take three terms of the series since $(1/6) \cdot (\pi/10)^6 < 0.0001$. Then we have

$$\cos 18^\circ \approx 1 - \frac{0.09870}{2} + \frac{0.00974}{24}; \quad \cos 18^\circ \approx 0.9511.$$

376. Calculate $\sqrt[5]{1.1}$ with an accuracy to within 0.0001.

Solution. We make use of the expansion of $(1 + x)^m$ into a series, putting $x = 0.1$, $m = 1/5$. We have

$$\sqrt[5]{1.1} = (1 + 0.1)^{1/5} = 1 + \frac{1}{5} \cdot 0.1 + \frac{(1/5)(1/5 - 1)}{2!} 0.01$$

$$+ \frac{(1/5)(1/5 - 1)(1/5 - 2)}{3!} 0.001 + \dots = 1 + 0.02 - 0.0008 + 0.000048 - \dots.$$

We discard all the terms beginning with the fourth since the fourth term is smaller than 0.0001. Thus, $\sqrt[5]{1.1} \approx 1.0192$.

377. Calculate $\sqrt[3]{130}$ with an accuracy to within 0.001.

Solution. Since 5^3 is a cube of an integer closest to 130, it is expedient to represent the number 130 as the sum of two summands: $130 = 5^3 + 5$. It follows that

$$\sqrt[3]{130} = \sqrt[3]{5^3 + 5} = 5\sqrt[3]{1 + \frac{1}{25}} = 5(1 + 0.04)^{1/3}$$

$$= 5\left[1 + \frac{1}{3} \cdot 0.04 + \frac{(1/3)(1/3 - 1)}{2!} \cdot 0.0016 + \frac{(1/3)(-2/3)(-5/3)}{3!} \cdot 0.000064 + \dots\right]$$

$$= 5 + \frac{1}{3} \cdot 0.2 - \frac{1}{9} \cdot 0.008 + \frac{5}{81} \cdot 0.00032 - \dots.$$

The fourth term is smaller than 0.001 and so we can discard it and all the terms following it. Then we have $\sqrt[3]{130} \approx 5 + 0.0667 - 0.0009$, i.e. $\sqrt[3]{130} \approx 5.066$.

378. Calculate ln 1.04 with an accuracy to within 0.0001.

Solution. We use the expansion of $\ln(1 + x)$ into a series:

$$\ln 1.04 = \ln(1 + 0.04) = 0.04 - \frac{0.04^2}{2} + \frac{0.04^3}{3} - \frac{0.04^4}{4} + \dots$$

or

$$\ln 1.04 = 0.04 - 0.0008 + 0.000021 - 0.00000064 + \dots,$$

whence $\ln 1.04 \approx 0.0392$.

379. The legs of a right triangle are equal to 1 cm and 5 cm. Determine the acute angle lying opposite the smaller leg with an accuracy to within 0.001 radian.

Solution. Since $\tan \alpha = 1/5$, it follows that $\alpha = \arctan(1/5)$. We use the expansion

$$\alpha = \arctan(1/5) = \frac{1}{5} - \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} - \ldots,$$

whence $\alpha \approx 0.2 - 0.0027$, i.e. $\alpha \approx 0.197$.

380. Evaluate the error of the approximate equality

$$\ln(t+1) \approx \ln t + 2\left[\frac{1}{2t+1} + \frac{1}{3(2t+1)^3} + \frac{1}{5(2t+1)^5} + \ldots + \frac{1}{(2n-1)(2t+1)^{2n-1}}\right].$$

Solution. The problem reduces to evaluating the sum of the remainder of the series

$$R_n = 2\left[\frac{1}{(2n+1)(2t+1)^{2n+1}} + \frac{1}{(2n+3)(2t+1)^{2n+3}} + \frac{1}{(2n+5)(2t+1)^{2n+5}} + \ldots\right].$$

Replacing each of the factors $2n+3$, $2n+5$, $2n+7$, $\ldots$ by the smaller number $2n+1$, we get the inequality

$$R_n < \frac{2}{2n+1}\left[\frac{1}{(2t+1)^{2n+1}} + \frac{1}{(2t+1)^{2n+3}} + \frac{1}{(2t+1)^{2n+5}} + \ldots\right].$$

Summing up the infinitely decreasing geometric progression in the square brackets, we get

$$R_n < \frac{2}{2n+1}\cdot\frac{1/(2t+1)^{2n+1}}{1-1/(2t+1)^2}$$

$$= \frac{2}{2n+1}\cdot\frac{1}{(2t+1)^{2n-1}[(2t+1)^2-1]} = \frac{2}{2n+1}\cdot\frac{1}{(2t+1)^{2n-1}\cdot 4t(t+1)},$$

i.e.

$$R_n < \frac{1}{2(2n+1)\cdot t(t+1)\cdot(2t+1)^{2n-1}}.$$

381. Calculate $\ln 2$ with an accuracy to within 0.0001.

Solution. We set $t = 1$ in the formula for determining $\ln(t+1)$ and in the inequality for evaluating R_n:

$$\ln 2 = 2\left(\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \frac{1}{7\cdot 3^7} + \ldots\right); R_n < \frac{1}{4(2n+1)\cdot 3^{2n-1}}.$$

Trying different values for n, we choose the value satisfying the inequality $R_n < 0.0001$. If $n = 2$, then $R_2 < 1/(4\cdot 5\cdot 3^3)$; $R_2 < 1/540$; if $n = 3$, then $R_3 < 1/(4\cdot 7\cdot 3^5)$; $R_3 < 1/6804$; now, if $n = 4$, then $R_4 < 1/(4\cdot 9\cdot 3^7)$; $R_4 < 1/10000$.

Thus, $n = 4$, and to determine ln 2 we obtain the following approximate equality:

$$\ln 2 \approx 2\left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7}\right).$$

Summing up the four summands, we get

$$\ln 2 \approx 0.66667 + 0.02469 + 0.00165 + 0.00013 = 0.69314 \approx 0.6931.$$

382. Calculate ln 5 with an accuracy to within 0.0001.

Solution. We set $t = 4$. Then we have

$$\ln 5 = 2 \ln 2 + 2\left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \dots\right); \quad R_n < \frac{1}{40 \cdot (2n+1) \cdot 9^{2n-1}}.$$

If $n = 1$, then $R_1 < 1/(40 \cdot 3 \cdot 9)$; $R_1 < 1/1080$; if $n = 2$, then $R_2 < 1/(40 \cdot 5 \cdot 9^3)$; $R_2 < 1/10000$. Hence it is sufficient to take two terms of the series. Consequently,

$$\ln 5 \approx 2 \ln 2 + 2\left(\frac{1}{9} + \frac{1}{3 \cdot 9^3}\right) \approx 1.38628 + 0.22222 + 0.00090 = 1.60940.$$

383. Prove the validity of the identity

$$\pi/4 = \arctan(1/2) + \arctan(1/3)$$

and compute π with an accuracy to within 0.001.

Solution. Setting $x = 1/2$, $y = 1/3$ in the equality

$$\arctan \frac{x+y}{1-xy} = \arctan x + \arctan y,$$

we obtain

$$\arctan 1 = \arctan \frac{1}{2} + \arctan \frac{1}{3}, \quad \text{or} \quad \pi = 4\left(\arctan \frac{1}{2} + \arctan \frac{1}{3}\right).$$

Using the expansion of arctan x into a series, we get

$$\pi = 4\left[\left(\frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \dots\right) + \left(\frac{1}{3} - \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} - \frac{1}{7 \cdot 3^7} + \dots\right)\right].$$

We perform the following operations:

2.00000	0.16667	3.36255
0.02500	0.00446	−
0.00087	+0.00018	0.22095
+0.00004	0.04938	3.14160
1.33333	0.00026	
0.00329	0.22095	
0.00002		
3.36255		

Thus we have $\pi \approx 3.1416$.

To calculate π we could have used the series which converge faster than the series presented above.

384. Calculate the number e with an accuracy to within 0.00001.

385. Calculate $1/\sqrt{e}$ to within 0.00001.

386. Calculate $\sin 9°$ to within 0.0001.

387. Calculate $\cosh 0.3$ to within 0.0001.

388. Calculate $\sqrt[3]{1.06}$ to within 0.0001.

389. Calculate $\sqrt{27}$ to within 0.001.

390. Calculate $\ln 0.98$ to within 0.0001.

391. Calculate $\ln 1.1$ to within 0.0001.

392. Calculate $\ln 3$ to within 0.0001.

393. Calculate $\ln 10$ to within 0.0001.

394. Find the smallest positive value of x satisfying the trigonometric equation $2 \sin x - \cos x = 0$.

395. Calculate π with an accuracy to within 0.001, setting $x = 1/\sqrt{3}$ in the expansion of $\arctan x$.

3.6. Application of Power Series in Calculating Limits and Determining Integrals

396. Find $\lim\limits_{x\to\infty} \dfrac{2e^x - 2 - 2x - x^2}{x - \sin x}$.

Solution. Replacing e^x and $\sin x$ by their expansions into power series, we obtain

$$\lim_{x\to 0} \frac{2e^x - 2 - 2x - x^2}{x - \sin x} = \lim_{x\to 0} \frac{2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - 2 - 2x - x^2}{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)}$$

$$= \lim_{x\to 0} \frac{\frac{2x^3}{3!} + \frac{2x^4}{4!} + \ldots}{\frac{x^3}{3!} - \frac{x^5}{5!} + \ldots} = \lim_{x\to 0} \frac{\frac{2}{3!} + \frac{2x}{4!} + \ldots}{\frac{1}{3!} - \frac{x^2}{5!} + \ldots} = 2.$$

397. Find $\lim\limits_{x\to 0}\dfrac{\sin x - \arctan x}{x^3}$.

Solution. We have

$$\lim_{x\to 0}\frac{\sin x - \arctan x}{x^3} = \lim_{x\to 0}\frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots - x + \dfrac{x^3}{3} - \dfrac{x^5}{5} + \ldots}{x^3}$$

$$= \lim_{x\to 0}\left[\left(\frac{1}{3} - \frac{1}{3!}\right) - \left(\frac{1}{5} - \frac{1}{5!}\right)x^2 + \ldots\right] = \frac{1}{6}.$$

398. Calculate $\int\limits_0^{1/2}\dfrac{1-\cos x}{x^2}\,dx$ with an accuracy to within 0.0001.

Solution. Replacing in the integrand $\cos x$ by its expansion into a power series, we obtain

$$\int_0^{1/2}\frac{1-\cos x}{x^2}\,dx = \int_0^{1/2}\frac{1 - 1 + \dfrac{x^2}{2!} - \dfrac{x^4}{4!} + \dfrac{x^6}{6!} - \ldots}{x^2}\,dx$$

$$= \int_0^{1/2}\left(\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \ldots\right)dx = \left[\frac{1}{2!}x - \frac{x^3}{4!\cdot 3} + \frac{x^5}{6!\cdot 5} - \ldots\right]_0^{1/2}$$

$$= \frac{1}{2!\cdot 2} - \frac{1}{4!\cdot 3\cdot 2^3} + \frac{1}{6!\cdot 5\cdot 2^5} - \ldots \approx 0.25 - 0.0017 = 0.2483.$$

399. Calculate $\int\limits_0^{0.1}\dfrac{\ln(1+x)}{x}\,dx$ with an accuracy to within 0.001.

Solution. We find

$$\int_0^{0.1}\frac{\ln(1+x)}{x}\,dx = \int_0^{0.1}\frac{x - \dfrac{1}{2}x^2 + \dfrac{1}{3}x^3 - \dfrac{1}{4}x^4 + \ldots}{x}\,dx$$

$$= \int_0^{0.1}\left(1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \ldots\right)dx = \left[x - \frac{1}{4}x^2 + \frac{1}{9}x^3 - \frac{1}{16}x^4 + \ldots\right]_0^{0.1}$$

$$= 0.1 - \frac{1}{4} \cdot 0.01 + \frac{1}{9} \cdot 0.001 - \ldots \approx 0.098.$$

400. Find $\lim\limits_{x \to 0} \dfrac{x - \arctan x}{x^3}$.

401. Find $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{e^x - 1 - x}$.

402. Calculate $\int\limits_0^{0.2} \dfrac{\sin x}{x} dx$ with an accuracy to within 0.0001.

403. Calculate $\int\limits_0^{0.1} \dfrac{e^x - 1}{x} dx$ with an accuracy to within 0.001.

3.7. Complex Numbers and Series with Complex Terms

3.7.1. Complex numbers. *Complex numbers* are those of the form $x + iy$, where x and y are real numbers, i is an *imaginary unit* specified by the equation $i^2 = -1$. The real numbers x and y are called, respectively, a *real* and an *imaginary* part of the complex number z. They are designated as $x = \operatorname{Re} z$; $y = \operatorname{Im} z$.

In geometry, each complex number $z = x + iy$ is represented by a point $M(x; y)$ of the coordinate plane xOy (Fig. 24).

In that case the xOy plane is called a complex number plane, or the *plane of the complex variable z*.

The polar coordinates r and φ of the point M of the plane, which is the representation of the complex number z, are called the *modulus* and the *argument* of the complex number z; they are designated as $r = |z|$, $\varphi = \operatorname{Arg} z$.

Since every point of the plane is associated with an infinitude of the values of the polar angle differing from one another by $2k\pi$ (k being a positive or negative integer), it follows that $\operatorname{Arg} z$ is an infinite-valued function of z.

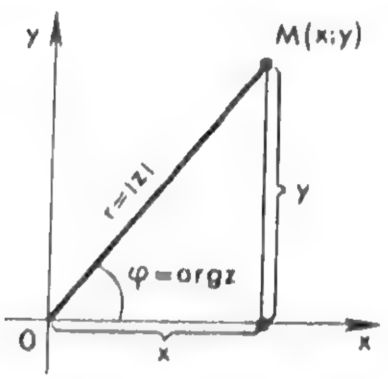


Fig. 24

The value of the polar angle φ which satisfies the inequality $-\pi < \varphi \leqslant \pi$ is called the *principal value* of the argument of z and is denoted by arg z.

In what follows we shall retain the symbol φ only for the principal value of the argument of z, that is, we shall put $\varphi = \arg z$; accordingly, for all the other values of the argument of z we shall obtain the equality

$$\operatorname{Arg} z = \arg z + 2k\pi = \varphi + 2k\pi.$$

The relationship between the modulus and the argument of the complex number z and its real and imaginary parts is established by means of the formulas

$$x = r\cos\varphi;\ y = r\sin\varphi.$$

Hence

$$r = |z| = \sqrt{x^2 + y^2};$$

$$\cos\varphi = x/|z| = x/\sqrt{x^2 + y^2};\quad \sin\varphi = y/|z| = y/\sqrt{x^2 + y^2}.$$

The argument of z can also be determined from the formula

$$\arg z = \arctan(y/x) + C,$$

where $C = 0$ for $x > 0$, $C = \pi$ for $x < 0, y > 0$; $C = -\pi$ for $x < 0, y < 0$.

Replacing x and y in the notation of the complex number $z = x + iy$ by their expressions in terms of r and φ, we obtain the so-called *trigonometric form of the complex number*:

$$z = r(\cos\varphi + i\sin\varphi).$$

The complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are considered to be *equal* if and only if their real parts are equal and their imaginary parts are equal:

$$z_1 = z_2 \quad \text{if} \quad x_1 = x_2, y_1 = y_2.$$

For the numbers presented in trigonometric form the equality takes place if their moduli are equal and the arguments differ by the integral multiple 2π:

$$z_1 = z_2 \quad \text{if} \quad |z_1| = |z_2| \quad \text{and} \quad \operatorname{Arg} z_1 = \operatorname{Arg} z_2 + 2k\pi.$$

Two complex numbers $z = x + iy$ and $\bar{z} = x - iy$ with equal real and opposite imaginary parts are known as *conjugate* numbers. The following relations hold for conjugate complex numbers:

$$|z_1| = |z_2|;\quad \arg z_1 = -\arg z_2$$

(the last equality can assume the form $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 = 2k\pi$).

The operations on complex numbers are governed by the following rules.

Addition. If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

Addition of complex numbers is commutative and associative:

$$z_1 + z_2 = z_2 + z_1;\quad (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) = z_1 + z_2 + z_3.$$

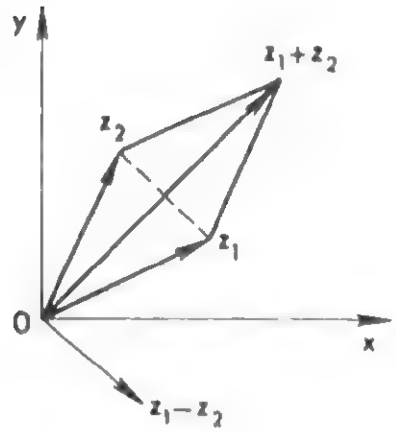


Fig. 25

Subtraction. If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

To explain addition and subtraction of complex numbers in geometric terms, it is more convenient to represent them not by points on the z-plane but by vectors: the number $z = x + iy$ is represented by the vector $\overline{OM}$, originating at the point O (the "zero" point of the plane, the origin) and terminating at the point $M(x; y)$. Then addition and subtraction of complex numbers are performed in accordance with the rules of addition and subtraction of vectors (Fig. 25).

Such geometric interpretation of the operations of addition and subtraction of vectors makes it easy to establish the theorems on the modulus of the sum and the difference of two complex numbers and the sum of several complex numbers expressed by the inequalities

$$\big||z_1| - |z_2|\big| \leqslant |z_1 \pm z_2| \leqslant |z_1| + |z_2|,$$

$$|z_1 + z_2 + \ldots + z_k| \leqslant |z_1| + |z_2| + \ldots + |z_k|.$$

Besides, it is useful to remember that the *modulus of the difference between two complex numbers z_1 and z_2 is equal to the distance between the points which are their representations on the z-plane*: $|z_1 - z_2| = d(z_1, z_2)$.

Multiplication. If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then

$$z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

Thus we see that complex numbers are multiplied as binomials, with i^2 being replaced by -1.

If $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$, $z_2 = r_2(\cos\varphi_2 - i\sin\varphi_2)$, then

$$z_1z_2 = r_1r_2[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)].$$

Thus, *the modulus of the product is equal to the product of the moduli of the factors, and the argument of the product is equal to the sum of the arguments of the factors.*

Multiplication of complex numbers is commutative, associative and distributive (relative to addition):

$$z_1 \cdot z_2 = z_2 \cdot z_1; \quad (z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3) = z_1 \cdot z_2 \cdot z_3;$$
$$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3.$$

Division. To find the quotient of two complex numbers given algebraically, it is necessary to multiply the dividend and the divisor by the number which is conjugate of the divisor:

$$z_1 \div z_2 = \frac{x_1 + iy_1}{z_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} =$$
$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

If z_1 and z_2 are in trigonometric form, then

$$z_1 \div z_2 = \frac{r_1}{r_2}[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)].$$

Thus, *the modulus of the quotient is equal to the quotient of the moduli of the dividend and the divisor and the argument of the quotient is equal to the difference between the arguments of the dividend and the divisor.*

Raising to a power. If $z = x + iy$, then

$$z^n = (x + iy)^n = x^n + C_n^1x^{n-1} \cdot iy + \ldots + (iy)^n$$

(n being a positive integer); in the expression obtained it is necessary to replace the degrees of i by their values:

$$i^2 = -1; \quad i^3 = -i; \quad i^4 = 1; \quad i^5 = i, \ldots$$

and, in a general case,

$$i^{4k} = 1; \quad i^{4k+1} = i; \quad i^{4k+2} = -1; \quad i^{4k+3} = -i.$$

If $z = r(\cos\varphi + i\sin\varphi)$, then

$$z^n = r^n(\cos n\varphi + i\sin n\varphi)$$

(here n may be either a positive integer or a negative integer).

In particular,

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi$$

(*De Moivre's formula*).

Root extraction. If n is a positive integer, $z = r(\cos\varphi + i\sin\varphi)$, then the nth root of the complex number z has n different values which can be found by the formula

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right),$$

where $k = 0, 1, 2, \ldots, n - 1$.

404. Find $(z_1z_2)/z_3$ if $z_1 = 3 + 5i$, $z_2 = 2 + 3i$, $z_3 = 1 + 2i$.

Solution. We have

$$z_1z_2 = (3 + 5i)(2 + 3i) = 6 + 9i + 10i - 15 = -9 + 19i,$$

$$\frac{z_1z_2}{z_3} = \frac{-9 + 19i}{1 + 2i} = \frac{(-9 + 19i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{-9 + 18i + 19i + 38}{1 + 4} = \frac{29}{5} + \frac{37}{5}i.$$

405. Represent the complex number $z = 2 + 5i$ in trigonometric form.

Solution. We find the modulus of the complex number: $r = \sqrt{4 + 25} = \sqrt{29} \approx 5.385$. Then we find the principal value of the argument: $\tan\varphi = 5/2 = 2.5$, $\varphi = 68°12'$. Consequently, $z = 5.385\,(\cos 68°12' + i\sin 68°12')$.

406. Represent the complex number $z = 2\sqrt{3} - 2i$ in trigonometric form.

Solution. We find

$$r = \sqrt{12 + 4} = 4, \quad \sin\varphi = -2/4 = -1/2; \ \cos\varphi = 2\sqrt{3}/4 = \sqrt{3}/2; \ \varphi = -\pi/6,$$

i.e.

$$z = 4[\cos(-\pi/6) + i\sin(-\pi/6)].$$

407. Represent the complex numbers $1, i, -1, -i$ in trigonometric form.

Solution. We have

$$1 = 1 + 0\cdot i = 1\cdot(\cos 0 + i\sin 0),$$
$$i = 0 + 1\cdot i = 1\cdot[\cos(\pi/2) + i\sin(\pi/2)],$$
$$-1 = -1 + 0\cdot i = 1\cdot(\cos\pi + i\sin\pi),$$
$$-i = 0 - 1\cdot i = 1\cdot[\cos(-\pi/2) + i\sin(-\pi/2)].$$

408. Represent the numbers $z_1 = 1 + i$, $z_2 = \sqrt{3} + i$, $z_3 = 1 + i\sqrt{3}$ in trigonometric form and then find the complex number $z_1/(z_2z_3)$.

Solution. We find

$$r_1 = |z_1| = \sqrt{1 + 1} = \sqrt{2}, \quad \tan\varphi_1 = 1, \quad \varphi_1 = \arg z_1 = \pi/4,$$
$$z_1 = \sqrt{2}\,[\cos(\pi/4) + i\sin(\pi/4)];$$
$$r_2 = |z_2| = \sqrt{3 + 1} = 2, \quad \tan\varphi_2 = 1/\sqrt{3}, \quad \varphi_2 = \arg z_2 = \pi/6,$$
$$z_2 = 2[\cos(\pi/6) + i\sin(\pi/6)];$$
$$r_3 = |z_3| = \sqrt{3 + 1} = 2, \quad \tan\varphi_3 = \sqrt{3}, \quad \varphi_3 = \arg z_3 = \pi/3,$$
$$z_3 = 2[\cos(\pi/3) + i\sin(\pi/3)].$$

Consequently,

$$z_2z_3 = 2\cdot 2\,[\cos(\pi/6 + \pi/3) + i\sin(\pi/6 + \pi/3)] = 4\,[\cos(\pi/2) + i\sin(\pi/2)],$$

$$\frac{z_1}{z_2z_3} = \frac{\sqrt{2}}{4}\cdot\frac{\cos(\pi/4)+i\sin(\pi/4)}{\cos(\pi/2)+i\sin(\pi/2)} = \frac{\sqrt{2}}{4}[\cos(-\pi/4)+i\sin(-\pi/4)] = \frac{1}{4}(1-i).$$

409. Find all the values of $\sqrt[3]{8+i}$.

Solution. We write the complex number $z = \sqrt[3]{8+i}$ in trigonometric form. We have $r = |z| = \sqrt{64+1} = \sqrt{65} \approx 8.062$, $\varphi = \arg z = \operatorname{argtan}(1/8) = 7°6'$, i.e. $z = 8.062\,(\cos 7°6' + i\sin 7°6')$. Consequently,

$$\sqrt[3]{8+i} = \sqrt[3]{8.062}\cdot\left(\cos\frac{7°6' + 360°k}{3} + i\sin\frac{7°6' + 360°k}{3}\right)$$

$$= 2.0052\,[\cos(2°22' + 120°k) + i\sin(2°22' + 120°k)].$$

If $k = 0$, then $w_0 = 2.0052\,(\cos 2°22' + i\sin 2°22')$;

If $k = 1$, then $w_1 = 2.0052\,(\cos 122°22' + i\sin 122°22')$;

If $k = 2$, then $w_2 = 2.0052\,(\cos 242°22' + i\sin 242°22')$;

It follows that $w_0 = 2.0034 + 0.0828i$; $w_1 = -1.0734 + 1.7120i$; $w_2 = -0.9300 - 17764i$.

410. Solve the binomial equation $w^5 + 32i = 0$.

Solution. Let us rewrite the equation in the form $w^5 = -32i$, and represent the number $-i$ in trigonometric form:

$$w^5 = 32[\cos(-90°) + i\sin(-90°)], \quad \text{or} \quad w = 2\sqrt[5]{\cos(-90°) + i\sin(-90°)},$$

i.e.

$$w = 2\left[\cos\frac{-90° + 360°k}{5} + i\sin\frac{-90° + 360°k}{5}\right]$$

$$= 2[\cos(18° + 72°k) + i\sin(-18° + 72°k)].$$

If $k = 0$, then $w_0 = 2[\cos(-18°) + i\sin(-18°)] = 1.9022 - 0.6180i$ (A),

if $k = 1$, then $w_1 = 2(\cos 54° + i\sin 54°) = 1.1756 + 1.6180i$ (B),

if $k = 2$, then $w_2 = 2(\cos 126° + i\sin 126°) = -1.1756 + 1.6180i$ (C),

if $k = 3$, then $w_3 = 2(\cos 198° + i\sin 198°) = -1.9022 - 0.6180i$ (D),

if $k = 4$, then $w_4 = 2(\cos 270° + i\sin 270°) = -2i$ (E).

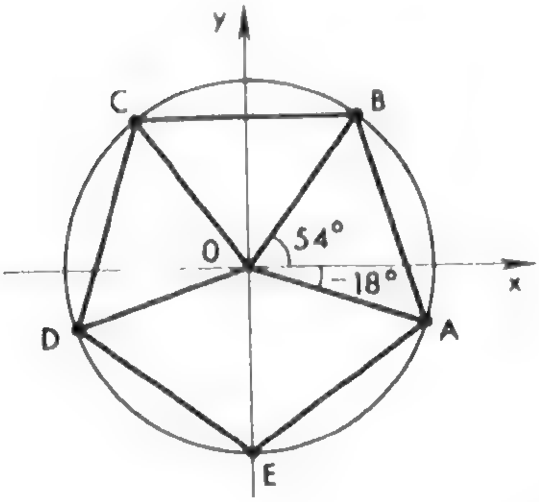


Fig. 26

To the roots of the binomial equation there correspond the vertices of a regular pentagon inscribed into a circle of radius $R = 2$ with centre at the origin (Fig. 26).

In general, to the roots of the binomial equation $w^n = a$, where a is a complex number, there correspond the vertices of a regular n-gon inscribed into a circle with centre at the origin and radius equal to $\sqrt[n]{|a|}$.

411. Using De Moivre's formula, express $\cos 5\varphi$ and $\sin 5\varphi$ in terms of $\cos\varphi$ and $\sin\varphi$.

Solution. We transform the left-hand side of the equality $(\cos\varphi + i\sin\varphi)^5 = \cos 5\varphi + i\sin 5\varphi$ by Newton's binomial formula:

$$\cos^5\varphi + 5i\cos^4\varphi\sin\varphi - 10\cos^3\varphi\sin^2\varphi - 10i\cos^2\varphi\sin^3\varphi + 5\cos\varphi\sin^4\varphi + i\sin^5\varphi = \cos 5\varphi + i\sin 5\varphi.$$

It remains to equate the real and imaginary parts of the equation:

$$\cos 5\varphi = \cos^5\varphi - 10\cos^3\varphi\sin^2\varphi + 5\cos\varphi\sin^4\varphi,$$
$$\sin 5\varphi = 5\cos^4\varphi\sin\varphi - 10\cos^2\varphi\sin^3\varphi + \sin^5\varphi.$$

412. Given the complex number $z = 2 - 2i$. Find $\operatorname{Re} z$, $\operatorname{Im} z$, $|z|$, $\arg z$.

413. Represent the complex number $z = -12 + 5i$ in trigonometric form.

414. Calculate the expression $(\cos 2^\circ + i\sin 2^\circ)^{45}$ by De Moivre's formula.

415. Calculate $\left(\frac{\sqrt{3} + i}{2}\right)^{12}$ by De Moivre's formula.

416. Represent the complex number $z = 1 + \cos 20^\circ + i\sin 20^\circ$ in trigonometric form.

417. Calculate the expression $(2 + 3i)^3$.

418. Calculate the expression $\dfrac{(1-2i)(2-3i)}{(3-4i)(4-5i)}$.

419. Calculate the expression $1/(3-2i)^2$.

420. Represent the complex number $5-3i$ in trigonometric form.

421. Represent the complex number $-1+i$ in trigonometric form.

422. Calculate the expression $\dfrac{(\cos 77^\circ + i\sin 77^\circ)(\cos 23^\circ + i\sin 23^\circ)}{\cos 55^\circ + i\sin 55^\circ}$.

423. Calculate the expression $\dfrac{(1+i)(-\sqrt{3}+i)}{(1-i)(\sqrt{3}+i)}$, first representing the factors in the numerator and the denominator in trigonometric form.

424. Find all the values of $\sqrt[4]{i}$.

425. Solve the binomial equation $w^3 - 4\sqrt{2}(1+i) = 0$.

426. Express $\cos 4\varphi$ and $\sin 4\varphi$ in terms of $\cos\varphi$ and $\sin\varphi$.

427. Show that the distance between the points z_1 and z_2 is equal to $|z_2 - z_1|$.

Solution. We have $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $z_2 - z_1 = (x_2 - x_1) + i(y_2 - y_1)$, whence

$$|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

i.e. $|z_2 - z_1|$ is equal to the distance between the given points.

428. What line is described by the point z satisfying the equation $|z-c| = R$, where c is a constant complex number and $R > 0$?

429. What is the geometric meaning of the inequalities (1) $|z-c| < R$; (2) $|z-c| > R$?

430. What is the geometric meaning of the inequalities (1) $\operatorname{Re} z > 0$; (2) $\operatorname{Im} z < 0$?

3.7.2. Series with complex terms. Let us consider the sequence of complex numbers $z_1, z_2, z_3, \ldots$, where $z_n = x_n + iy_n$ $(n = 1, 2, 3, \ldots)$.

The constant number $c = a + bi$ is called the *limit* of the sequence $z_1, z_2, z_3, \ldots$ if for any arbitrarily small number $\varepsilon > 0$ there is a number N such that all the values of z_n with the numbers $n > N$ satisfy the inequality $|z_n - c| < \varepsilon$. In that case, we write $\lim\limits_{n\to\infty} z_n = c$.

The necessary and sufficient condition of the existence of a limit for a sequence of complex numbers consists in the following: the number $c = a + bi$ is the limit of the sequence of the complex numbers $x_1 + iy_1, x_2 + iy_2, x_3 + iy_3, \ldots$ if and only if $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$.

The series

$$w_1 + w_2 + w_3 + \ldots, \tag{1}$$

whose terms are complex numbers, is said to be *convergent* if the nth partial sum S_n

of the series tends to a definite limit as $n \to \infty$. Otherwise, series (1) is said to be *divergent*.

Series (1) converges if and only if the series with real numbers

$$\operatorname{Re} w_1 + \operatorname{Re} w_2 + \operatorname{Re} w_3 + \dots \tag{2}$$

and with imaginary numbers

$$\operatorname{Im} w_1 + \operatorname{Im} w_2 + \operatorname{Im} w_3 + \dots \tag{3}$$

converge.

If the sum of series (2) is the number S' and the sum of series (3) is the number S'', then the sum of series (1) is the complex number $S = S' + iS''$.

If the series

$$w_1 + w_2 + w_3 + \dots \quad (\text{where } w_n = u_n + iv_n)$$

converges, then $\lim\limits_{n\to\infty} w_n = 0$ (i.e. $\lim\limits_{n\to\infty} u_n = 0$, $\lim\limits_{n\to\infty} v_n = 0$).

If the series

$$|w_1| + |w_2| + |w_3| + \dots$$

converges, so does the series

$$w_1 + w_2 + w_3 + \dots .$$

In that case, the last series is said to be *absolutely convergent*.

Suppose we are given a power series

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \dots,$$

where $z_0, a_0, a_1, a_2, a_3, \dots$ are complex numbers, with the coefficients of the series being nonzero, and z is a complex variable.

The series converges in the circle $|z - z_0| < R$, where $R = \lim\limits_{n\to\infty} |a_n/a_{n+1}|$, and diverges in the exterior of the indicated circle, that is, at the values of z complying with the inequality $|z - z_0| > R$.

431. Test the series

$$(1 + i) + \left(\frac{1}{2} + \frac{1}{3}i\right) + \left(\frac{1}{4} + \frac{1}{9}i\right) + \left(\frac{1}{8} + \frac{1}{27}i\right) + \dots$$

for convergence.

Solution. The series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \quad \text{and} \quad 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$$

converge since they are set up from the terms of infinitely decreasing geometric progressions. Consequently, the given series with complex terms converges as well.

Let us find the sums of these progressions:

$$S_1 = \frac{1}{1 - 1/2} = 2, \quad S_2 = \frac{1}{1 - 1/3} = \frac{3}{2}.$$

It follows that the sum of the series under discussion is the complex number $S = 2 + (3/2)i$.

432. Test the series

$$(1 + 0.1i) + \left(\frac{1}{2} + 0.01i\right) + \left(\frac{1}{3} + 0.001i\right) + \ldots$$

for convergence.

Solution. Let us consider the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \quad \text{and} \quad 0.1 + 0.01 + 0.001 + \ldots$$

The first of them diverges, consequently, the given series with the complex terms diverges too.

433. Test the series

$$\left(\frac{1}{2} + \frac{2}{3}i\right) + \left(\frac{2}{3} + \frac{3}{4}i\right) + \left(\frac{3}{4} + \frac{4}{5}i\right) + \ldots$$

for convergence.

Solution. The series converges since its general term $w_n = \dfrac{n}{n+1} + \dfrac{n+1}{n+2}i$ does not tend to zero (the reader is recommended to make sure of this fact).

434. Show that the series

$$\frac{1+i}{2} + \left(\frac{1+i}{2}\right)^2 + \left(\frac{1+i}{2}\right)^3 + \ldots$$

is absolutely convergent.

Solution. Since $1 + i = \sqrt{2}\,[\cos(\pi/4) + i\sin(\pi/4)]$, it follows that

$$w_n = \left(\frac{1+i}{2}\right)^n = \left[\frac{\cos(\pi/4) + i\sin(\pi/4)}{2}\right]^n = \frac{1}{2^{n/2}}\left(\cos\frac{\pi n}{4} + i\sin\frac{\pi n}{4}\right).$$

Consequently, $|w_n| = 1/2^{n/2}$. Let us derive a series from the moduli

$$\frac{1}{2^{1/2}} + \frac{1}{2} + \frac{1}{2^{3/2}} + \frac{1}{2^2} + \ldots.$$

The series whose terms form an infinitely decreasing geometric progression converges; therefore, the given series with complex terms is absolutely convergent.

435. Find the domain of convergence of the series

$$\frac{\sqrt{3}+i}{3}(z-i) + \left(\frac{\sqrt{3}+i}{3}\right)^2 (z-i)^2 + \left(\frac{\sqrt{3}+i}{3}\right)^3 (z-i)^3 + \ldots.$$

Solution. We have

$$a_n = \left(\frac{\sqrt{3}+i}{3}\right)^n, \quad a_{n+1} = \left(\frac{\sqrt{3}+i}{3}\right)^{n+1}, \quad \frac{a_n}{a_{n+1}} = \frac{3}{\sqrt{3}+i},$$

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{3}{|\sqrt{3}+i|} = \frac{3}{\sqrt{3+1}} = \frac{3}{2}, \quad R = \frac{3}{2}.$$

The domain of convergence of the series is the circle $|z - i| < 3/2$.

436. Show that the series

$$\left(\frac{1}{5} - \frac{1}{2}i\right) + \left(\frac{1}{25} - \frac{1}{4}i\right) + \left(\frac{1}{125} - \frac{1}{8}i\right) + \ldots$$

converges and find its sum.

437. Test the series

$$\left(1 + \frac{1}{2}i\right) + \left(\frac{1}{2\sqrt{2}} + \frac{1}{2^2}i\right) + \left(\frac{1}{3\sqrt{3}} + \frac{1}{2^3}i\right) + \ldots$$

for convergence.

438. Test the convergence of the series with the general term $w_n = \dfrac{1}{n!} + \dfrac{i}{n}$.

439. Show that the series

$$1 + \frac{1}{2!}(1+i) + \frac{1}{3!}(1+i)^2 + \frac{1}{4!}(1+i)^3 \ldots$$

is absolutely convergent.

440. Find the domain of convergence of the series

$$z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots .$$

441. Find the domain of convergence of the series

$$(z - 1 - i) + 2!(z - 1 - i)^2 + 3!(z - 1 - i)^3 + \ldots .$$

3.7.3. The exponential function and the trigonometric functions of a complex variable. The *exponential function* and the *trigonometric functions of the complex variable* z are specified by the equations

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots;$$

$$\sin z = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots;$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \ldots .$$

The following relations exist between the indicated functions:

$$e^{zi} = \cos z + i \sin z, \tag{1}$$

$$e^{-zi} = \cos z - i \sin z, \tag{2}$$

$$\cos z = \frac{e^{zi} + e^{-zi}}{2}, \tag{3}$$

$$\sin z = \frac{e^{zi} - e^{-zi}}{2i}, \tag{4}$$

These relations are known as *Euler's formulas*.

With the aid of formula (1) the complex number given in trigonometric form $z = r(\cos\varphi + i \sin \varphi)$ can be represented in the exponential form $z = re^{\varphi i}$.

442. Represent the complex number $z = 3 + i\sqrt{3}$ in the trigonometric form and the exponential form.

Solution. We find $r = \sqrt{9 + 3} = 2\sqrt{3}$, $\varphi = \arctan(\sqrt{3}/3) = \pi/6$. Consequently, the trigonometric form of the given number is

$$z = 2\sqrt{3}\,[\cos(\pi/6) + i \sin(\pi/6)],$$

and the exponential form is

$$z = 2\sqrt{3}e^{\pi i/6}.$$

443. Represent the number $z = \sqrt{2} - i\sqrt{2}$ in exponential form.

Solution. We have $r = \sqrt{2 + 2} = 2$, $\tan \varphi = -\sqrt{2}/\sqrt{2} = -1$, $\varphi = -\pi/4$, i.e. $z = 2e^{-\pi i/4}$.

444. Find the numerical value of $e^{\pi i/2}$.

Solution. We make use of formula (1):

$$e^{\pi i/2} = \cos(\pi/2) + i \sin(\pi/2) = i.$$

445. Prove with the aid of Euler's formula that

$$\cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x.$$

Solution. Since $\cos x = (e^{xi} + e^{-xi})/2$, we have

$$\cos^3 x = \frac{e^{3xi} + 3e^{xi} + 3e^{-xi} + e^{-3xi}}{8}$$

$$= \frac{1}{4}\cdot\frac{e^{3xi} + e^{-3xi}}{2} + \frac{3}{4}\cdot\frac{e^{xi} + e^{-xi}}{2} = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x.$$

446. Represent the complex number $z = \sqrt{3} + i$ in exponential form.

447. Represent the number $-i$ in exponential form.

448. What is the numerical value of $e^{\pi i/2}$?

449. Show that

$$\cos^5 x = \frac{1}{16}\cos 5x + \frac{1}{16}\cos 3x + \frac{5}{8}\cos x.$$

450. Express $\sin^3 x$ linearly in terms of $\sin x$ and $\sin 3x$.

451. Show with the aid of Euler's formula that i^i possesses infinitely many values which are all real.

3.8. Fourier Series

The *Fourier series* of the function $f(x)$ defined on the closed interval $[-\pi, \pi]$ is the series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx),$$

where

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx \,(m = 0, 1, 2, \ldots),$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx \,(m = 1, 2, \ldots).$$

If x_0 is discontinuity point of the 1st kind of the function $f(x)$, then the sum of the Fourier series will determine the function which coincides with the function $f(x)$ at the continuity points and is equal to $(1/2)\,[f(x_0 - 0) + f(x_0 + 0)]$ at the indicated point of discontinuity of the 1st kind.

We shall agree to assume $(1/2)[f(-\pi + 0) + f(\pi - 0)]$ as the value of the function $f(x)$ at each end point of the closed interval $[-\pi, \pi]$.

If the function $f(x)$ possesses a finite number of extrema on the interval $[-\pi, \pi]$ and is continuous except, perhaps, a finite number of points of discontinuity of the 1st kind (that is, satisfies the so-called Dirichlet conditions), then at each point of the interval $[-\pi, \pi]$ the Fourier series converges to the function $f(x)$ (**the Dirichlet theorem**).

It is easy to see that the sum of the Fourier series of the function $f(x)$ is a periodic function with period 2π.

If the function $f(x)$ is given on the interval $[-l, l]$, where l is an arbitrary number, then, the Dirichlet conditions being fulfilled on the interval $[-l, l]$, the indicated function can be represented as the sum of the Fourier series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{l} + b_m \sin \frac{m\pi x}{l} \right),$$

where

$$a_m = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{m\pi x}{l}\, dx, \quad b_m = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{m\pi x}{l}\, dx.$$

If $f(x)$ is an even function, then its Fourier series involves only a constant term and cosines, that is,

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos \frac{m\pi x}{l},$$

where

$$a_m = \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{m\pi x}{l}\, dx.$$

If $f(x)$ is an odd function, then its Fourier series involves only sines, that is,

$$f(x) = \sum_{m=1}^{\infty} b_m \cos \frac{m\pi x}{l},$$

where

$$b_m = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{m\pi x}{l}\, dx.$$

If the function $f(x)$ is given on the closed interval $[0, l]$, then to make the Fourier expansion, it is sufficient to extend the definition to the interval $[-l, 0]$ in an arbitrary way, and then expand it into the Fourier series, assuming it to be given on the interval $[-l, l]$. It is more expedient to extend the definition of the function so that its values at the points of $[-l, 0]$ could be found from the condition $f(x) = f(-x)$ or $f(x) = -f(-x)$. In the first case, the function $f(x)$ on the interval $[-l, l]$ will be even, in the second case, it will be odd. In that case, the coefficients in the expansion of such a function (a_m in the first case and b_m in the second) can be found from the above-given formulas for the coefficients of even and odd functions.

452. Expand into the Fourier series the function defined on the interval $[-\pi, \pi]$ by the equation $f(x) = \pi + x$.

Solution. The graph of the function is a line segment connecting the points $(-\pi;$

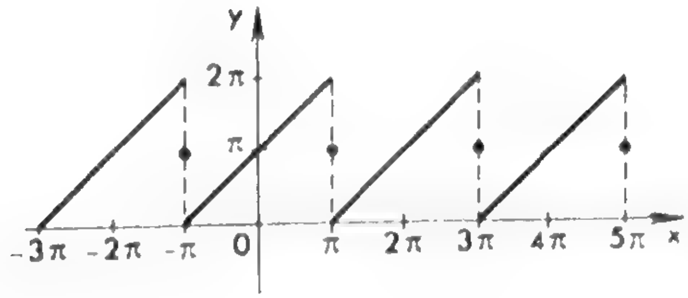


Fig. 27

0) and $(\pi; 2\pi)$. Figure 27 shows the graph of the function $y = S(x)$, where $S(x)$ is the sum of the Fourier series of the function $f(x)$. This sum is a periodic function with period 2π and coincides with the function $f(x)$ on the interval $[-\pi, \pi]$.

Let us determine the coefficients of the Fourier series. First we find

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi + x)dx = \int_{-\pi}^{\pi} dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx.$$

The second integral is zero as an integral of an odd function taken over the interval symmetric about the origin. Thus we have

$$a_0 = \int_{-\pi}^{\pi} dx = 2\pi.$$

Next we find the coefficient a_m

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi + x)\cos mx\,dx$$

$$= \int_{-\pi}^{\pi}\cos mx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos mx\,dx.$$

It is easy to see that both integrals are zero (the integrand of the second integral is odd as the product of an even function by an odd function). Thus we have $a_m = 0$, i.e. $a_1 = a_2 = a_3 = \ldots = 0$.

Now we determine the coefficients b_m:

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi + x)\sin mx\,dx$$

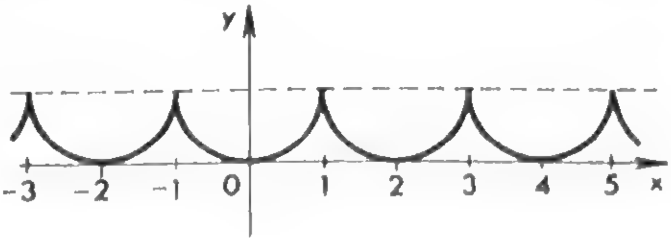


Fig. 28

$$= \int_{-\pi}^{\pi} \sin mx\, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin mx\, dx.$$

The first integral is zero. The integrand of the second integral is even as the product of two odd functions. Thus it follows that

$$b_m = \frac{2}{\pi} \int_0^{\pi} x \sin mx\, dx.$$

Integrating by parts, we get $u = x$, $dv = \sin mx\, dx$, $du = dx$, $v = -(1/m) \cos mx$, i.e.

$$b_m = -\frac{2x}{m\pi} \cos mx \Big|_0^{\pi} + \frac{2}{m\pi} \int_0^{\pi} \cos mx\, dx = -\frac{2}{m} \cdot \cos m\pi + \frac{2}{\pi m^2} \sin mx \Big|_0^{\pi}$$

$$= -\frac{2}{m} (-1)^m = \frac{2}{m} (-1)^{m+1}.$$

Consequently, the expansion of the function $f(x)$ into the Fourier series has the form

$$f(x) = \pi + 2 \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sin mx = \pi + 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots \right).$$

453. Expand into the Fourier series the function defined on the closed interval $[-1, 1]$ by the equation $f(x) = x^2$ (Fig. 28).

Solution. The function in question is even. Its graph is the arc of the parabola contained between the points $(-1; 1)$ and $(1; 1)$. Here $l = 1$; therefore,

$$a_0 = \frac{2}{l} \int_0^{l} f(x)\, dx = 2 \int_0^{1} x^2\, dx = \frac{2}{3},$$

$$a_m = \frac{2}{l}\int_0^l f(x)\cos\frac{m\pi x}{l}\,dx = 2\int_0^1 x^2\cos m\pi x\,dx.$$

Here we must perform a double integration by parts:

(1) $u = x^2$, $dv = \cos m\pi x\,dx$, $du = 2x\,dx$, $v = \frac{1}{\pi m}\sin m\pi x$;

$$a_m = \frac{2x^2}{m\pi}\sin m\pi x\Big|_0^1 - \frac{4}{m\pi}\int_0^1 x\sin m\pi x\,dx = -\frac{4}{m\pi}\int_0^1 x\sin m\pi x\,dx;$$

(2) $u = x$, $dv = \sin m\pi x\,dx$, $du = dx$, $v = -\frac{1}{\pi m}\cos m\pi x$;

$$a_m = \frac{4x}{m^2\pi^2}\cos m\pi x\Big|_0^1 - \frac{4}{m^2\pi^2}\int_0^1 \cos m\pi x\,dx = -\frac{4}{m^2\pi^2}(-1)^m.$$

Since the function being considered is even, we have $b_m = 0$. Consequently,

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^m}{m^2}\cos m\pi x$$

$$= \frac{1}{3} - \frac{4}{\pi^2}\left(\cos\pi x - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - \frac{\cos 4\pi x}{4^2} + \cdots\right).$$

454. Expand into the Fourier series the function defined on the half-period within the closed interval [0, 2] by the equation $f(x) = x - x^2/2$.

Solution. The function can be expanded into the Fourier series by infinitely many methods. Here we present two most important variants of the expansion.

(1) We extend the definition of the function $f(x)$ to the closed interval $[-2, 0]$ in an even way (Fig. 29) and have $l = 2$,

$$a_0 = \int_0^2\left(x - \frac{1}{2}x^2\right)dx = \left[\frac{x^2}{2} - \frac{1}{6}x^3\right]_0^2 = \frac{2}{3},$$

Fig. 29

$$a_m = \int_0^2 \left(x - \frac{1}{2}x^2\right)\cos\frac{m\pi x}{2}\,dx.$$

Next we integrate by parts

$$u = x - \frac{1}{2}x^2,\ dv = \cos\frac{m\pi x}{2}\,dx,\ du = (1-x)\,dx,\ v = \frac{2}{m\pi}\sin\frac{m\pi x}{2};$$

$$a_m = \frac{2}{m\pi}\left(x - \frac{1}{2}x^2\right)\sin\frac{m\pi x}{2}\Bigg|_0^2 - \frac{2}{m\pi}\int_0^2 (1-x)\sin\frac{m\pi x}{2}\,dx$$
$$= -\frac{2}{m\pi}\int_0^2 (1-x)\sin\frac{m\pi x}{2}\,dx.$$

We again integrate by parts:

$$u = 1 - x,\quad dv = \sin\frac{m\pi x}{2}\,dx,\quad du = -dx,\quad v = -\frac{2}{m\pi}\cos\frac{m\pi x}{2};$$

$$a_m = \frac{4}{m^2\pi^2}(1-x)\cdot\cos\frac{m\pi x}{2}\Bigg|_0^2 + \frac{4}{m^2\pi^2}\int_0^2 \cos\frac{m\pi x}{2}\,dx$$

$$= -\frac{4}{m^2\pi^2}\cos m\pi - \frac{4}{m^2\pi^2} = -\frac{4}{m^2\pi^2}[1 + (-1)^m],\quad b_m = 0.$$

and obtain

$$f(x) = \frac{1}{3} - \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{1+(-1)^m}{m^2}\cos\frac{m\pi x}{2}$$

$$= \frac{1}{3} - \frac{8}{\pi^2}\left(\frac{1}{2^2}\cos\pi x + \frac{1}{4^2}\cos 2\pi x + \frac{1}{6^2}\cos 3\pi x + \dots\right).$$

(2) We extend the definition of the function $f(x)$ to the interval $[-2, 0]$ in an odd way (Fig. 30):

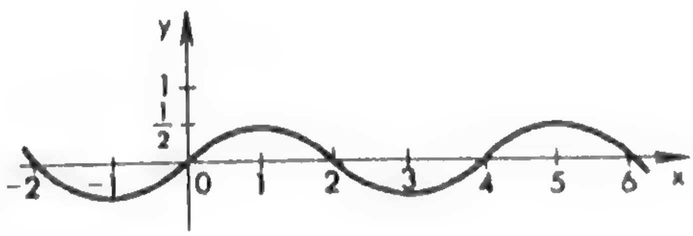


Fig. 30

$$b_m = \int_0^2 \left(x - \frac{1}{2}x^2\right) \sin\frac{m\pi x}{2} dx; \quad u = x - \frac{1}{2}x^2, \quad dv = \sin\frac{m\pi x}{2} dx,$$

$$du = (1-x)dx, \quad v = -\frac{2}{m\pi}\cos\frac{m\pi x}{2};$$

$$b_m = -\frac{2}{m\pi}\left(x - \frac{1}{2}x^2\right)\cos\frac{m\pi x}{2}\bigg|_0^2 + \frac{2}{m\pi}\int_0^2 (1-x)\cos\frac{m\pi x}{2} dx$$

$$= \frac{2}{m\pi}\int_0^2 (1-x)\cos\frac{m\pi x}{2} dx;$$

$$u = 1 - x, \quad dv = \cos\frac{m\pi x}{2} dx, \quad du = -dx, \quad v = \frac{2}{m\pi}\sin\frac{m\pi x}{2};$$

$$b_m = \frac{4}{m^2\pi^2}(1-x)\sin\frac{m\pi x}{2}\bigg|_0^2 + \frac{4}{m^2\pi^2}\int_0^2 \sin\frac{m\pi x}{2} dx$$

$$= -\frac{8}{m^3\pi^3}\cdot\cos\frac{m\pi x}{2}\bigg|_0^2 = -\frac{8}{m^3\pi^3}\cos m\pi + \frac{8}{m^3\pi^3} = \frac{8}{m^3\pi^3}[1-(-1)^m];$$

$$a_m = 0 \quad (m = 0, 1, 2, \ldots).$$

Thus we have

$$f(x) = \frac{8}{\pi^3}\sum_{m=1}^{\infty}\frac{1-(-1)^m}{m^3}\sin\frac{m\pi x}{2}$$

$$= \frac{16}{\pi^3}\left(\sin\frac{\pi x}{2} + \frac{1}{3^3}\cdot\sin\frac{3\pi x}{2} + \frac{1}{5^3}\cdot\sin\frac{5\pi x}{2} + \ldots\right).$$

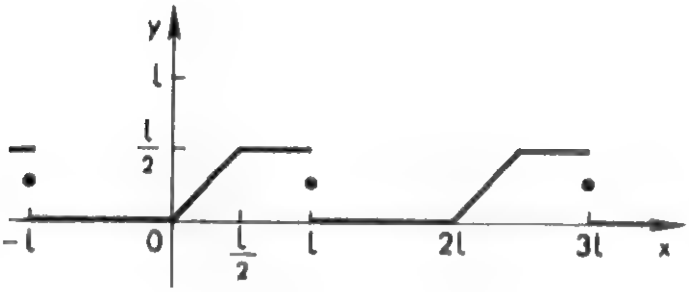


Fig. 31

455. Expand into the Fourier series the function (Fig. 31) defined on the closed interval $[-l, l]$ in the following way:

$$f(x) = \begin{cases} 0, & \text{if} \quad -l \leqslant x \leqslant 0; \\ x, & \text{if} \quad 0 \leqslant x \leqslant l/2; \\ l/2, & \text{if} \quad l/2 \leqslant x \leqslant l. \end{cases}$$

Solution. We find

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{l}\int_{-l}^{0} f(x)\,dx + \frac{1}{l}\int_{0}^{l/2} f(x)\,dx + \frac{1}{l}\int_{l/2}^{l} f(x)\,dx$$

$$= \frac{1}{l}\int_{0}^{l/2} x\,dx + \frac{1}{l}\int_{l/2}^{l} \frac{l}{2}\,dx = \frac{1}{l}\cdot\frac{x^2}{2}\bigg|_0^{l/2} + \frac{1}{2}x\bigg|_{l/2}^{l} = \frac{l}{8} + \frac{l}{4} = \frac{3}{8}l;$$

$$a_m = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{m\pi x}{2}\,dx = \frac{1}{l}\int_{-l}^{0} f(x)\cos\frac{m\pi x}{l}\,dx$$

$$+ \frac{1}{l}\int_{0}^{l/2} f(x)\cos\frac{m\pi x}{l}\,dx + \frac{1}{l}\int_{l/2}^{l} f(x)\cos\frac{m\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_{0}^{l/2} x\cos\frac{m\pi x}{l}\,dx + \frac{1}{l}\int_{l/2}^{l} \frac{l}{2}\cos\frac{m\pi x}{l}\,dx.$$

We integrate the first integral by parts

$$u = x, \quad dv = \cos\frac{m\pi x}{l}\,dx; \quad du = dx, \quad v = \frac{l}{m\pi}\sin\frac{m\pi x}{l},$$

whence we get

$$a_m = \frac{x}{m\pi}\sin\frac{m\pi x}{l}\bigg|_0^{l/2} - \frac{1}{m\pi}\int_0^{l/2}\sin\frac{m\pi x}{l}\,dx + \frac{l}{2m\pi}\sin\frac{m\pi x}{l}\bigg|_{l/2}^{l}$$

$$= \frac{l}{2m\pi}\cdot\sin\frac{m\pi}{2} + \frac{l}{m^2\pi^2}\cdot\cos\frac{m\pi x}{l}\bigg|_0^{l/2}$$

$$+\frac{l}{2m\pi}\left(\sin m\pi-\sin\frac{m\pi}{2}\right)=\frac{l}{m^2\pi^2}\left(\cos\frac{m\pi}{2}-1\right).$$

We determine the coefficients b_m:

$$b_m=\frac{1}{l}\int_0^{l/2}x\sin\frac{m\pi x}{l}\,dx+\frac{1}{2}\int_{l/2}^{l}\sin\frac{m\pi x}{l}\,dx.$$

We integrate the first integral by parts

$$u=x,\quad dv=\sin\frac{m\pi x}{l}\,dx;\quad du=dx,\quad v=-\frac{l}{m\pi}\cos\frac{m\pi x}{l}.$$

and obtain

$$b_m=-\frac{x}{m\pi}\cos\frac{m\pi x}{l}\bigg|_0^{l/2}+\frac{1}{m\pi}\int_0^{l/2}\cos\frac{m\pi x}{l}\,dx-\frac{l}{2m\pi}\cos\frac{m\pi x}{l}\bigg|_{l/2}^{l}$$

$$=\frac{l}{2m\pi}\cos\frac{m\pi}{2}+\frac{l}{m^2\pi^2}\sin\frac{m\pi x}{l}\bigg|_0^{l/2}-\frac{l}{2m\pi}\left(\cos m\pi-\cos\frac{m\pi}{2}\right)$$

$$=\frac{1}{m^2\pi^2}\sin\frac{m\pi}{2}-\frac{l}{2m\pi}(-1)^m.$$

If $m=1$, then $a_1=-\frac{l}{\pi^2}$, $b_1=\frac{l}{\pi^2}+\frac{l}{2\pi}=l\cdot\frac{2+\pi}{2\pi^2}$,

If $m=2$, then $a_2=-\frac{l}{2\pi^2}$, $b_2=-\frac{l}{4\pi}$;

If $m=3$, then $a_3=-\frac{l}{9\pi^2}$, $b_3=-\frac{l}{9\pi^2}+\frac{l}{6\pi}=l\cdot\frac{3\pi-2}{18\pi^2}$;

If $m=4$, then $a_4=0$, $b_4=-\frac{l}{8\pi}$;

If $m=5$, then $a_5=-\frac{l}{25\pi^2}$, $b_5=\frac{l}{25\pi^2}+\frac{l}{10\pi}=l\cdot\frac{2+5\pi}{50\pi^2}$;

. . . .

Consequently,

$$f(x) = l\left[\frac{3}{16} + \left(-\frac{1}{\pi^2}\cos\frac{\pi x}{l} + \frac{2+\pi}{2\pi^2}\sin\frac{\pi x}{l}\right)\right.$$

$$+ \left(-\frac{1}{2\pi^2}\cos\frac{2\pi x}{l} - \frac{1}{4\pi}\sin\frac{2\pi x}{l}\right)$$

$$\left. + \left(-\frac{1}{9\pi^2}\cos\frac{3\pi x}{l} + \frac{-2+3\pi}{18\pi^2}\sin\frac{3\pi x}{l}\right) + \ldots\right].$$

456. Expand into Fourier's series the function defined on the closed interval $[-\pi, \pi]$ by the equation $f(x) = x$.

457. Expand into Fourier's series the function defined on the closed interval $[-1, 1]$ by the equation $f(x) = |x|$.

458. Expand into Fourier's series the function defined on the closed interval $[-\pi, \pi]$ by the equation $f(x) = e^x$.

459. Expand into Fourier's series the function defined on the closed interval $[-\pi, \pi]$ by the equation $f(x) = x^3$.

460. Expand into Fourier's series the function defined on the closed interval $[0, \pi]$ by the equation $f(x) = \pi - 2x$, extending it to the interval $[-\pi, 0]$; (1) in an even way, (2) in an odd way.

461. Expand into Fourier's series the function defined on the closed interval $[-\pi, \pi]$ so that

$$f(x) = \begin{cases} -h, & \text{if } -\pi \leqslant x \leqslant 0; \\ h, & \text{if } 0 < x \leqslant \pi. \end{cases}$$

462. Expand into Fourier's series the function defined on the closed interval $[-\pi, \pi]$ so that

$$f(x) = \begin{cases} -2x, & \text{if } -\pi \leqslant x \leqslant 0; \\ 3x, & \text{if } 0 < x \leqslant \pi. \end{cases}$$

463. Expand into Fourier's series the function defined on the closed interval $[0, \pi]$ by the equation $f(x) = x^2$, by means of an odd extension to the interval $[-\pi, 0]$.

464. Expand into Fourier's series, on the interval $[-\pi, \pi]$, the function

$$f(x) = \begin{cases} -x, & \text{if } -\pi < x < 0; \\ 0, & \text{if } 0 < x < \pi. \end{cases}$$

465. Determine the sine series of the function $f(x) = \cos 2x$ on the closed interval $[0, \pi]$.

466. Determine the sine series of the function $f(x) = x$ on the interval $[0, 1]$.

467. Determine the cosine series, on the closed interval $[0, 2]$, of the function

$$f(x) = \begin{cases} x, & \text{if } 0 < x \leqslant 1; \\ 2 - x, & \text{if } 1 < x \leqslant 2. \end{cases}$$

3.9. Fourier Integral

If the function $f(x)$ complies with the Dirichlet conditions on any finite intercept of the Ox-axis and is absolutely integrable throughout the axis (i.e. $\int_{-\infty}^{+\infty} |f(x)|\,dx$ converges), then the *integral Fourier's formula* (obtained from the Fourier series by a limiting process al $l \to \infty$)

$$f(x) = \frac{1}{\pi} \int_0^{+\infty} dz \int_{\infty}^{+\infty} f(u) \cos z(u - x)\,du$$

is valid for it (at the points of discontinuity of the 1st kind we assume, as before, $1/2[f(x_0 - 0] + f(x_0 + 0)]$ to be the value of $f(x)$; here x_0 is the abscissa of the point of discontinuity).

The Fourier integral can be represented in *complex form:*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} f(u) e^{iz(u-x)}\,du = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-izx}\,dz \int_{-\infty}^{+\infty} e^{izu} f(u)\,du.$$

For an even function the Fourier integral can be represented in the form

$$f(x) = \frac{2}{\pi} \int_0^{+\infty} \cos zx\,dz \int_0^{+\infty} f(u) \cos zu\,du,$$

and for an odd function, in the form

$$f(x) = \frac{2}{\pi} \int_0^{+\infty} \sin zx\,dz \int_0^{+\infty} f(u) \sin zu\,du.$$

The last three formulas are related to the so-called Fourier integral transformation.

1. *Fourier's transformation of the general form:*

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{izx} f(x)\,dx \text{ (direct)},$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-izx} F(z)\,dz \text{ (inverse)},$$

2. *Fourier's cosine transformation* (for even functions):

$$f_c(z) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \cos zx\,dx \quad \text{(direct)},$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f_c(z)\cos zx\, dz \quad (\textit{inverse}).$$

3. *Fourier's sine transformation* (for odd functions):

$$f_s(z) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x)\sin zx\, dx \quad (\textit{direct}),$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f_s(z)\sin zx\, dz \quad (\textit{inverse}).$$

Sine and cosine Fourier transformations can only be applied to functions specified on the positive semi-axis Ox if they are absolutely integrable along this semi-axis and comply with the Dirichlet conditions on any of its finite intercepts. The sine transformation performs an odd extension of the function $f(x)$ to the negative semi-axis and the cosine transformation performs an even extension.

Note. In integral Fourier's forms, all the integrals of the form $\int_{-\infty}^{+\infty} f(u)du$ are understood in the sense of the *principal value*, that is,

$$\int_{-\infty}^{\infty} f(u)du = \lim_{N\to\infty} \int_{-N}^{N} f(u)du.$$

468. Find the sine and cosine transformations of the function $f(x) = e^{-x}$ $(x \geqslant 0)$.

Solution. We have

$$f_c(z) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} e^{-u}\cos zu\, du.$$

Since $\int_0^{+\infty} e^{-u}\cos zu\, du = \frac{1}{z^2+1}$, it follows that

$$f_c(z) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{z^2+1}.$$

Similarly we obtain

$$f_s(z) = \sqrt{\frac{2}{\pi}} \cdot \frac{z}{z^2+1}.$$

Now, applying Fourier's cosine and sine transformations to the functions $f_c(z)$ and $f_s(z)$, we get the function $f(x)$, that is,

$$\frac{2}{\pi} \int_0^{+\infty} \frac{\cos zx}{z^2+1} dz = e^{-x}, \quad \frac{2}{\pi} \int_0^{+\infty} \frac{z\sin zx}{z^2+1} dz = e^{-x}.$$

Hence we obtain the *Laplace integrals:*

$$\int_0^{+\infty} \frac{\cos zx}{z^2+1}\,dz = \frac{\pi}{2}e^{-x}, \qquad \int_0^{+\infty} \frac{z\sin zx}{z^2+1}\,dz = \frac{\pi}{2}e^{-x}.$$

469. Suppose the function $f(x)$ is specified by the equalities

$$f(x) = \begin{cases} 1, & \text{if } 0 \leqslant x < a; \\ 1/2, & \text{if } x = a; \\ 0, & \text{if } x > a. \end{cases}$$

Find its cosine and sine transformations (Fig. 32),

Solution. We find the cosine transformation of the given function:

$$f_c(z) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(u)\cos zu\,du$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{a} \cos zu\,du + \sqrt{\frac{2}{\pi}} \int_a^{+\infty} 0\cdot\cos zu\,du$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{a} \cos zu\,du = \sqrt{\frac{2}{\pi}}\,\frac{\sin az}{z}.$$

Now we find the sine transformation:

$$f_s(z) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(u)\sin zu\,du$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{a} \sin zu\,du + \sqrt{\frac{2}{\pi}} \int_0^{\infty} 0\cdot\sin zu\,du$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{a} \sin zu\,du = \sqrt{\frac{2}{\pi}}\cdot\frac{1-\cos az}{z}.$$

This yields

$$\frac{2}{\pi}\int_0^{+\infty} \frac{\sin az}{z}\cos xz\,dz = \begin{cases} 1, & \text{if } 0 \leqslant x < a; \\ 1/2, & \text{if } x = a; \\ 0, & \text{if } x > a, \end{cases}$$

(*Dirichlet's discontinuous factor*) and

$$\frac{2}{\pi}\int_0 \frac{1-\cos az}{z}\sin xz\,dz = \begin{cases} 1, & \text{if } 0 \leqslant x < a; \\ 1/2, & \text{if } x = a; \\ 0, & \text{if } x > a. \end{cases}$$

470. Find the Fourier transformation of the function

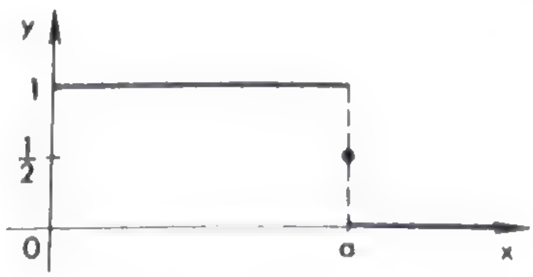


Fig. 32

$$f(x) = \begin{cases} x+1, & \text{if } -1 \leqslant x \leqslant -1/2; \\ 1, & \text{if } |x| < 1/2; \\ -x+1, & \text{if } 1/2 \leqslant x \leqslant 1; \\ 0, & \text{if } 1 < |x|. \end{cases}$$

Solution. By the Fourier transformation formula

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)e^{izu}du,$$

making use of the form of the function $f(x)$, we find that

$$\sqrt{2\pi}F(z) = \int_{-\infty}^{-1} 0 \cdot e^{izu}du + \int_{-1}^{-1/2} (u+1)e^{izu}du$$
$$+ \int_{-1/2}^{1/2} 1 \cdot e^{izu}du + \int_{1/2}^{1} (-u+1)\, e^{izu}du + \int_{1}^{+\infty} 0 \cdot e^{izu}du.$$

Evidently, the first and third integrals are zero. Let us designate the rest of the integrals by I_1, I_2 and I_3, respectively, and evaluate them:

$$I_1 = \int_{-1}^{-1/2} (u+1)e^{izu}du = \left[\frac{1}{zi}(u+1)e^{izu} - \frac{1}{i^2z^2}e^{izu}\right]_{-1}^{-1/2}$$
$$= \frac{1}{zi}\cdot\frac{1}{2}e^{-iz/2} - \frac{1}{i^2z^2}e^{-iz/2} + \frac{1}{i^2z^2}e^{-iz} = \frac{1}{2zi}e^{-zi/2}$$
$$+ \frac{1}{z^2}e^{-zi/2} - \frac{1}{z^2}e^{-zi};$$

$$I_2 = \int_{-1/2}^{1/2} e^{izu}du = \frac{1}{zi}e^{izu}\Big|_{-1/2}^{1/2} = \frac{1}{zi}\left(e^{zi/2} - e^{-zi/2}\right) = \frac{2\sin(z/2)}{z};$$

$$I_3 = \int_{1/2}^{1} (-u+1)e^{izu}du = \left[\frac{1}{zi}(-u+1)e^{izu} + \frac{1}{i^2z^2}e^{izu}\right]_{1/2}^{1}$$

$$= -\frac{1}{z^2}e^{zi} - \frac{1}{2zi}e^{zi/2} + \frac{1}{z^2}e^{zi/2}.$$

Thus we have

$$F(z) = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{2zi}e^{-zi/2} + \frac{1}{z^2}e^{-zi/2} - \frac{1}{z^2}e^{-zi} + \frac{2\sin(z/2)}{z}\right.$$

$$\left.-\frac{1}{z^2}e^{zi} - \frac{1}{2zi}e^{zi/2} + \frac{1}{z^2}e^{zi/2}\right] = \frac{1}{\sqrt{2\pi}}\left[-\frac{2\cos z}{z^2} + \frac{\sin(z/2)}{z} + \frac{2\cos(z/2)}{z^2}\right].$$

471. Find the Fourier transformation of the function

$$f(x) = \begin{cases} \cos(x/2), & \text{if } |x| \leqslant \pi; \\ 0, & \text{if } |x| > \pi. \end{cases}$$

472. Find the Fourier transformation of the function

$$f(x) = \begin{cases} -e^{x}, & \text{if } -1 \leqslant x < 0; \\ e^{-x}, & \text{if } 0 \leqslant x \leqslant 1; \\ 0, & \text{if } |x| > 1. \end{cases}$$

473. Find Fourier's sine and cosine transformations of the function

$$f(y) = \begin{cases} -1, & \text{if } -1 \leqslant x \leqslant -1/2; \\ 0, & \text{if } -1/2 \leqslant x < 1/2; \\ 1, & \text{if } 1/2 \leqslant x \leqslant 1. \end{cases}$$

Chapter 4

Ordinary Differential Equations

4.1. First-Order Differential Equations

4.1.1. Principal notions. A *differential equation* is an equation relating independent variables, their function and the derivatives (or differentials) of that function. If the equation contains one independent variable, it is called an *ordinary* equation; now if there are two or more independent variables in the equation, then it is called a differential *equation in partial derivatives.*

The order of the highest derivative appearing in the equation is called the *order of the differential equation*, for example,

(1) $x^2y' + 5xy = y^2$ is an ordinary first-order differential equation;

(2) $\frac{d^2y}{dx^2} - 4xy\frac{dy}{dz} = x^2$ is an ordinary second-order differential equation;

(3) $y'^3 + y''y''' = x$ is an ordinary third-order differential equation;

(4) $F(x, y, y', y'') = 0$ is the general form of an ordinary second-order differential equation;

(5) $x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = 0$ is a first-order differential equation in partial derivatives.

Considered in this section are ordinary differential equations of the first order, that is, equations of the form $F(x, y, y') = 0$ or (in the form solved for y') $y' = f(x, y)$.

The *solution* of a differential equation is a differentiable function $y = \varphi(x)$ such that being substituted into the equation for the unknown function, it turns the equation into an identity.

The *general solution* of the first-order differential equation $y' = f(x, y)$ in the domain D is the function $y = \varphi(x, C)$, possessing the following properties: (1) it is a solution of the given equation for any values of the arbitrary constant C belonging to a certain set; (2) for any initial condition $y(x_0) = y_0$ such that $(x_0; y_0) \in D$, there is a single value $C = C_0$ for which the solution $y = \varphi(x, C_0)$ satisfies the given initial condition.

Every solution $y = \varphi(x, C_0)$ obtained from the general solution $y = \varphi(x, C)$ at a concrete value $C = C_0$ is called a *particular solution.*

The problem in which it is required to find a particular solution of the equation $y' = f(x, y)$, satisfying the initial condition $y(x_0) = y_0$, is known as Cauchy's problem.

The graph of any solution $y = \varphi(x)$ of the given differential equation constructed on the xOy plane is called an *integral curve* of that equation. Thus, the general solu-

tion $y = \varphi(x, C)$ is associated with a family of integral curves constructed on the xOy plane and dependent on one parameter, an arbitrary constant C, and the particular solution satisfying the initial condition $y(x_0) = y_0$ is associated with a curve belonging to that family and passing through the given point $M_0(x_0; y_0)$.

However, there are differential equations possessing solutions which cannot be obtained from the general solution for any values of C (even for $C = \pm\infty$). Such solutions are called *singular*. For instance, it can be verified that the equation $y' = \sqrt{1 - y^2}$ possesses a general solution $y = \sin(x + C)$, and at the same time the function $y = 1$ is also a solution of that equation, but this solution cannot be obtained from the general solution for any values of C, that is, it is a singular solution.

The graph of a singular solution is an integral curve which possesses, at its every point, a common tangent with one of the integral curves defined by the general solution. Such a curve is called an *envelope* of the family of the integral curves.

The process of finding solutions of a differential equation is called the ***integration* of the differential equation**.

4.1.2. Differential equations with variables separable. The differential equation of the form

$$f_1(x)\varphi_1(y)dx + f_2(x)\varphi_2(y)dy = 0$$

is an equation with variables separable. If neither of the functions $f_1(x)$, $f_2(x)$, $\varphi_1(y)$, $\varphi_2(y)$ is identically zero, then the division of the original equation by $f_2(x)\varphi_1(y)$ yields

$$\frac{f_1(x)}{f_2(x)}dx + \frac{\varphi_2(y)}{\varphi_1(y)}dy = 0.$$

A term-by-term integration of the last equation leads to the relation

$$\int \frac{f_1(x)}{f_2(x)}dx + \int \frac{\varphi_2(y)}{\varphi_1(y)}dy = C,$$

which defines (in an implicit form) the solution of the original equation. (The solution of a differential equation expressed in an implicit form is called the *integral* of that equation.)

474. Find the particular solution of the equation $y' \cos x = y/\ln y$, satisfying the initial condition $y(0) = 1$.

Solution. Setting $y' = \dfrac{dy}{dx}$, we rewrite the given equation as

$$\cos x \frac{dy}{dx} = \frac{y}{\ln y}.$$

Next we separate the variables,

$$\frac{\ln y}{y}dy = \frac{dx}{\cos x},$$

and integrate both parts of the equation:

$$\int \frac{\ln y}{y} dy = \int \frac{dx}{\cos x} + C, \text{ or } \frac{1}{2} \ln^2 y = \ln \tan \left(\frac{x}{2} + \frac{\pi}{4}\right) + C.$$

Using the initial condition $y = 1$ for $x = 0$, we find $C = 0$. The final result is

$$\frac{1}{2} \ln^2 y = \ln \tan \left(\frac{x}{2} + \frac{\pi}{4}\right).$$

475. Find the general solution of the equation $y' = \tan x \tan y$.

Solution. Setting $y' = \dfrac{dy}{dx}$ and separating the variables, we arrive at the equation $\cot y\,dy = \tan x\,dx$. Integration yields

$$\int \cot y\, dy = \int \tan x\, dx, \text{ or } \ln|\sin y| = -\ln|\cos x| + \ln C$$

(here it is more convenient to designate the constant of integration as $\ln C$). From this we find $\sin y = C/\cos x$, or $\sin y \cos x = C$ (the general solution).

476. Find the particular solution of the differential equation $(1 + x^2)dy + y\,dx = 0$ subject to the initial condition $y(1) = 1$.

Solution. Let us reduce the given equation to the form $\dfrac{dy}{y} = -\dfrac{dx}{1 + x^2}$. Integration yields

$$\int \frac{dy}{y} = -\int \frac{dx}{1 + x^2} \text{ or } \ln|y| = -\arctan x + C.$$

This is precisely the general solution of the given equation.

Using now the initial condition, we find the arbitrary constant C; we have $\ln 1 = -\arctan 1 + C$, i.e. $C = \pi/4$.

Consequently,

$$\ln y = -\arctan x + \pi/4,$$

whence we get the desired particular solution

$$y = e^{\pi/4 - \arctan \alpha}$$

Let us solve geometrical and physical problems leading to differential equations of the form in question.

477. Find the curves the lengths of whose normals and subnormals, when summed up, give a constant quantity equal to a.

Solution. The length of the subnormal is $|yy'|$, and the length of the normal is $|y\sqrt{1 + y'^2}|$. Thus, the equation which should be satisfied by the sought curves has the form

$$|yy'| + |y\sqrt{1 + y'^2}| = a.$$

Solving it for y', we find (taking into account both possible signs):

$$y' = \pm \frac{a^2 - y^2}{2ay}.$$

We separate the variables:

$$\frac{2ydy}{a^2 - y^2} = \pm \frac{dx}{a}.$$

Integration gives the general solution $\ln|a^2 - y^2| = \pm x/a + \ln C$. The operation of taking antilogarithms reduces the equation of the desired curves to the form

$$y^2 = a^2 - Ce^{\pm x/a}.$$

The condition of the problem is satisfied only at the values $C > 0$. Indeed, we find from the equation of the family of curves:

$$|yy'| = \frac{|a^2 - y'^2|}{2a}, \quad |y\sqrt{1 + y'^2} = \frac{a^2 + y^2}{2a}.$$

Therefore, to satisfy the condition $|yy'| + |y\sqrt{1 + y'^2}| = a$, it is necessary that $|a^2 - y^2| = a^2 - y^2$, i.e. $y^2 < a^2$; hence it follows that C assumes only positive values.

478. A cylindrical reservoir 6 m high, with the base diameter of 4 m, is positioned vertically and filled with water. It is open at the top. What time is required for the water filling the reservoir to escape from it through a round orifice of 1/12 m radius made in the bottom of the reservoir?

Solution. We make use of Bernoulli's equation to determine the velocity v (in m/s) of the escape of the liquid from the orifice in the reservoir which is made at the level h m lower than the free level of the liquid:

$$v = \sigma\sqrt{2gh}.$$

Here $g = 9.8\,\mathrm{m/s^2}$ is the acceleration due to gravity, σ is a constant (dimensionless) coefficient dependent on the properties of the liquid (for water $\sigma \approx 0.6$).

Suppose t seconds after the beginning of the escape of the water the level of the water r remaining in the reservoir was h m, and during the time interval dt seconds it became dh m lower ($dh < 0$). Let us compute the volume of the water that escaped during this infinitesimal time interval dt using two different methods.

On one hand, this volume $d\omega$ is equal to the volume of the cylindrical layer $|dh|$ in height, with the radius equal to the radius r of the base reservoir ($r = 2$ m). Thus we have $d\omega = \pi r^2|dh| = -\pi r^2 dh$.

On the other hand, this volume is equal to the volume of the cylinder whose base is the orifice in the bottom of the reservoir and whose height is vdt (where v is the velocity of the escape). If the radius of the orifice is ρ ($\rho = 1/12$ m), then $d\omega = \pi\rho^2 v dt = \pi\rho^2\sigma\sqrt{2gh}\,dt$.

Equating two expressions for the same volume, we arrive at the equation

$$-r^2dh = \sigma\rho^2\sqrt{2gh}\,dt.$$

Separating the variables and integrating, we get

$$dt = -\frac{r^2}{\sigma\rho^2\sqrt{2g}} \cdot \frac{dh}{\sqrt{h}}; \quad t = C - \frac{2r^2}{\sigma\rho^2\sqrt{2g}}\sqrt{h}.$$

At $t = 0$, we have $h = h_0 = 6$ m. From this we find

$$C = \frac{2r^2}{\sigma\rho^2\sqrt{2g}}\sqrt{h_0}.$$

Thus, t and h are related by the equation

$$t = \frac{2r^2}{\sigma\rho^2\sqrt{2g}}(\sqrt{h_0} - \sqrt{h}),$$

and the total time of the escape T can be found by setting $h = 0$ in the last formula:

$$T = \frac{2r^2\sqrt{h_0}}{\sigma\rho^2\sqrt{2g}}.$$

Using the data given in the hypothesis ($r = 2$ m, $h_0 = 6$ m, $\sigma = 0.6$, $\rho = 1/12$ m, $g = 9.8\,\text{m/s}^2$), we find $T \approx 1062$ s ≈ 17.7 min.

479. In a room where the temperature is 20°C a certain body cooled down from 100°C to 60°C in a time period of 20 minutes. Find the law according to which the body cooled down. How many minutes does it need to cool down to 30°C? The raising of temperature in the room may be neglected.

Solution. By virtue of Newton's law (the rate of cooling is proportional to the difference between the temperatures), we can write

$$\frac{dT}{dt} = k(T - 20), \quad \text{or} \quad \frac{dT}{T-20} = k\,dt, \quad \text{i.e.} \quad \ln(T - 20) = kt + \ln C.$$

If $t = 0$, then $T = 100°$; hence $C = 80$. If $t = 20$, then $T = 60°$. Thus, $\ln 40 = 20k + \ln 80$, whence $k = -(1/20)\ln 2$. So, the law governing the cooling of the body has the form

$$T - 20 = 80 \cdot e^{-(1/20)t\cdot\ln 2} = 80(1/2)^{t/20}, \quad \text{or} \quad T = 20 + 80(1/2)^{t/20}.$$

At $T = 30°$ we have $10 = 80(1/2)^{t/20}$, or $(1/2)^{t/20} = 1/8$. Thus we have $t/20 = 3$, whence $t = 60$ minutes.

480. Determine the time necessary for the liquid in two communicating vessels to reach the same level. The small orifice between the vessels has the area of ω sq.m. The surface areas of the horizontal sections of the first and the second vessel are $S_1\text{m}^2$ and $S_2\text{m}^2$ respectively. At the initial moment the level of the liquid in the first vessel was at the height of h_1 m from the orifice and in the second, at the height of h_2 m ($h_2 < h_1$).

Solution. Suppose that t seconds after the beginning of the escape of the liquid, the water level in the first vessel dropped to z_1 m, and in the second it rose to z_2 m. During the following infinitesimal time interval dt seconds the level of the liquid in the first vessel fell down by dz_1 m ($dz_1 < 0$) and in the second it rose by dz_2 m ($dz_2 > 0$).

Since the decrease in the volume of the liquid in the first vessel is equal to its increase in the second, we have $S_1|dz_1| = S_2|dz_2|$, or $-S_1dz_1 = S_2dz_2$, whence $dz_2 = -(S_1/S_2)dz_1$.

If we introduce the designation $u = z_1 - z_2$, then the velocity of the liquid flow through the orifice between the vessels can be found by the formula $v = \sigma\sqrt{2gu}$; it is

specified by Bernoulli's equation (see Problem 478), in which we must assume that the orifice is at the depth $u = z_1 - z_2$ under the free level of the liquid.

Therefore, the volume of the liquid, flowing during the time interval dt, which is equal to $-S_1dz_1$ in accordance with the above-said, is also equal to $v\omega dt = \sigma\omega\sqrt{2gu}\,dt$. Equating these expressions for one and the same volume, we arrive at the equation

$$-S_1dz_1 = \sigma\omega\sqrt{2gu}\,dt.$$

But $du = dz_1 - dz_2 = dz_1 + (S_1/S_2)dz_1$, i.e. $dz_1 = S_2du/(S_1 + S_2)$. Substituting the expression obtained for dz_1 into the preceding equation, we find the differential equation relating u and t:

$$-\frac{S_1S_2}{S_1+S_2}du = \sigma\omega\sqrt{2gu}\,dt, \quad \text{or} \quad dt = -\frac{S_1S_2}{(S_1 + S_2)\sigma\omega\sqrt{2g}}\cdot\frac{du}{\sqrt{u}}.$$

Integrating, we find

$$t = C - \frac{S_1S_2}{(S_1 + S_2)\sigma\omega\sqrt{2g}}\cdot 2\sqrt{u}.$$

At $t = 0$ we have $u = h_1 - h_2$, whence $C = \dfrac{S_1S_2\sqrt{2(h_1 - h_2)}}{(S_1 + S_2)\sigma\omega\sqrt{2g}}$. We can find the desired time T necessary for levelling off the heads of the liquid in the vessels by setting $u = 0$:

$$T = C = \frac{S_1S_2\sqrt{2(h_1 - h_2)}}{(S_1 + S_2)\sigma\omega\sqrt{2g}}.$$

Solve the following equations:

481. $\ln\cos y\,dx + x\tan y\,dy = 0$.

482. $\dfrac{yy'}{x} + e^y = 0;\ y(1) = 0$.

483. $3e^x\tan y\,dx + (1 + e^x)\sec^2 y\,dy = 0;\ y(0) = \pi/4$.

484. $e^{1+x^2}\tan y\,dx - \dfrac{e^{2x}}{x-1}dy = 0;\ y(1) = \pi/2$.

485. $(1 + e^{2x})y^2dy = e^xdx;\ y(0) = 0$.

486. $y' + \cos(x + 2y) = \cos(x - 2y);\ y(0) = \pi/4$.

487. $y' = 2^{x-y};\ y(-3) = -5$.

488. $y\ln^3 y + y'\sqrt{x + 1} = 0;\ y(-15/16) = e$.

489. $y/y' = \ln y;\ y(2) = 1$.

490. $x\sqrt{1 + y^2}\,dx + y\sqrt{1 + x^2}\,dy = 0$.

491. $\dfrac{x\,dy}{\sqrt{1 - y^2}} + \dfrac{y\,dx}{\sqrt{1-x^2}} = 0$.

492. $y' + \sin(x+y) = \sin(x-y)$.
493. $yy' = -2x\sec y$.
494. $y' = e^{x+y} + e^{x-y}$; $y(0) = 0$.
495. $y' = \sinh(x+y) + \sinh(x-y)$.
496. $y' = \sqrt{(a^2 - y^2)/(a^2 - x^2)}$.
497. $\dfrac{dx}{x(y-1)} + \dfrac{dy}{y(x+2)} = 0$; $y(1) = 1$.
498. $x(y^6 + 1)dx + y^2(x^4 + 1)dy = 0$; $y(0) = 1$.
499. $(\sqrt{xy} - \sqrt{x})dx + (\sqrt{xy} + \sqrt{y})dy = 0$.
500. $\sqrt{\dfrac{1 + \cos 2x}{1 + \sin y}} + y' = 0$; $y(\pi/4) = 0$.
501. $y' = \dfrac{\cos y - \sin y' - 1}{\cos x - \sin x + 1}$.
502. $\dfrac{4 + y^2}{\sqrt{x^2 + 4x + 13}} = \dfrac{3y + 2}{x + 1}y'$.
503. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$; $y(\pi/4) = \pi/4$.
504. $5e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$.
505. Find the curve for which the segment of the tangent contained between the coordinate axes is divided in two at the point of tangency.
506. The rate of devaluation of the machinery due to service wear is proportional, at each time moment, to its actual cost. The initial cost is A_0. Find the cost of the machinery after the lapse of t years.
507. A certain substance is transformed into another substance at the rate proportional to the quantity of the substance remaining untransformed. It is known that after the lapse of one hour the quantity of the first substance is 31.4 g and after three hours, 9.7 g. Determine: (1) the quantity of the substance at the beginning of the process; (2) what time it takes for the untransformed substance to reach 1% the initial quantity.
508. A cylindrical reservoir 6 m in length and of 4 m diameter occupies a horizontal position. What time is needed for the water to escape from the reservoir if an orifice with the radius of 1/12 m is at the level of the lowermost generating element of the cylinder?
509. A conical funnel with an orifice in area of ω cm^2 and an angle of 2α at the vertex of the cone is filled with water whose level is H cm above the orifice. Find the relationship between the variable level of the water h in the cone and the escape time t. Determine the time of the complete escape. Calculate it at $\omega = 0.1$ cm^2, $\alpha = 45°$, $H = 20$ cm.
510. Find the time during which all the water will escape from the conical funnel if it is known that 2 minutes are required for half the water to flow out.

4.1.3. Homogeneous differential equations. The equation of the form $P(x,y)dx + Q(x,y)dy = 0$ is called *homogeneous* if $P(x,y)$ and $Q(x,y)$ are homo-

geneous functions of the same degree. The function $f(x, y)$ is said to be ***homogeneous of degree m***, if

$$f(\lambda x, \lambda y) = \lambda^m f(x, y).$$

A homogeneous equation can be reduced to the form $y' = f(y/x)$. By means of the substitution $y = tx$, a homogeneous equation can be reduced to the equation with variables separable with respect to the new unknown function t.

511. Find the general solution of the equation

$$(x^2 + 2xy)dx + xy\,dy = 0.$$

Solution. Here $P(x, y) = x^2 + 2xy$, $Q(x, y) = xy$. Both functions are homogeneous, of the second degree. We introduce the substitution $y = tx$, whence we have $dy = x\,dt + t\,dx$. Then the equation assumes the form

$$(x^2 + 2x^2 t)dx + tx^2(x\,dt + t\,dx) = 0, \quad \text{or} \quad (x^2 + 2x^2 t + t^2x^2)dx + tx^3 dt = 0.$$

Separating the variables and integrating, we obtain

$$\frac{dx}{x} + \frac{t\,dt}{(t + 1)^2} = 0; \qquad \int \frac{dx}{x} + \int \frac{t\,dt}{(t + 1)^2} = C.$$

We transform the second integral:

$$\ln|x| + \int \frac{t + 1 - 1}{(t + 1)^2}dt = C, \quad \text{or} \quad \ln|x| + \ln|t + 1| + \frac{1}{t + 1} = C.$$

Returning to the old unknown function $y(t = y/x)$, we get the final result:

$$\ln|x + y| + \frac{x}{x + y} = C.$$

512. Find the particular solution of the equation $y' = \dfrac{y}{x} + \sin\dfrac{y}{x}$, subject to the initial condition $y(1) = \pi/2$.

Solution. We perform the substitution $y/x = t$, whence we have $y = tx$, $dy = x\,dt + t\,dx$. As a result, we get

$$x\,dt + t\,dx = (t + \sin t)dx; \quad x\,dt = \sin t\,dx; \quad \frac{dt}{\sin t} = \frac{dx}{x}.$$

By means of integration, we obtain

$$\ln|\tan(t/2)| = \ln|x| + \ln C, \quad \text{whence } t/2 = \arctan(Cx).$$

Performing the reverse substitution $t = y/x$, we find the general solution of the original equation $y = 2x \arctan(Cx)$. Using the given initial condition, we get $\pi/2 = 2 \arctan C$, whence we have $C = 1$. Thus we see that the sought-for particular solution has the form $y = 2x \arctan x$.

513. Find the curve passing through the point $A(0; 1)$, for which the triangle formed by the Oy axis, the tangent line to the curve at its arbitrary point and the radius vector of the point of tangency is isosceles (its base being the segment of the tangent from the point of tangency to the Oy axis).

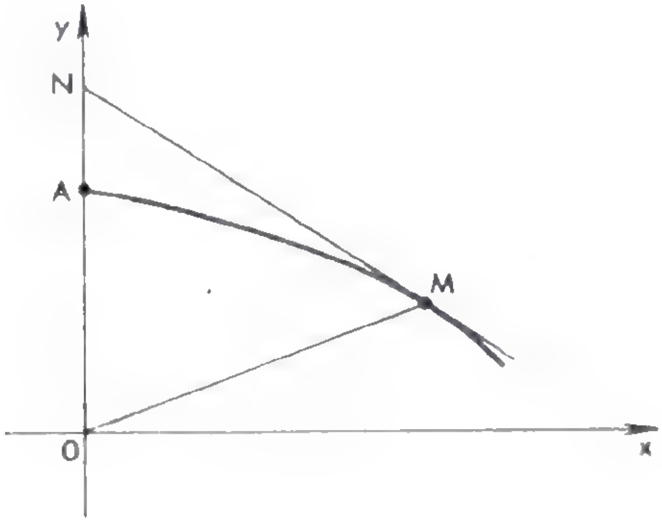


Fig. 33

Solution. Suppose $y = f(x)$ is the desired equation of the curve. Let us draw a tangent MN to the curve at an arbitrary point $M(x; y)$ till the intersection with the Oy axis at a point N (Fig. 33). By the hypothesis, the equality $|ON| = |OM|$ must be satisfied. But $|OM| = \sqrt{x^2 + y^2}$, and $|ON|$ will be found from the equation of the tangent line $Y - y = y'(X - x)$ setting $X = 0$, i.e. $Y = |ON| = y - xy'$.

Thus we arrive at the homogeneous equation

$$\sqrt{x^2 + y^2} = y - xy'.$$

Setting $y = tx$, we obtain, after the substitution and the separation of the variables,

$$\frac{dt}{\sqrt{1 + t^2}} = -\frac{dx}{x}, \quad \text{or} \quad \ln(t + \sqrt{1 + t^2}) = \ln C - \ln x,$$

whence

$$x^2 = C(C - 2y)$$

(a family of parabolas whose axis is the Oy axis).

Substituting the coordinates of the point A into the general solution obtained, we get $0 = C(C - 2)$. Only the second of the two values $C = 0$ and $C = 2$ is valid, since at $C = 0$ the parabola degenerates into the Oy axis. Thus, the desired curve is the parabola

$$x^2 = 4(1 - y), \quad \text{or} \quad y = 1 - x^2/4.$$

514. Find the shape of the mirror gathering all the parallel rays into a single point.

Solution. Evidently, the mirror must have the shape of a surface of revolution, whose axis is parallel to the direction of the incident rays. Let us assume Ox as such an axis and find the equation of the curve $y = f(x)$, whose rotation generates the desired surface.

We put the origin at the point which gathers the reflected rays. Let us designate the incident ray as KM and the reflected ray as MO (Fig. 34). We draw a tangent

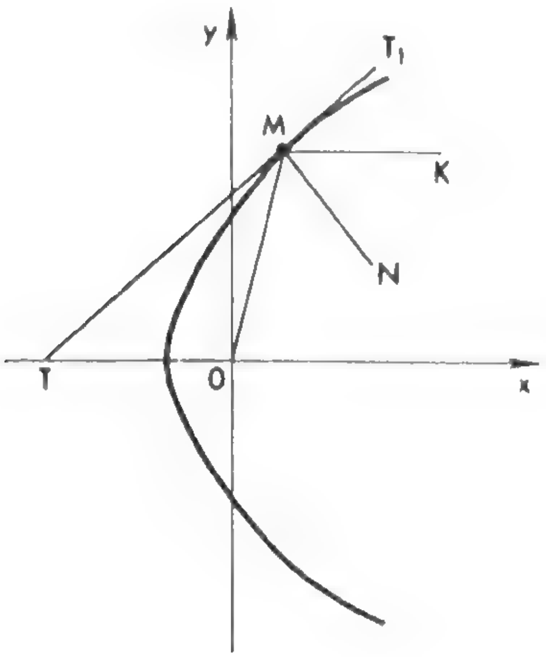


Fig. 34

TT_1 and a normal MN to the desired curve at the point M. Then the triangle OMT is isosceles with vertex at the point O (since $\widehat{OMT} = \widehat{KMT_1} = \widehat{OTM} = \alpha$). Consequently, $|OM| = |OT|$; but $|OM| = \sqrt{x^2 + y^2}$, and $|OT|$ will be found from the equation of the tangent line $Y - y = y'(X - x)$ by setting $Y = 0$; we have $X = x - \frac{y}{y'}$, whence $|OT| = |X| = -X = -x + \frac{y}{y'}$.

Thus we obtain a differential equation

$$\sqrt{x^2 + y^2} = -x + \frac{y}{y'}, \quad \text{or} \quad (x + \sqrt{x^2 + y^2})y' = y, \quad \text{i.e.}$$

$$(x + \sqrt{x^2 + y^2})dy - y\,dx = 0.$$

This differential equation is homogeneous. To integrate it, it is expedient to introduce the substitution $x = ty$, assuming y to be the argument and x (and t) to be the unknown functions of that argument. Then we get

$$(\sqrt{t^2y^2 + y^2} + ty)dy - y(t\,dy + y\,dt) = 0, \quad \text{or} \quad \sqrt{t^2 + 1}\,dy - y\,dt = 0.$$

We separate the variables and integrate:

$$\frac{dy}{y} - \frac{dt}{\sqrt{1 + t^2}} = 0; \quad \ln y = \ln(t + \sqrt{1 + t^2}) + \ln C.$$

Hence it follows that $y = C(t + \sqrt{1 + t^2})$ or, returning to the original variables x and y, we have

$$x + \sqrt{x^2 + y^2} = \frac{y^2}{C}.$$

After simplification, we obtain the final solution in the form

$$y^2 = 2C\left(x + \frac{C}{2}\right).$$

The desired curve is a parabola and the mirror is shaped as a paraboloid of revolution.

515. Find the orthogonal trajectories of the family of the parabolas $x = ay^2$ (a being the parameter of the family).

Solution. The *orthogonal trajectories* of the given family of curves are curves of another family each of which intersects each of the curves of the first family at right angles.

If the equation of the given family is $F(x, y, a) = 0$, then, to find the orthogonal trajectories, it is necessary:

(1) to set up a differential equation of the given family $f(x, y, y') = 0$;

(2) proceeding from the condition of orthogonality ($y'_{\mathrm{I}} y'_{\mathrm{II}} = -1$), to replace y' by $-1/y'$ in this differential equation;

(3) to integrate the equation obtained $f(x, y, -1/y') = 0$.

To solve the problem posed, we differentiate the equation of the given family of parabolas: $1 = 2ayy'$. Eliminating the parameter a from the equations $x = ay^2$ and $1 = 2ayy'$, we find the differential equation of the given family of the parabolas: $2xy' = y$. Replacing y' by $-1/y'$, we get the differential equation of the family of the orthogonal trajectories:

$$2x + yy' = 0, \quad \text{or} \quad 2x\,dx + y\,dy = 0.$$

Integrating the equation obtained, we find the equation of the family of the orthogonal trajectories:

$$x^2 + \frac{1}{2}y^2 = C, \quad \text{or} \quad \frac{x^2}{C} + \frac{y^2}{2C} = 1.$$

Thus, the orthogonal trajectories of the given family of parabolas are similar ellipses whose semi-major axis (vertical) is twice the semi-minor axis.

Solve the following equations:

516. $xy'\sin(y/x) + x = y\sin(y/x)$.

517. $xy + y^2 = (2x^2 + xy)y'$.

518. $xy'\ln(y/x) = x + y\ln(y/x)$.

519. $xyy' = y^2 + 2x^2$.

520. $xy' - y = x\tan(y/x)$; $y(1) = \pi/2$.

521. $y' = (y/x) + \cos(y/x)$.

522. $y' = 4 + y/x + (y/x)^2$; $y(1) = 2$.

523. $(x^2 + y^2)dx - xy\,dy = 0$.

524. $y' = (x + y)/(x - y)$.

525. $xy' = xe^{y/x} + y$; $y(1) = 0$.

526. $xy' - y = \dfrac{x}{\arctan(y/x)}$.

527. $(x^4 + 6x^2y^2 + y^4)dx + 4xy(x^2 + y^2)dy = 0$; $y(1) = 0$.

528. $xy' = 2(y - \sqrt{xy})$.

529. $3y\sin(3x/y)dx + [y - 3x\sin(3x/y)]dy = 0$.

530 Find the curve for which the product of the abscissa of any point belonging

to the curve by the intercept cut by the normal on the Ox axis is equal to double the square of the distance between that point and the origin.

531. Find the orthogonal trajectories of the family of circles $(x-1)^2 + (y-1)^2 = R^2$.

4.1.4. Differential equations reducible to homogeneous equations. Equations of the form

$$y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right)$$

for $a_1b_2 - a_2b_1 \neq 0$ can be reduced to homogeneous equations by means of the substitution $x = u + \alpha, y = v + \beta$, where $(\alpha; \beta)$ is the point of intersection of the straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

Now if $a_1b_2 - a_2b_1 = 0$, then the substitution $a_1x + b_1y = t$ allows separation of the variables.

532. Find the general solution of the equation

$$(2x + y + 1)dx + (x + 2y - 1)dy = 0.$$

Solution. The equation is of the first type since $y' = -\dfrac{2x + y + 1}{x + 2y - 1}$ and $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \neq 0$. We find the intersection point of the straight lines $2x + y + 1 = 0$ and $x + 2y - 1 = 0$; we have $x = \alpha = -1; y = \beta = 1$.

By performing a change of variables in the original equation, setting $x = u + \alpha = u - 1, y = v + \beta = v + 1; dx = du, dy = dv$, we reduce the equation to the form

$$(2u + v)du + (u + 2v)dv = 0.$$

In the homogeneous equation obtained, we put $v = ut$, whence $dv = u\,dt + t\,du$; we arrive at the equation with variables separable:

$$2(t^2 + t + 1)u\,du + u^2(1 + 2t)dt = 0,$$

whose general integral is $u\sqrt{t^2 + t + 1} = C$, or (after the substitution $t = v/u$ and squaring),

$$u^2 + uv + v^2 = C^2.$$

Returning to the variables x and $y (u = x + 1, v = y - 1)$, we get, after some elementary transformations, the general solution of the original equation:

$$x^2 + y^2 + xy + x - y = C_1$$

(here we have put $C_1 = C^2 - 1$).

533. Find the general solution of the equation

$$(x + y + 2)dx + (2x + 2y - 1)dy = 0.$$

Solution. The equation is of the second type since $\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$. Therefore, we put

$y + x = t$, $dy = dt - dx$. The equation assumes the form

$$(t + 2)dx + (2t - 1)(dt - dx) = 0, \quad \text{or} \quad (3 - t)dx + (2t - 1)dt = 0.$$

Separating the variables and integrating, we get

$$\int \frac{2t - 1}{3 - t} dt + \int dx = C, \quad \text{or} \quad -2t - 5\ln|t - 3| + x = -C.$$

Returning to the old variables ($t = x + y$), we get the final result:

$$x + 2y + 5\ln|x + y - 3| = C.$$

Solve the following equations:

534. $2(x + y)dy + (3x + 3y - 1)dx = 0$; $y(0) = 2$.
535. $(x - 2y + 3)dy + (2x + y - 1)dx = 0$.
536. $(x - y + 4)dy + (x + y - 2)dx = 0$.
537. Find the integral curve of the differential equation $y' = (x + y - 2)/(y - x - 4)$ passing through the point $M(1; 1)$.

4.1.5. **Differential equations in total differentials.** The differential equation

$$P(x, y)\, dx + Q(x, y)\, dy = 0,$$

where $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ is called an *equation in total differentials,* that is, the left-hand side of such an equation is the total differential of a certain function $u(x, y)$. If we rewrite this equation in the form $du = 0$, then its general solution will be determined by the equality $u = C$. The function $u(x, y)$ can be found by the formula

$$u = \int_{x_0}^{x} P(x, y)\, dx + \int_{y_0}^{y} Q(x_0, y)\, dy.$$

In the last formula, the lower limits of the integrals, x_0 and y_0, are arbitrary; their choice is restricted by the following single condition: the integrals on the right-hand side of the formula must be meaningful (that is, they must not be divergent improper integrals of the 2nd kind). If the condition $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ is not satisfied, then, in certain cases, the equation in question can be reduced to the indicated type by multiplying it by the so-called *integrating factor,* which in the general case is a function of x and y: $\mu(x, y)$. If a given equation possesses an integrating factor dependent only on x, it can be found from the formula

$$\mu = e^{\int \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)/Q dx},$$

where the relation $\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\Big/Q$ must only be a function of x. By analogy, the integrating factor dependent only on y can be found by the formula

$$\mu = e^{-\int\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)/P dy},$$

where $\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\Big/P$ must only be a function of y (the absence of y in the first case and of x in the second case, in these relations, is the test for the existence of the integrating factor of the kind being considered).

538. Find the general solution of the equation

$$(e^x + y + \sin y)dx + (e^y + x + x\cos y)dy = 0.$$

Solution. Here $P(x, y) = e^x + y + \sin y$, $Q(x, y) = e^y + x + x\cos y$, $\frac{\partial P}{\partial y} = 1 + \cos y$, $\frac{\partial Q}{\partial x} = 1 + \cos y$. Consequently, the left-hand side of the equation is the total differential of a certain function $u(x, y)$, i.e.

$$\frac{\partial u}{\partial x} = e^x + y + \sin y, \quad \frac{\partial u}{\partial y} = e^y + x + x\cos y.$$

Let us integrate $\frac{\partial u}{\partial x}$ for x:

$$u = \int(e^x + y + \sin y)dx + C(y) = e^x + xy + x\sin y + C(y).$$

We find the function $C(y)$ differentiating the last expression with respect to y:

$$\frac{\partial u}{\partial y} = x + x\cos y + C'(y).$$

We obtain the equation

$$x + x\cos y + C'(y) = x + x\cos y + e^y,$$

whence we find $C'(y) = e^y$, i.e. $C(y) = e^y$. Thus, the general integral of the equation has the form

$$e^x + xy + x\sin y + e^y = C.$$

539. Find the general solution of the equation

$$(x + y - 1)\,dx + (e^y + x)\,dy = 0.$$

Solution. Here $P(x, y) = x + y - 1$, $Q(x, y) = e^y + x$, $\dfrac{\partial P}{\partial y} = 1$, $\dfrac{\partial Q}{\partial x} = 1$. Thus, the condition of the total differential is fulfilled, that is, the given equation is an equation in total differentials.

Let us find the general solution by the formula

$$\int_{x_0}^{x} P(x, y)\, dx + \int_{y_0}^{y} Q(x_0, y) dy = C.$$

Taking $x_0 = 0$, $y_0 = 0$, we get

$$\int_0^x (x + y - 1) dx + \int_0^y e^y dy = C_1, \text{ or } \left[\frac{1}{2}x^2 + xy - x\right]_0^x + e^y \Big|_0^y = C_1.$$

Substituting the limits, we find

$$\frac{1}{2}x^2 + xy - x + e^y - 1 = C_1, \quad \text{or} \quad e^y + \frac{1}{2}x^2 + xy - x = C,$$

where $C = C_1 + 1$.

540. Find the general solution of the equation

$$(x \cos y - y \sin y) dy + (x \sin y + y \cos y) dx = 0.$$

Solution. We have

$$P(x, y) = x \sin y + y \cos y, \quad Q(x, y) = x \cos y - y \sin y,$$

$$\frac{\partial P}{\partial y} = x \cos y + \cos y - y \sin y, \quad \frac{\partial Q}{\partial x} = \cos y,$$

$$\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\Big/ Q = \frac{x \cos y - y \sin y}{x \cos y - y \sin y} = 1.$$

Therefore, the given equation possesses an integrating factor dependent only on x. Let us find this integrating factor:

$$\mu = e^{\int \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)/Q dx} = e^{\int dx} = e^x.$$

Multiplying the original equation by e^x, we obtain the equation

$$e^x(x \cos y - y \sin y) dy + e^x(x \sin y + y \cos y) dx = 0,$$

which, as can be easily proved, is an equation in total differentials; indeed, we have $P_1(x, y) = e^x(x \sin y + y \cos y)$, $Q_1(x, y) = e^x(x \cos y - y \sin y)$. Hence we have

$$\frac{\partial P_1}{\partial y} = \frac{\partial}{\partial y}[e^x(x \sin y + y \cos y)] = e^x(x \cos y + \cos y - y \sin y);$$
$$\frac{\partial Q_1}{\partial x} = \frac{\partial}{\partial x}[e^x(x \cos y - y \sin y)] = e^x[x \cos y - y \sin y + \cos y].$$

These derivatives are equal and, consequently, the left-hand side of the equation obtained is equal to $du(x, y)$.

Thus we have

$$\frac{\partial u}{\partial y} = e^x(x \cos y - y \sin y), \quad \frac{\partial u}{\partial x} = e^x(x \sin y + y \cos y).$$

Integrating the first equality with respect to y, we get

$$u = \int e^x(x \cos y - y \sin y)dy + C(x) = xe^x \sin y + e^x y \cos y - e^x \sin y + C(x).$$

Let us now find the derivative with respect to x of the function obtained:

$$\frac{\partial u}{\partial x} = e^x \sin y + xe^x \sin y - e^x \sin y + e^x y \cos y + C'(x)$$
$$= e^x(x \sin y + y \cos y) + C'(x).$$

Comparing the value of $\frac{\partial u}{\partial x}$ we have found with $P(x, y)$, we get $C'(x) = 0$, i.e. $C(x) = 0$. Consequently, the general solution of the original equation has the form

$$u(x, y) = xe^x \cdot \sin y + e^x y \cdot \cos y - e^x \cdot \sin y = C, \text{ or}$$
$$e^x(x \sin y + y \cos y - \sin y) = C.$$

Solve the following equations:

541. $(x + \sin y)dx + (x \cos y + \sin y)dy = 0.$

542. $(y + e^x \sin y)dx + (x + e^x \cos y)dy = 0.$

543. $(xy + \sin y)dx + (0.5x^2 + x \cos y)dy = 0.$

544. $(x^2 + y^2 + y)dx + (2xy + x + e^y)dy = 0; y(0) = 0.$

545. $(2xye^{x^2} + \ln y)dx + \left(e^{x^2} + \frac{x}{y}\right)dy = 0; y(0) = 1.$

546. $[\sin y + (1 - y)\cos x]dx + [(1 + x) \cos y - \sin x]dy = 0.$

547. $(y + x \ln y)dx + \left(\frac{x^2}{2y} + x + 1\right)dy = 0.$

548. $(x^2 + \sin y)dx + (1 + x\cos y)dy = 0.$
549. $ye^x dx + (y + e^x)dy = 0.$
550. $(e^x \sin y + x)dx + (e^x \cos y + y)dy = 0.$

551. $(\ln y - 5y^2 \sin 5x)dx + \left(\frac{x}{y} + 2y\cos 5x\right)dy = 0;\ y(0) = e.$

552. $(\arcsin x + 2xy)dx + (x^2 + 1 + \arctan y)dy = 0.$
553. $(3x^2y + \sin x)dx + (x^3 - \cos y)dy = 0.$
554. $(e^{x+y} + 3x^2)dx + (e^{x+y} + 4y^3)dy = 0;\ y(0) = 0.$
555. $(\tan y - y\operatorname{cosec}^2 x)dx + (\cot x + x\sec^2 y)dy = 0$
556. $\left(\frac{y}{x^2+y^2} - y\right)dx + \left(e^y - x - \frac{x}{x^2+y^2}\right)dy = 0.$

Integrate the following equations possessing an integrating factor dependent only on x or only on y:

557. $y\,dx - x\,dy + \ln x\,dx = 0\ (\mu = \varphi(x)).$
558. $(x^2\cos x - y)dx + x\,dy = 0\ (\mu = \varphi(x)).$
559. $y\,dx - (x + y^2)dy = 0\ (\mu = \varphi(y)).$
560. $y\sqrt{1-y^2}dx + (x\sqrt{1-y^2} + y)dy = 0\ (\mu = \varphi(y)).$
561. Prove that the equation $P(x, y)dx + Q(x, y)dy = 0$, which is simultaneously a homogeneous equation and an equation in total differentials, possesses the general integral $Px + Qy = C$.

Hint. Make use of Euler's theorem on homogeneous functions, in accordance with which

$$x \cdot \frac{\partial P}{\partial x} + y \cdot \frac{\partial P}{\partial y} = tP(x, y),$$

where t is the degree of the homogeneous functions $P(x, y)$ and $Q(x, y)$.

4.1.6. First-order linear differential equations. Bernoulli's equations. An equation of the form

$$y' + P(x)y = Q(x)$$

is said to be *linear* (y and y' appear in the equation in the first order without being multiplied by each other). If $Q(x) \not\equiv 0$, then the equation is linear *nonhomogeneous,* and if $Q(x) \equiv 0$, it is linear *homogeneous.*

The general solution of the homogeneous linear equation $y' + P(x)y = 0$ can be easily obtained by separating the variables:

$$\frac{dy}{y} = -P(x)dx;\ \int\frac{dy}{y} = -\int P(x)dx;\ \ln y = -\int P(x)dx + \ln C,$$

or, finally,

$$y = Ce^{-\int P(x)dx},$$

where C is an arbitrary constant.

The general solution of a nonhomogeneous linear equation can be found on the basis of the general solution of the corresponding homogeneous equation using *Lagrange's method* of varying the arbitrary constant, that is, putting $y = C(x)e^{-\int P(x)dx}$, where $C(x)$ is a certain differentiable function of x which is to be determined.

To find $C(x)$, y must be substituted into the original equation, which operation will lead to the equation

$$C'(x)e^{-\int P(x)dx} = Q(x),$$

which yields

$$C(x) = \int Q(x)e^{\int P(x)dx}dx + C,$$

where C is an arbitrary constant. Then the desired general solution of the nonhomogeneous equation has the form

$$y = e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx + C \right).$$

First-order linear equations can also be integrated by *Bernoulli'a method,* which consists in the following. By means of the substitution $y = uv$, where u and v are two unknown functions, the original equation is transformed as follows:

$$u'v + uv' + P(x)uv = Q(x), \quad \text{or} \quad u\,[v' + P(x)v] + vu' = Q(x).$$

Using the fact that one of the unknown functions (say, v) may be chosen quite arbitrarily (because only the product uv must satisfy the original equation), we can take as v *any particular solution* of the equation $v' + P(x)v = 0$ (for instance, $v = e^{-\int P(x)dx}$, which thus turns into zero the coefficient in u in the last equation.

Then the previous equation is reduced to the equation

$$vu' = Q(x), \quad \text{or} \quad u' = \frac{Q(x)}{v} = Q(x)e^{\int P(x)dx},$$

whence it follows that

$$u = C + \int Q(x)e^{\int P(x)dx}dx.$$

The general solution of the original equation can be found by multiplying u by v:

$$y = e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx + C \right).$$

A (nonlinear) equation of the form

$$y' + P(x)y = Q(x)y^m,$$

where $m \neq 0$, $m \neq 1$, is known as *Bernoulli's equation*. It can be reduced to a linear equation by changing the unknown function by means of the substitution $z = y^{1-m}$, as a result of which the original equation is transformed as follows:

$$\frac{1}{1-m} z' + P(x)z = Q(x).$$

When integrating concrete Bernoulli's equations, it is not necessary to begin with transforming them into linear equations. You can directly apply either Bernoulli's method or the method of varying an arbitrary constant.

562. Integrate the equation $y' \cos^2 x + y = \tan x$ subject to the initial condition $y(0) = 0$.

Solution. We integrate the corresponding homogeneous equation $y' \cos^2 x + y = 0$; separating the variables, we get

$$\frac{dy}{y} + \frac{dx}{\cos^2 x} = 0, \quad \ln y + \tan x = \ln C, \quad y = Ce^{-\tan x}.$$

We seek the solution of the original nonhomogeneous equation in the form $y = C(x)e^{-\tan x}$, where $C(x)$ is an unknown function. Substituting $y = C(x)e^{-\tan x}$ and $y' = C'(x)e^{-\tan x} - C(x)e^{-\tan x}\sec^2 x$ into the original equation, we arrive at the equation

$$\cos^2 x\, C'(x)e^{-\tan x} - C(x)e^{-\tan x}\sec^2 x \cos^2 x + C(x)e^{-\tan x} = \tan x,$$

or

$$C'(x) \cos^2 x\, e^{-\tan x} = \tan x,$$

whence we have

$$C(x) = \int \frac{e^{\tan x}\tan x}{\cos^2 x}\, dx = e^{\tan x}(\tan x - 1) + C.$$

Thus we obtain the general solution of the given equation:

$$y = \tan x - 1 + Ce^{-\tan x}.$$

Using the initial condition $y(0) = 0$, we get $0 = -1 + C$, whence $C = 1$. Consequently, the desired particular solution has the form

$$y = \tan x - 1 + e^{-\tan x}.$$

563. Integrate the equation $y' - y \tanh x = \cosh^2 x$.

Solution. This is a linear equation. We shall solve it by Bernoulli's method. Setting $y = uv$, we get

$$u'v + v'u - uv \tanh x = \cosh^2 x, \quad \text{or} \quad u(v' - v \tanh x) + u'v = \cosh^2 x.$$

We set $v' - v \tanh x = 0$, whence we get $\dfrac{dv}{v} = \tanh x\, dx$; integration leads to $\ln v = \ln \cosh x$ or $v = \cosh x$ (we do not introduce an integration constant since it is sufficient to find some particular solution of this auxiliary equation).

To define u, we have the equation $u'v = \cosh^2 x$ or $u' \cosh x = \cosh^2 x$, from which we find $u = \int \cosh x\, dx = \sinh x + C$. Multiplying u by v, we obtain the general solution

$$y = \cosh x(\sinh x + C).$$

564. Integrate the equation

$$y' + \frac{xy}{1 - x^2} = \arcsin x + x.$$

Solution. We integrate the corresponding homogeneous equation:

$$y' + \frac{xy}{1 - x^2} = 0; \quad \frac{dy}{y} = -\frac{x\,dx}{1 - x^2}; \quad \ln y = \frac{1}{2} \ln(1 - x^2) + \ln C,$$

i.e. $y = C\sqrt{1 - x^2}$. Setting now $y = C(x)\sqrt{1 - x^2}$, we get

$$y' = C'(x)\sqrt{1 - x^2} - \frac{xC(x)}{\sqrt{1 - x^2}}.$$

After substitution into the original nonhomogeneous equation, we get

$$C'(x)\sqrt{1 - x^2} - \frac{xC(x)}{\sqrt{1 - x^2}} + \frac{x}{1 - x^2} C(x)\sqrt{1 - x^2} = \arcsin x + x,$$

i.e.

$$C'(x) = \frac{\arcsin x}{\sqrt{1 - x^2}} + \frac{x}{\sqrt{1 - x^2}}.$$

Integration gives

$$C(x) = \int \left[\frac{\arcsin x}{\sqrt{1 - x^2}} + \frac{x}{\sqrt{1 - x^2}}\right] dx = \frac{1}{2} (\arcsin x)^2 - \sqrt{1 - x^2} + C.$$

Thus, the general solution of the given equation has the form

$$y = \sqrt{1 - x^2}\left[\frac{1}{2} (\arcsin x)^2 - \sqrt{1 - x^2} + C\right].$$

565. Solve the equation $y' + \frac{y}{x} = x^2y^4$.

Solution. This is Bernoulli's equation. We integrate it using the method of variation of an arbitrary constant. For that purpose, we first integrate the corresponding homogeneous linear equation $y' + \frac{y}{x} = 0$, whose solution is $y = \frac{C}{x}$.

We seek the solution of the original Bernoulli equation, setting $y = \frac{C(x)}{x}$, $y' = \frac{C'(x)}{x} - \frac{C(x)}{x^2}$. The substitution of y and y' into the original equation leads to

$$\frac{C'(x)}{x} - \frac{C(x)}{x^2} + \frac{C(x)}{x^2} = x^2\left[\frac{C(x)}{x}\right]^4, \quad \text{or} \quad \frac{C'(x)}{x} = \frac{[C(x)]^4}{x^2}.$$

We integrate the equation obtained:

$$\frac{dC(x)}{[C(x)]^4} = \frac{dx}{x}; \; -\frac{1}{3[C(x)]^3} = \ln x - \ln C; \; C(x) = \sqrt[3]{\frac{1}{3\ln(C/x)}}.$$

Thus, the general solution of the original equation is

$$y = \frac{C(x)}{x} = \frac{1}{x\sqrt[3]{3\ln(C/x)}}.$$

566. Integrate the equation

$$y' - \frac{2xy}{1+x^2} = 4\frac{\sqrt{y}}{\sqrt{1+x^2}}\arctan x.$$

Solution. This is Bernoulli's equation. We integrate it by Bernoulli's method for which purpose we put $y = uv$. We substitute $y = uv$, $y' = u'v + uv'$ into the original equation and collect the terms containing u in the first degree:

$$u'v + u\left(v' - \frac{2xv}{1+x^2}\right) = 4\frac{\sqrt{uv}}{\sqrt{1+x^2}}\arctan x.$$

We take as v some particular solution of the equation:

$$v' - \frac{2xv}{1+x^2} = 0.$$

Separating the variables appearing in it, we find

$$\frac{dv}{v} = \frac{2x\,dx}{1 + x^2};\ \ln v = \ln(1 + x^2);\ v = 1 + x^2$$

(we do not introduce an integration constant).

To find u, we use the equation

$$u'v = 4\frac{\sqrt{uv}}{\sqrt{1 + x^2}}\arctan x,$$

or (since $v = 1 + x^2$)

$$u' = \frac{4\sqrt{u}\arctan x}{1 + x^2}.$$

We separate the variables and integrate:

$$\frac{du}{2\sqrt{u}} = \frac{2\arctan x}{1 + x^2}dx;\quad \sqrt{u} = \arctan^2 x + C.$$

Thus, $u + (\arctan^2 x + C)^2$ and $y = uv = (1 + x^2)(\arctan^2 x + C)^2$ is the general solution of the original equation.

567. Integrate the equation $y = xy' + y'\ln y$.

Solution. This equation is easy to integrate by interchanging the roles of x and y: taking y as the argument and x as the unknown function. For that purpose it is only necessary to put $y'_x = 1/x'_y$ (using the formula for differentiating an inverse function). Then the given equation is transformed as follows:

$$yx'_y = x + \ln y.$$

This is a linear equation with respect to x. Integrating the corresponding homogeneous equation $yx' = x$, we get

$$\frac{dx}{x} = \frac{dy}{y};\quad x = Cy.$$

We seek the solution of the original nonhomogeneous equation, setting $x = C(y)y$, whence it follows that $x'_y = C'(y)y + C(y)$. Substitution into the equation gives

$$C'(y)y^2 + C(y)y = C(y)y + \ln y,\ \text{whence}\ C'(y) = \frac{\ln y}{y^2},\ C(y) = C - \frac{1 + \ln y}{y}.$$

Multiplying $C(y)$ by y, we find the solution of the original equation:

$$x = Cy - 1 - \ln y.$$

568. Integrate the equation $(x^2\ln y - x)y' = y$.

Solution. This equation can be integrated by means of the same transformation as was used in the preceding problem. Taking y to be the argument and x to be the unknown function, we reduce the equation to the form

$$x^2\ln y - x = yx',\quad \text{or}\quad yx' + x = x^2\ln y.$$

This is Bernoulli's equation with respect to x. Integrating the corresponding homogeneous linear equation $yx' + x = 0$, we find $x = C/y$.

We set $x = \dfrac{C(y)}{y}$ in the original equation, whence we get $x' = \dfrac{C'(y)}{y} - \dfrac{C(y)}{y^2}$; we arrive at the following equation which can be used to determine $C(y)$:

$$C'(y) - \frac{C(y)}{y} + \frac{C(y)}{y} = \frac{[C(y)]^2}{y}\ln y, \quad \text{or} \quad C'(y) = \frac{[C(y)]^2 \ln y}{y^2}.$$

We separate the variable and perform integration:

$$\frac{dC(y)}{[C(y)]^2} = \frac{\ln y}{y^2}\,dy; \quad -\frac{1}{C(y)} = C - \frac{\ln y + 1}{y}; \quad C(y) = \frac{y}{\ln y + 1 - Cy}.$$

Multiplying $C(y)$ by $1/y$, we find the general solution of the original equation

$$x = \frac{1}{\ln y + 1 - Cy}.$$

Solve the following equations:

569. $xy' - y = x^2\cos x$.
570. $y' + 2xy = xe^{-x^2}$.
571. $y'\cos x + y = 1 - \sin x$.
572. $y' + \dfrac{n}{x}y = \dfrac{a}{x^n}$; $y(1) = 0$.
573. $(1 + x^2)y' + y = \arctan x$.
574. $y'\sqrt{1 - x^2} + y = \arcsin x$; $y(0) = 0$.
575. $y' - \dfrac{y}{\sin x} = \cos^2 x \ln\tan\dfrac{x}{2}$.
576. $y' - \dfrac{y}{x\ln x} = x\ln x$; $y(e) = e^2/2$.
577. $y'\sin x - y\cos x = 1$; $y(\pi/2) = 0$.
578. $y'(x + y^2) = y$.

Hint. Assume x to be the unknown function.

579. $y' + 3y\tan 3x = \sin 6x$; $y(0) = 1/3$.
580. $(2xy + 3)dy - y^2dx = 0$.

Hint. Assume x to be the unknown function.

581. $(y^4 + 2x)y' = y$.

Hint. Assume x to be the unknown function.

582. $y' + \dfrac{2y}{x} = 3x^2y^{4/3}$.

583. $y' - \dfrac{y}{x-1} = \dfrac{y^2}{x-1}$.

584. $y' + \dfrac{2y}{x} = \dfrac{2\sqrt{y}}{\cos^2 x}$.

585. $4xy' + 3y = -e^x x^4 y^5$.

586. $y' + y = e^{x/2}\sqrt{y}$; $y(0) = 9/4$.

587. $y' + \dfrac{3x^2y}{x^3+1} = y^2(x^3+1)\sin x$; $y(0) = 1$.

588. $y\,dx + (x + x^2y^2)dy = 0$.

Hint. Assume x to be the unknown function.

589. $y' - 2y\tan x + y^2\sin^2 x = 0$.

590. $(y^2 + 2y + x^2)y' + 2x = 0$; $y(1) = 0$.

Hint. Assume x to be the unknown function.

4.1.7. Equations of the form $x = \varphi(y')$ and $y = \varphi(y')$. We can easily integrate these equations in parametric form if we put $y' = p$ and assume p to be the parameter in terms of which both x and y should be expressed. Indeed, setting $y' = p$ in the equation $x = \varphi(y')$, we directly obtain the expression for x in terms of the parameter p: $x = \varphi(p)$. Differentiating, we find $dx = \varphi'(p)dp$, and since $dy = y'dx = p\,dx$, it follows that $dy = p\varphi'(p)dp$. Integrating it we can find y: $y = \int p\varphi'(p)dp + C$.

Thus, we can write the solution of the equation $x = \varphi(y')$ in parametric form:

$$\begin{cases} x = \varphi(p), \\ y = \int p\varphi'(p)dp + C. \end{cases}$$

By analogy, setting $y' = p$ in the equation $y = \varphi(y')$, we find $y = \varphi(p)$. Differentiating y, we get $dy = \varphi'(p)dp$. But $dy = p\,dx$ as before. Thus, $p\,dx = \varphi'(p)dp$, whence $dx = \dfrac{\varphi'(p)dp}{p}$, and x can be found by means of integration: $x = \displaystyle\int \frac{\varphi'(p)dp}{p} + C$. The general solution of the equation $y = \varphi(y')$ has the form

$$\begin{cases} x = \displaystyle\int \frac{\varphi'(p)dp}{p} + C, \\ y = \varphi(p). \end{cases}$$

When it is possible, we eliminate the parameter p in both cases and find the general solution of the equation.

591. Integrate the equation $x = y'\sin y' + \cos y'$.
Solution. We put $y' = p$. Then $x = p\sin p + \cos p$. We differentiate this equality:

$$dx = (\sin p + p\cos p - \sin p)dp = p\cos p\,dp,$$

and substitute the expression for dx into the equality $dy = p\,dx$:

$$dy = p^2\cos p\,dp,$$

i.e.

$$y = \int p^2\cos p\,dp = (p^2 - 2)\sin p + 2p\cos p + C.$$

Thus, the general solution in parametric form is

$$\begin{cases} x = p\sin p + \cos p, \\ y = (p^2 - 2)\sin p + 2p\cos p + C. \end{cases}$$

592. Integrate the equation $y' = \arctan(y/y'^2)$.
Solution. First we find $y = y'^2\tan y'$. Putting $y' = p$, we get $y = p^2\tan p$. Let us differentiate this equality: $dy = (2p\tan p + p^2\sec^2 p)dp$. Replacing dy by $p\,dx$, we obtain $p\,dx = p(2\tan p + p\sec^2 p)dp$, whence, cancelling by p and integrating, we find

$$x = \int(2\tan p + p\sec^2 p)dp = p\tan p - \ln\cos p + C.$$

The general solution of the given equation has the form

$$\begin{cases} y = p^2\tan p, \\ x = p\tan p - \ln\cos p + C. \end{cases}$$

593. Integrate the equation $x = y' + \ln y'$.
Solution. We put $y' = p$. Thus we get $x = p + \ln p$; differentiating, we find $dx = dp + \dfrac{dp}{p}$. Since $dy = p\,dx$, it follows that

$$dy = p\left(dp + \frac{dp}{p}\right) = (p + 1)dp.$$

Integrating, we get

$$y = \frac{1}{2}(p + 1)^2 + C.$$

The general solution of the given equation, written in parametric form, is

$$\begin{cases} x = p + \ln p, \\ y = \dfrac{1}{2}(p + 1)^2 + C. \end{cases}$$

It is easy to eliminate the parameter p; from the second equality we get $p = \sqrt{2(y - C)} - 1$ ($p > 0$ and, therefore, the root sign must be preceded by the "plus" sign). Substituting the expression obtained for p into the first equality, we

find the general solution of the equation in the following form:

$$x = \sqrt{2(y - C)} - 1 + \ln[\sqrt{2(y - C)} - 1].$$

Solve the following equations:

594. $\arcsin(x/y') = y'$.
595. $y = e^{y'}(y' - 1)$.
596. $x = 2(\ln y' - y')$.
597. $y(1 + y'^2)^{1/2} = y'$.
598. $x = 2y' + 3y'^2$.
599. $x = y'(1 + e^{y'})$.
600. $x = e^{2y'}(2y'^2 - 2y' + 1)$.
601. $y = y' \ln y'$.

4.1.8. Equations of Lagrange and Clairaut. The Lagrange equation is a first-order differential equation, linear with respect to x and y, whose coefficients are the functions of y':

$$P(y')x + Q(y')y + R(y') = 0.$$

Lagrange's equation can be integrated as follows. We solve it for y and assume y' to be the parameter, putting $y' = p$:

$$y = xf(p) + \varphi(p).$$

(We have introduced the notation $f(y') = -P(y')/Q(y')$, $\varphi(y') = -R(y')/Q(y')$.) Differentiating the equation obtained and replacing dy by $p\,dx$ on the left-hand side, we arrive at the equation

$$p\,dx = f(p)dx + xf'(p)dp + \varphi'(p)dp.$$

The equation obtained is linear with respect to x (as a function of p) and can, therefore, be integrated. If its solution is $x = F(p, C)$, then the general solution of the original Lagrangian equation can be written as

$$\begin{cases} x = F(p, C), \\ y = xf(p) + \varphi(p) = F(p, C)f(p) + \varphi(p). \end{cases}$$

Clairaut's equation is an equation of the form

$$y = xy' + \varphi(y'),$$

which is a particular case of Lagrange's equation. Integrating it by the indicated method, it is easy to obtain the general solution $y = Cx + \varphi(C)$, which defines the family of straight lines on a plane.

Besides the general solution, however, Clairaut's equation also possesses a singular solution specified by the following parametric equations:

$$\begin{cases} x = -\varphi'(p), \\ y = -p\varphi'(p) + \varphi(p). \end{cases}$$

The singular solution of Clairaut's equation (it exists if $\varphi'(p) \neq$ const) is the envelope of the family of straight lines defined by the general solution (in other words, *the general solution of Clairaut's equation is the family of tangents to the singular solution*).

Lagrange's equation can also possess singular solutions, these solutions (if they exist) being the common tangents to all the integral curves defined by the general solution.

602. Integrate the equation $y = xy' - e^{y'}$.

Solution. This is Clairaut's equation. We put $y' = p$ and rewrite the equation in the form $y = px - e^p$. We differentiate it: $dy = pdx + xdp - e^p dp$; but $dy = pdx$ and, therefore, the last equation assumes the form $xdp - e^p dp = 0$, or $(x - e^p)dp = 0$. Thus, either $dp = 0$ or $x = e^p$. If we set $dp = 0$, then $p = C$; substituting this value of p into the equality $y = px - e^p$, we obtain the general solution of the given equation:

$$y = Cx - e^C.$$

If we put $x = e^p$, then $y = pe^p - e^p = (p - 1)e^p$, and we arrive at a singular solution of the original equation:

$$\begin{cases} x = e^p, \\ y = (p - 1)e^p. \end{cases}$$

Eliminating the parameter p ($p = \ln x$ in the given case), we find the singular solution in the explicit form:

$$y = x(\ln x - 1).$$

Let us verify that the collection of straight lines defined by the general solution is a family of tangents to the singular integral curve.

Differentiating the singular solution, we find $y' = \ln x$. The equation of the tangent to the singular integral curve at the point $M(x_0; y_0)$ (where $y_0 = x_0(\ln x_0 - 1)$) can be written as

$$y - y_0 = y_0'(x - x_0), \quad \text{or} \quad y - x_0(\ln x_0 - 1) = \ln x_0(x - x_0),$$

which, after simplification, yields $y = x\ln x_0 - x_0$. If we put $\ln x_0 = C$, then the equation of the family of tangents to the singular integral curve assumes the form $y = Cx - e^C$, and that is what we had to prove.

603. Integrate the equation $y = xy'^2 + y'^2$.

Solution. This is Lagrange's equation. We do as before, that is, put $y' = p$, then $y' = xp^2 + p^2$. Let us differentiate the last equation: $dy = p^2dx + 2pxdp + 2pdp$. Performing the substitution $dy = pdx$, we arrive at the equation $pdx = p^2dx + 2pxdp + 2pdp$. Then, cancelling by p, we get an equation with the variables separable:

$$(1 - p)dx = 2(x + 1)dp, \quad \text{or} \quad \frac{dx}{x + 1} = \frac{2dp}{1 - p}.$$

Integrating, we find

$$\ln(x + 1) = -2\ln|1 - p| + \ln C; \quad x + 1 = C/(p - 1)^2.$$

Using the given equation $y = p^2(x + 1)$, we obtain

$$y = Cp^2/(1 - p^2).$$

The cancellation by p we have carried out could lead (and did lead in the given case) to a loss of the singular solution; putting $p = 0$, we find $y = 0$ from the given equation; this is a singular solution.

Thus,

$$\begin{cases} x + 1 = C/(p - 1)^2, & \text{is the general solution; } y = 0 \text{ is a} \\ y = Cp^2/(p - 1)^2 & \text{singular solution.} \end{cases}$$

In the general solution, the parameter p can be eliminated and the solution can be reduced to the form $(\sqrt{y} + \sqrt{x + 2})^2 = C$.

Solve the following equations:

604. $y = xy' + \sqrt{b^2 + a^2y'^2}$.

605. $x = \dfrac{y}{y'} + \dfrac{1}{y'^2}$.

606. $y = xy' + y' - y'^2$.

607. $y = x\left(\dfrac{1}{x} + y'\right) + y'^2$.

608. $2y(y' + 1) = xy'^2$.

4.2. Differential Equations of Higher Orders

4.2.1. Principal notions. A *differential equation of the nth order* is an equation of the form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0.$$

The solution of this equation is any, n times differentiable function $y = \varphi(x)$, which turns the given equation into identity, i.e.

$$F[x, \varphi(x), \varphi'(x), \varphi''(x), \ldots, \varphi^{(n)}(x)] = 0.$$

Cauchy's problem for this equation consists in finding a solution to this equation which will satisfy the conditions $y = y_0, y' = y'_0, \ldots, y^{(n-1)} = y_0^{(n-1)}$ for $x = x_0$, where $x_0, y_0, y'_0, \ldots, y_0^{(n-1)}$ are given numbers which are called the *initial data* or *initial conditions*.

The function $y = \varphi(x, C_1, C_2, \ldots, C_n)$ is called the *general solution* of the given differential equation of the nth order if, with the requisite choice of the arbitrary constants $C_1, C_2, \ldots, C_n$, this function is the solution of any Cauchy's problem stated for the given equation.

Any solution obtained from the general solution at concrete values of the constants $C_1, C_2, \ldots, C_n$ (in particular, every solution of Cauchy's problem) is called a *particular solution* of that equation. To separate some particular solution from the set of solutions of a differential equation, use is sometimes made of the so-called *boundary conditions*. These conditions (whose number must not exceed the order of the equation) are imposed not at a single point but at the end points of some interval. It is evident that the boundary conditions are imposed only on equations of the order higher than the first.

Integration of differential equations of the nth order (in the final form) turns out to be possible only in some particular cases.

4.2.2. Equation of the form $y^{(n)} = f(x)$. A solution of this equation can be found by means of n-fold integration, namely,

$$y^{(n)} = f(x),\ y^{(n-1)} = \int f(x)dx + C_1 = f_1(x) + C_1,$$

$$y^{(n-2)} = \int [f_1(x) + C_1]dx = f_2(x) + C_1x + C_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y = f_n(x) + \frac{C_1}{(n-1)!}x^{n-1} + \frac{C_2}{(n-2)!}x^{n-2} + \ldots + C_{n-1}x + C_n,$$

where

$$f_n(x) = \underbrace{\int\int\int\ldots\int}_{n \text{ times}} f(x)dx^n.$$

Since $\frac{C_1}{(n-1)!}, \frac{C_2}{(n-2)!}, \ldots$ are constant quantities, the general solution can also be written in the form

$$y = f_n(x) + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_{n-1}x + C_n.$$

609. Find the particular solution of the equation $y'' = xe^{-x}$ satisfying the initial conditions $y(0) = 1$, $y'(0) = 0$.

Solution. We find the general solution by means of successive integration of the given equation:

$$y' = \int xe^{-x}dx = -xe^{-x} - e^{-x} + C_1,$$

$$y = \int(-xe^{-x} - e^{-x} + C_1)dx = xe^{-x} + 2e^{-x} + C_1x + C_2,$$

or

$$y = (x + 2)e^{-x} + C_1x + C_2.$$

We use the initial conditions: $1 = 2 + C_2$; $C_2 = -1$; $0 = -1 + C_1$; $C_1 = 1$. Con-

sequently, the desired particular solution has the form

$$y = (x + 2)e^{-x} + x - 1.$$

The same solution can also be found by using directly the initial conditions:

$$y' = y'(0) + \int_0^x xe^{-x}dx = [-xe^{-x} - e^{-x}]_0^x = -xe^{-x} - e^{-x} + 1;$$

$$y = y(0) + \int_0^x [-xe^{-x} - e^{-x} + 1]dx = 1 + [(x + 2)e^{-x} + x]_0^x = (x + 2)e^{-x} + x - 1.$$

Solve the following equations:

610. $y^{IV} = \cos^2 x; y(0) = 1/32; y'(0) = 0, y''(0) = 1/8, y'''(0) = 0.$
611. $y''' = x\sin x; y(0) = 0, y'(0) = 0, y''(0) = 2.$
612. $y''\sin^4 x = \sin 2x.$
613. $y'' = 2\sin x\cos^2 x - \sin^3 x.$
614. $y''' = xe^{-x}; y(0) = 0, y'(0) = 2, y''(0) = 2.$

4.2.3. Differential equations of the form $F(x, y^{(k)}, y^{(k+1)}, \ldots, y^{(n)}) = 0$, which do not contain the desired function. The order of such an equation can be reduced by taking the lowermost derivative of the given equation for the new unknown function, that is, putting $y^{(k)} = z$. Then we obtain an equation

$$F(x, z, z', \ldots, z^{(n-k)}) = 0.$$

Thus we reduce the order of the equation by k units.

615. Find the general solution of the equation $xy'' = y'\ln(y'/x)$.
Solution. Putting $y' = z$, we reduce the equation to the form

$$xz' = z\ln(z/x), \quad \text{or} \quad z' = (z/x)\ln(z/x).$$

This is a 1st-order homogeneous equation. Putting $z/x = t$, whence $z = tx$, $z' = t'x + t$, we obtain an equation

$$t'x + t = t\ln t, \quad \text{or} \quad \frac{dt}{t(\ln t - 1)} = \frac{dx}{x}.$$

Integrating it, we find

$$\ln(\ln t - 1) = \ln x + \ln C_1, \quad \text{or} \quad \ln t - 1 = C_1 x,$$

whence $t = e^{1 + C_1 x}$; returning to the variable y, we arrive at the equation $y' = xe^{1 + C_1 x}$. Consequently,

$$y = \int xe^{1 + C_1 x}dx = \frac{1}{C_1}xe^{1 + C_1 x} - \frac{1}{C_1^2}e^{1 + C_1 x} + C_2.$$

616. A body of mass m falls vertically from some height with zero initial velocity and experiences air drag proportional to the square of the velocity of the body. Find the law of motion of the body.

Solution. We introduce the following notation: s is the path traversed by the body, $v = ds/dt$ is its velocity, $w = d^2s/dt^2$ is its acceleration. The following forces act on the body: its weight $P = mg$ (in the direction of the motion) and the air drag $F = kv^2 = k(ds/dt)^2$ (opposite to the direction of the motion).

Proceeding from Newton's second law, we arrive to the following differential equation of the motion of the body:

$$mw = P - kv^2, \quad \text{or} \quad m\frac{d^2s}{dt^2} = mg - k\left(\frac{ds}{dt}\right)^2.$$

We make use of the initial conditions: if $t = 0$, then $s = 0$, $v = ds/dt = 0$. Replacing ds/dt by v, we rewrite the equation as

$$\frac{dv}{dt} = g - \frac{k}{m}v^2,$$

whence, setting $mg/k = a^2$, we have $\dfrac{dv}{a^2 - v^2} = \dfrac{k}{m}dt$. Integrating, we find ($v \leqslant a$):

$$\frac{1}{2a}\ln\frac{a+v}{a-v} = \frac{k}{m}t + C_1.$$

If $t = 0$, then $v = 0$, whence $C_1 = 0$. Thus we have

$$\ln\frac{a+v}{a-v} = \frac{2ak}{m}t.$$

It follows that

$$v = a\frac{e^{2akt/m} - 1}{e^{2akt/m} + 1} = a\frac{e^{akt/m} - e^{-akt/m}}{e^{akt/m} + e^{-akt/m}} = a\tanh(akt/m).$$

But $\dfrac{ak}{m} = \sqrt{\dfrac{mg}{k}} \cdot \dfrac{k}{m} = \sqrt{\dfrac{kg}{m}}$; replacing v by ds/dt, we obtain the following equation specifying s:

$$\frac{ds}{dt} = a\tanh\sqrt{\frac{kg}{m}}\,t,$$

integrating it, we find

$$s = \sqrt{\frac{m}{kg}}\,a\ln\cosh\sqrt{\frac{kg}{m}}\,t + C_2 = \frac{m}{k}\ln\cosh\sqrt{\frac{kg}{m}}\,t + C_2.$$

Since $s = 0$ for $t = 0$, we have $C_2 = 0$.

Thus, the law of free fall of a body experiencing an air drag, proportional to the square of its velocity, is specified by the formula

$$s = \frac{m}{k}\ln\cosh\sqrt{\frac{kg}{m}}\,t,$$

and the velocity of the body, by the formula $v = a \tanh \sqrt{\frac{kg}{m}}\, t$. Here $a = \sqrt{\frac{mg}{k}}$; it is of use to note that the velocity of the falling body does not increase indefinitely since $\lim_{t \to \infty} v = a = \sqrt{\frac{P}{k}}$ $\left(\text{because } \lim_{t \to \infty} \tanh \sqrt{\frac{kg}{m}}\, t = 1\right)$, where P is the weight of the body, with the velocity practically reaching its limit very soon, differing from it by a rather small quantity. Such is precisely the picture observed in practical free-fall drops with a parachute from high altitudes.

Solve the following equations:

617. $y'' - \dfrac{y'}{x - 1} = x(x - 1)$; $y(2) = 1$, $y'(2) = -1$.

618. $(1 - x^2)y'' - xy' = 2$.

619. $2xy'''y'' = y''^2 - a^2$.

620. $(1 + x^2)y'' + 1 + y'^2 = 0$.

621. $y'''(x - 1) - y'' = 0$; $y(2) = 2$; $y'(2) = 1$, $y''(2) = 1$.

4.2 4. Differential equations of the form $F(y, y', y'', \ldots, y^{(n)}) = 0$, without an independent variable. Equations of this kind admit of reducing the order, if we put $y' = z$ and assume y to be the new argument. In this case, $y'', y''', \ldots$ are expressed by the formulas (which are derived in accordance with the rule of differentiating composite function) $y'' = z\dfrac{dz}{dy}$, $y''' = z\left[z\dfrac{d^2z}{d^2y} + \left(\dfrac{dz}{dy}\right)^2\right]$, ... in terms of z and in terms of the derivatives of z with respect to y, the order of integration being reduced by unity.

622. Solve the equation $1 + y'^2 = yy''$.

Solution. We put $y' = z$, $y'' = z\dfrac{dz}{dy}$. The equation assumes the form $1 + z^2 = yz\dfrac{dz}{dy}$. This is an equation of the first order with respect to z with variables separable. We separate the variables and integrate:

$$\frac{z\,dz}{1 + z^2} = \frac{dy}{y}; \quad \ln(1 + z^2) = 2\ln y + 2\ln C_1; \; 1 + z^2 = C_1^2y^2; \quad z = \pm\sqrt{C_1^2y^2 - 1}.$$

Returning to the variable y, we obtain

$$y' = \pm\sqrt{C_1^2y^2 - 1}, \quad \frac{dy}{\sqrt{C_1^2y^2 - 1}} = \pm dx,$$

$$\frac{1}{C_1}\ln(C_1 y + \sqrt{C_1^2 y^2 - 1}) = \pm(x + C_2),$$

or

$$y = \frac{1}{2C_1}(e^{\pm(x + C_2)C_1} + e^{\mp(x + C_2)C_1}) = \frac{1}{C_1}\cosh C_1(x + C_2) = C_1^*\cosh\frac{x + C_2}{C_1^*}.$$

623. Find the curve whose radius of curvature is equal to the cube of the normal; the desired curve must pass through the point $M(0; 1)$ and possess at that point a tangent making an angle of 45° with the x-axis.

Solution. Since the radius of curvature of a plane curve is expressed by the formula $R = (1 + y'^2)^{3/2}/y''$, and the length of the normal $N = y\sqrt{1 + y'^2}$, the differential equation of the problem assumes the form

$$\frac{(1 + y'^2)^{3/2}}{y''} = (y\sqrt{1 + y'^2})^3.$$

Hence, cancelling by $(1 + y'^2)^{3/2}$, we arrive at the equation $y''\cdot y^3 = 1$.

Putting $y' = z$, $y'' = z\cdot\frac{dz}{dy}$, we obtain for z the equation $z\cdot\frac{dz}{dy}\cdot y^3 = 1$. Integrating it, we find

$$zdz = y^{-3}\cdot dy, \quad \text{or} \quad \frac{1}{2}z^2 = -\frac{1}{2}y^{-2} + \frac{1}{2}C_1, \quad \text{i.e.} \quad z^2 = C_1 - y^{-2};$$

returning to the variable y, we arrive at the equation $y'^2 = C_1 - y^{-2}$.

We find the arbitrary constant C_1 from the condition stating that the tangent makes an angle of 45° with the x-axis at the point $M(0; 1)$, i.e. $\tan 45° = y'_M = 1$, or $y'(0) = 1$. Consequently, $1 = C_1 - 1$, i.e. $C_1 = 2$.

Thus we have obtained the 1st-order equation $y'^2 = 2 - y^{-2}$ for determining y, whence we have $y' = \frac{\sqrt{2y^2 - 1}}{y}$; we separate the variables and integrate:

$$\frac{ydy}{\sqrt{2y^2 - 1}} = dx; \quad \frac{1}{2}\sqrt{2y^2 - 1} = x + \frac{1}{2}C_2; \quad y^2 = \frac{1}{2}[(x + C_2)^2 + 1].$$

The arbitrary constant C_2 is found from the condition that the curve passes through the point $M(0; 1)$, i.e. $1 = \frac{1}{2}[(2\cdot 0 + C_2)^2 + 1]$; $C_2 = 1$. Consequently, the desired curve is specified by the equation

$$y^2 = 2x^2 + 2x + 1.$$

Solve the following equations:

624. $y''(2y + 3) - 2y'^2 = 0$.
625. $yy'' - y'^2 = 0$; $y(0) = 1$, $y'(0) = 2$.

626. $a^2y''^2 = 1 + y'^2$.
627. $yy'' - y'^2 = y^2 \ln y$.
628. $y(1 - \ln y)y'' + (1 + \ln y)y'^2 = 0$.
629. $y''(1 + y) = y'^2 + y'$.
630. $y'' = y'/\sqrt{y}$.

4.2.5. Equations of the form $F(x, y, y', y'', \ldots, y^{(n)}) = 0$, homogeneous with respect to $y, y', y'', \ldots, y^{(n)}$. Equations of this kind admit of reducing the order by unity upon the substitution $y'/y = z$, where z is a new unknown function.

631. Solve the equation $3y'^2 = 4yy'' + y^2$.

Solution. Let us divide both sides of the equation by y^2:

$$3\left(\frac{y'}{y}\right)^2 - 4 \cdot \frac{y''}{y} = 1.$$

We put $y'/y = z$, whence $\dfrac{y''}{y} - \dfrac{y'^2}{y^2} = z'$, or $y''/y = z' + z^2$. As a result we obtain an equation

$$3z^2 - 4z^2 - 4z' = 1, \quad \text{or} \quad -4z' = 1 + z^2, \quad \text{i.e.} \quad \frac{dz}{1 + z^2} = -\frac{1}{4}\,dx.$$

Integrating, we find

$$\arctan z = C_1 - \frac{1}{4}x, \quad \text{or} \quad z = \tan\left(C_1 - \frac{x}{4}\right), \quad \text{or} \quad \frac{y'}{y} = \tan\left(C_1 - \frac{x}{4}\right).$$

Integrating the last equation, we obtain

$$\ln y = 4 \ln \cos\left(C_1 - \frac{x}{4}\right) + \ln C_2, \quad \text{or} \quad y = C_2 \cdot \cos^4\left(C_1 - \frac{x}{4}\right).$$

632. Solve the equation $y'^2 + yy'' = yy'$.

Solution. Although the equation belongs to the previous kind, it can be integrated much simpler. The left-hand side of this equation is $(yy')'$, and, hence, the equation assumes the form $(yy')' = yy'$, or $\dfrac{d(yy')}{yy'} = dx$, whence we find

$$\ln(yy') = x + \ln C_1, \quad \text{or} \quad yy' = C_1e^x, \quad \text{i.e.} \quad ydy = C_1e^x dx.$$

Integration yields the final answer:

$$y^2/2 = C_1e^x + C_2.$$

Solve the following equations:

633. $yy'' - y'^2 = 0$.

634. $(y + y')y'' + y'^2 = 0.$
635. $2xy'''y'' = y'^2 - a^2.$
636. $y'' = y'e^y$; $y(0) = 0$, $y'(0) = 1$.
637. Find the curve if the projection of the radius of curvature of the y-axis is constant and is equal to a, and the x-axis touches the desired curve at the origin.
638. Find the curve whose radius of curvature at any point is $\sec \alpha$, where α is the angle formed by the x-axis and the tangent at the corresponding point. The sought curve passes through the point $M(0; 1)$ and the tangent to the curve at that point is parallel to the x-axis.
639. The body being immersed in a liquid at the initial moment goes down due to gravity without initial velocity. The resistance of the liquid is directly proportional to the velocity of the body. Find the law of motion of the body if its mass is m.

4.3. Higher-Order Linear Equations

4.3.1. Principal notions. A *linear differential equation of the nth order* is an equation of the form

$$y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \ldots + a_{n-1}(x)y' + a_n(x)y = f(x). \quad (1)$$

Here the functions $a_1(x), a_2(x), \ldots, a_n(x)$ and $f(x)$ are defined and continuous in a certain interval (a, b).

Equation (1) is called *nonhomogeneous linear*, or an equation *with a right-hand side*. Now if $f(x) \equiv 0$, then the equation is called *homogeneous linear*. A homogeneous equation with the same left-hand side as the given nonhomogeneous equation is said *to correspond* to it.

Knowing one particular solution y_1 of a homogeneous linear equation, we can use the linear substitution of the desired function $y = y_1 \cdot \int zdx$ to reduce its order, and, consequently, the order of the corresponding nonhomogeneous equation, by unity. The resulting equation of the $(n - 1)$th order for z is also linear.

640. Given the equation $y''' + \frac{2}{x}y'' - y' + \frac{1}{x \ln x}y = x$, and the particular solution $y_1 = \ln x$ of the corresponding homogeneous equation is known. Reduce the order of the equation.

Solution. We use the substitution $y = \ln x \cdot \int zdx$, where z is a new unknown function. Then, substituting the appropriate derivatives

$$y' = \frac{1}{x}\int zdx + z \ln x, \quad y'' = -\frac{1}{x^2}\int zdx + \frac{2z}{x} + z'\ln x,$$

$$y''' = \frac{2}{x^3}\int zdx - \frac{3z}{x^2} + \frac{3z'}{x} + z''\ln x$$

into the given equation, we obtain the second-order equation

$$z''\ln x + \frac{2\ln x + 3}{3} \cdot z' + \left(\frac{1}{x^2} - \ln x\right)z = x.$$

Note. Pay attention to the fact that applying the indicated substitution to a second-order linear equation and taking into account that a first-order linear equation can be integrated by quadratures, we can integrate by quadratures any second-order linear equation if we know one particular solution of the corresponding homogeneous equation.

641. Integrate the equation $y'' + \frac{2}{x}y' + y = 0$ possessing a particular solution $y_1 = \frac{\sin x}{x}$.

Solution. We perform a change of the variable $y = \frac{\sin x}{x} \cdot \int z\,dx$; then we have

$$y' = \frac{x\cos x - \sin x}{x^2} \cdot \int z\,dx + \frac{\sin x}{x} z,$$

$$y'' = \frac{\sin x}{x} z' + \frac{2(x\cos x - \sin x)}{x^2} \cdot z - \frac{(x^2 - 2)\sin x + 2x\cos x}{x^3} \cdot \int z\,dx.$$

We obtain the equation

$$\sin x \cdot z' + 2\cos x \cdot z = 0, \quad \text{i.e.} \quad z = \frac{C_1}{\sin^2 x}.$$

Consequently,

$$y = \frac{\sin x}{x} \cdot \int \frac{C_1 dx}{\sin^2 x} = \frac{\sin x}{x}(C_2 - C_1 \cot x) = C_2 \cdot \frac{\sin x}{x} - C_1 \frac{\cos x}{x}.$$

642. Reduce the order and integrate the equation $y''\sin^2 x = 2y$ possessing a particular solution $y = \cot x$.

643. The equation $y'' - \frac{y'}{x} + \frac{y}{x^2} = 0$ possesses a particular solution $y = x$. Reduce the order of the equation and integrate it.

644. The equation $y'' + (\tan x - 2\cot x)y' + 2\cot^2 x \cdot y = 0$ possesses a particular solution $y = \sin x$. Reduce the order of this equation and integrate it.

4.3.2. Homogeneous linear equations. A linear equation possesses a remarkable property consisting in the fact that its general solution can be found from a certain number of its particular solutions. Here is a theorem on the structure of the general solution of a homogeneous linear equation.

Theorem. *If $y_1, y_2, \ldots, y_n$ are linearly independent particular solutions of the equation*

$$y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \ldots + a_n(x)y = 0,$$

then $y = C_1y_1 + C_2y_2 + \ldots + C_ny_n$ is the general solution of that equation ($C_1, C_2, \ldots, C_n$ being arbitrary constants).

Note. The functions $y_1(x), y_2(x), \ldots, y_n(x)$ are said to be *linearly independent in the interval* (a, b) if they are not connected by any identity

$$\alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_n y_n = 0,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are some constants which are not simultaneously zero. For the case of two functions, this condition can also be formulated as follows: two functions $y_1(x)$ and $y_2(x)$ are linearly independent if their ratio is not a constant quantity; $y_1/y_2 \neq$ const. For example: (1) $y_1 = x$, $y_2 = x^2$ are linearly independent; (2) $y_1 = e^x$, $y_2 = e^{-x}$ are linearly independent; (3) $y_1 = 2e^{3x}$, $y_2 = 5e^{3x}$ are linearly dependent.

A sufficient condition for linear independence of n functions, continuous together with their partial derivatives up to the $(n-1)$th order in the interval (a, b), is the fact that the *Wronskian* $W[y_1, y_2, \ldots, y_n]$ of these functions is not zero at any point of the interval (a, b), i.e.

$$W[y_1, y_2, \ldots, y_n] = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y'_1(x) & y'_2(x) & \ldots & y'_n(x) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix} \neq 0.$$

If the given n functions are particular solutions of a homogeneous linear differential nth-order equation, then this condition (not turning to zero) is not only the sufficient but also the necessary condition of linear independence of these n solutions.

The Wronskian of the n solutions of the homogeneous linear nth-order equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = 0$$

is connected with the first coefficient $a_1(x)$ of that equation by the *formula of Liouville-Ostrogradsky*:

$$W[y_1, y_2, \ldots, y_n] = W[y_1, y_2, \ldots, y_n]\big|_{x=x_0} \cdot e^{-\int_{x_0}^{x} a_1(x)dx}.$$

The set of n solutions of a homogeneous linear equation of the nth order defined and linearly independent on the interval (a, b) is called a *fundamental system* of solutions of that equation.

For the homogeneous linear differential equation of the second order

$$y'' + a_1(x)y' + a_2(x)y = 0$$

the fundamental system consists of two linearly independent solutions $y_1(x)$ and $y_2(x)$; its general solution can be found by the formula

$$y = C_1y_1(x) + C_2y_2(x).$$

If we know one particular solution $y_1(x)$ for such an equation, its second solution, linearly independent of the first, can be found by the formula (which is a consequence of the Liouville-Ostrogradsky formula)

$$y_2(x) = y_1(x)\int \frac{e^{-\int a_1(x)dx}}{y_1^2(x)}\,dx.$$

This makes it possible to integrate directly homogeneous linear equations of the second order, for which one particular solution is known, without reducing their order.

Thus, in Problem 641 we know one particular solution $y_1(x) = \frac{\sin x}{x}$ for the equation $y'' + \frac{2}{x}y' + y = 0$. Proceeding from the formula given above, we find the second solution:

$$y_2(x) = \frac{\sin x}{x} \int \frac{e^{-2\int \frac{dx}{x}}}{\left(\frac{\sin x}{x}\right)^2} dx = \frac{\sin x}{x} \int \frac{dx}{\sin^2 x} = -\frac{\cos x}{x}.$$

Therefore, the general solution of the given equation has the form

$$y = C_1 \frac{\sin x}{x} - C_2 \frac{\cos x}{x}.$$

We recommend the reader to solve problems 642-644 by this method.

645. Show that $y = C_1 e^{3x} + C_2 e^{-3x}$ is the general solution of the equation $y'' - 9y = 0$.

Solution. By making a substitution, it is easy to verify that the functions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are its solutions. These particular solutions are linearly independent since $y_1/y_2 = e^{3x}/e^{-3x} = e^{6x} \neq$ const, and, therefore, they constitute a fundamental system of solutions and, consequently, $y = C_1e^{3x} + C_2e^{-3x}$ is the general solution.

646. Given the equation $y'' - y' = 0$. Do the functions e^x, e^{-x}, $\cosh x$, which are easily verifiable solutions of this equation, constitute a fundamental system of solutions?

Solution. To verify the linear independence of these solutions, let us calculate the Wronskian:

$$W(x) = \begin{vmatrix} e^x & e^{-x} & \cosh x \\ e^x & -e^{-x} & \sinh x \\ e^x & e^{-x} & \cosh x \end{vmatrix}.$$

It is equal to zero since the elements of the first and third rows are the same.

Consequently, the given funçtions are linearly dependent, and, therefore, it is impossible to derive the general solution from these particular solutions. The same result can be attained much faster; since $\cosh x = (e^x + e^{-x})/2$, it follows that the given three functions are linearly dependent.

647. The equation $y'' - y = 0$ is satisfied by two particular solutions $y_1 = \sinh x$, $y_2 = \cosh x$. Do they constitute a fundamental system?

648. Can the general solution of the equation $y'' + \frac{1}{x} y' + \left(1 - \frac{1}{4x^2}\right) y = 0$ $(x \neq 0)$ be derived from its two particular solutions $y' = \frac{1}{\sqrt{x}} \cdot \sin x$, $y_2 = \frac{1}{\sqrt{x}} \cdot \cos x$?

649. Are the functions $x + 1, 2x + 1, x + 2$ linearly independent?

650. The same question relating to the functions $2x^2 + 1, x^2 - 1, x + 2$.

651. The same question relating to the functions $\sqrt{x}, \sqrt{x + a}, \sqrt{x + 2a}$.

652. The same question relating to the functions $\ln(2x)$, $\ln(3x)$, $\ln(4x)$.

4.3.3. Homogeneous linear equations with constant coefficients. *An nth-order homogeneous linear equation with constant coefficients* is an equation of the form

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_{n-1} y' + a_n y = 0, \tag{1}$$

where the coefficients $a_1, a_2, \ldots, a_{n-1}, a_n$ are some real numbers. To find the particular solutions of equation (1), the *characteristic equation*

$$k^n + a_1 k^{n-1} + a_2 k^{n-2} + \ldots + a_{n-1} k + a_n = 0, \tag{2}$$

is derived, which is obtained from (1) by replacing in it the derivatives of the desired function by the respective powers of k, the function itself being replaced by unity. Equation (2) is an equation of the nth order and possesses n roots (real or complex, some of which may even be equal).

Then, the construction of the general solution of the differential equation (1) depends on the character of the roots of equation (2):

(1) each simple real root k is associated in the general solution with a summand of the form Ce^{kx};

(2) each real root of multiplicity m is associated in the general solution with a summand of the form $(C_1 + C_2 x + \ldots + C_m x^{m-1}) e^{kx}$;

(3) each pair of complex conjugate simple roots $k^{(1)} = \alpha + \beta i$ and $k^{(2)} = \alpha - \beta i$ is associated in the general solution with a summand of the form $e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$;

(4) each pair of complex conjugate roots $k^{(1)} = \alpha + \beta i$ and $k^{(2)} = \alpha - \beta i$ of multiplicity m is associated in the general solution with a summand of the form

$$e^{\alpha x}[(C_1 + C_2 x + \ldots + C_{m-1} x^{m-1}) \cos \beta x] + [(C_1' + C_2' x + \ldots + C_{m-1}' x^{m-1}) \sin \beta x].$$

653. Find the general solution of the equation $y'' - 7y' + 6y = 0$.

Solution. Let us derive the characteristic equation $k^2 - 7k + 6 = 0$; its roots are $k_1 = 6$; $k_2 = 1$. Consequently, e^{6x} and e^x are particular, linear independent solutions, and the general solution has the form

$$y = C_1 e^{6x} + C_2 e^x.$$

654. Find the general solution of the equation $y^{\mathrm{IV}} - 13y'' + 36y = 0$.

Solution. The characteristic equation has the form $k^4 - 13k^2 + 36 = 0$; its

roots $k_{1,2} = \pm 3$, $k_{3,4} = \pm 2$ are associated with linearly independent particular solutions e^{3x}, e^{-3x}, e^{2x} and e^{-2x}. Consequently, the general solution is

$$y = C_1e^{3x} + C_2e^{-3x} + C_3e^{2x} + C_4e^{-2x}.$$

655. Find the solution of the equation $\ddot{x} - \dot{x} - 2x = 0$, satisfying the initial conditions $x = 0$, $\dot{x} = 3$ for $t = 0$.

Solution. The characteristic equation $k^2 - k - 2 = 0$ has the roots $k_1 = 2$, $k_2 = -1$. Consequently, the general solution is $x = C_1e^{2t} + C_2e^{-t}$. Substituting the initial conditions into the gerenal solution and its derivative, we obtain a system of equations for C_1 and C_2:

$$\begin{cases} 0 = C_1 + C_2, \\ 3 = 2C_1 - C_2, \end{cases}$$

whence we get $C_1 = 1$, $C_2 = -1$. It follows that the solution satisfying the initial conditions has the form $y = e^{2t} - e^{-t}$.

656. Find the solution of the equation $\ddot{x} - 2\dot{x} = 0$, satisfying the boundary conditions $x = 0$ for $t = 0$ and $x = 3$ for $t = \ln 2$.

Solution. The characteristic equation $k^2 - 2k = 0$ has the roots $k_1 = 0$, $k_2 = 2$. Consequently, the general solution is written as $x = C_1 + C_2e^{2t}$. Substituting the boundary conditions into the general solution obtained, we get

$$\begin{cases} C_1 + C_2 = 0, \\ C_1 + C_2e^{2\ln 2} = 3, \end{cases} \quad \text{or} \quad \begin{cases} C_1 + C_2 = 0, \\ C_1 + 4C_2 = 3. \end{cases}$$

Hence, $C_1 = -1$, $C_2 = 1$. Thus it follows that $x = e^{2t} - 1$ is the sought particular solution satisfying the boundary conditions.

657. Find the gerenal solution of the equation $y''' - 2y'' - y' = 0$.

Solution. The characteristic equation $k^3 - 2k^2 + k = 0$ has the roots $k_1 = 0$, $k_2 = k_3 = 1$. Here 1 is a double root and, therefore, $1, e^x, xe^x$ are linearly independent particular solutions. The general solution is

$$y = C_1 + C_2e^x + C_3xe^x.$$

658. Find the general solution of the equation $y'' - 4y' + 13y = 0$.

Solution. The characteristic equation $k^2 - 4k + 13 = 0$ has the roots $k = 2 \pm 3i$. The roots of the characteristic equation are complex conjugate and, therefore, they are associated with the particular solutions $e^{2x}\cos 3x$ and $e^{2x} \cdot \sin 3x$. Consequently, the general solution is

$$y = e^{2x}(C_1 \cos 3x + C_2 \sin 3x).$$

659. A particle of mass m moves along the x-axis under the action of a restoring force directed to the origin and proportional to the distance of the moving particle from the origin; the resistance offered by the medium to the motion of the particle is proportional to the particle velocity. Find the law of motion.

Solution. Suppose $\dot{x}$ is the velocity of the particle, $\ddot{x}$ is its acceleration. Two forces act on the particle: the restoring force $f_1 = -ax$ and the medium resistance

force $f_2 = -b\dot{x}$. In accordance with Newton's second law, we have

$$m\ddot{x} = -b\dot{x} - ax, \quad \text{or} \quad m\ddot{x} + b\dot{x} + ax = 0.$$

We have obtained a homogeneous linear differential equation of the second order. Its characteristic equation $mk^2 + bk + a = 0$ has the roots

$$k_{1,2} = (-b \pm \sqrt{b^2 - 4ma})/(2m).$$

(1) If $b^2 - 4ma > 0$, then the roots are real, distinct and both negative; introducing for them the notation

$$k_1 = (-b + \sqrt{b^2 - 4ma})/(2m) = -r_1, k_2 = -(b + \sqrt{b^2 - 4ma})/(2m) = -r_2,$$

we find the general solution of the equation of motion in the form

$$x = C_1 e^{-r_1 t} + C_2 e^{-r_2 t}$$

(this is the case of the so-called *aperiodic* motion).

(2) If $b^2 - 4ma = 0$, then the roots of the characteristic equation are real and equal:

$$k_1 = k_2 = -b/(2m) = -r.$$

In this case, the general solution of the equation of motion has the form

$$x = (C_1 + C_2 t)e^{-rt}.$$

(3) If, finally, $b^2 - 4ma < 0$, then the characteristic equation possesses complex conjugate roots

$$k_1 = -\alpha + \beta i, \quad k_2 = -\alpha - \beta i,$$

where

$$\alpha = b/(2m), \quad \beta = (\sqrt{4am - b^2})/(2m).$$

The general solution of the equation of motion has the form

$$x = e^{-\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t), \quad \text{or} \quad x = Ae^{-\alpha t} \sin (\beta t + \varphi_0),$$

where

$$A = \sqrt{C_1^2 + C_2^2}, \quad \sin \varphi_0 = C_1/A, \quad \cos \varphi_0 = C_2/A$$

(*damped* or *convergent oscillations*).

Find the general solutions of the following equations:

660. $y'' - y' - 2y = 0.$

661. $y'' + 25y = 0.$

662. $y'' - y' = 0.$

663. $y'' - 4y' + 4y = 0.$

664. $y^{\mathrm{IV}} - 2y''' + y'' = 0.$

665. $y^{\mathrm{IV}} + a^4 y = 0.$

666. $y^{\mathrm{IV}} + 5y'' + 4y = 0.$

Find the solutions of equations satisfying the given initial or boundary conditions:

667. $y'' + 5y' + 6y = 0;\ y(0) = 1,\ y'(0) = -6.$
668. $y'' - 10y' + 25y = 0;\ y(0) = 0,\ y'(0) = 1.$
669. $y'' - 2y' + 10y = 0;\ y(\pi/6) = 0,\ y'(\pi/6) = e^{\pi/6}.$
670. $9y'' + y = 0;\ y(3\pi/2) = 2,\ y'(3\pi/2) = 0.$
671. $y'' + 3y' = 0;\ y(0) = 1,\ y'(0) = 2.$
672. $y'' + 9y = 0;\ y(0) = 0,\ y(\pi/4) = 1.$
673. $y'' + y = 0;\ y'(0) = 1,\ y'(\pi/3) = 0.$
674. Solve Problem 659 if the force of resistance of the medium is zero.

4.3.4. Nonhomogeneous linear equations. The structure of the general solution of a nonhomogeneous linear equation, that is, equation with a right-hand side

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = f(x),$$

is defined by the following theorem:

If $u = u(x)$ is a particular solution of a nonhomogeneous equation, and $y_1, y_2, \ldots, y_n$ is a fundamental system of solutions of the corresponding homogeneous equation, then the general solution of the nonhomogeneous linear equation has the form $y = u + C_1y_1 + C_2y_2 + \ldots + C_ny_n$; in other words, the general solution of a nonhomogeneous equation is equal to the sum of its any particular solution and the general solution of the corresponding homogeneous equation.

Consequently, to construct the general solution of a nonhomogeneous equation, it is necessary to find one of its particular solutions (assuming the general solution of the corresponding homogeneous equation to be known).

We shall consider two methods of finding a particular solution of a nonhomogeneous linear equation.

The method of variation of arbitrary constants. This method is used in seeking a particular solution of a nonhomogeneous linear equation of the nth order, both with variable and with constant coefficients, provided the general solution of the corresponding homogeneous equation is known.

The method of variation consists in the following. Suppose the fundamental system of solutions $y_1, y_2, \ldots, y_n$ of the corresponding homogeneous equation is known. Then the general solution of the nonhomogeneous equation must be sought in the form

$$u(x) = C_1(x)y_1 + C_2(x)y_2 + \ldots + C_n(x)y_n,$$

where the functions $C_1(x), C_2(x), \ldots, C_n(x)$ can be found from the system of equations

$$\begin{cases} C_1'(x)y_1 + C_2'(x)y_2 + \ldots + C_n'(x)y_n = 0, \\ C_1'(x)y_1' + C_2'(x)y_2' + \ldots + C_n'(x)y_n' = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C_1'(x)y_1^{(n-2)} + C_2'(x)y_2^{(n-2)} + \ldots + C_n'(x)y_n^{(n-2)} = 0, \\ C_1'(x)y_1^{(n-1)} + C_2'(x)y_2^{(n-1)} + \ldots + C_n'(x)y_n^{(n-1)} = f(x) \end{cases}$$

($f(x)$ being the right-hand side of the given equation).

For the second-order equation $y'' + a_1(x)y' + a_2(x)y = f(x)$, the corresponding system has the form

$$\begin{cases} C_1'(x)y_1 + C_2'(x)y_2 = 0, \\ C_1'(x)y_1' + C_2'(x)y_2' = f(x). \end{cases}$$

The solution of this system can be found by the formulas

$$C_1(x) = -\int \frac{y_2 \cdot f(x)dx}{W(y_1, y_2)}; \quad C_2(x) = \int \frac{y_1 \cdot f(x)dx}{W(y_1, y_2)},$$

and, accordingly, $u(x)$ can be directly determined by the formula

$$u(x) = -y_1 \int \frac{y_2 \cdot f(x)dx}{W(y_1, y_2)} + y_2 \int \frac{y_1 \cdot f(x)dx}{W(y_1, y_2)}.$$

(Here $W(y_1, y_2)$ is the Wronskian of the solutions y_1 and y_2.)

Suppose it is required to integrate the equation

$$y'' + \frac{2}{x}y' + y = \frac{\cot x}{x}.$$

We have found the particular solutions $y_1 = \dfrac{\sin x}{x}$ and $y_2 = \dfrac{\cos x}{x}$ for the corresponding homogeneous equation (see p. 170); their Wronskian is $W(y_1, y_2) = -1/x^2$.

Consequently, $u(x)$ can be found by the formula

$$u(x) = -\frac{\sin x}{x}\int \frac{\frac{\cos x}{x} \cdot \frac{\cot x}{x}}{(-1/x^2)}dx + \frac{\cos x}{x}\int \frac{\frac{\sin x}{x} \cdot \frac{\cot x}{x}}{(-1/x^2)}dx$$

$$= \frac{\sin x}{x} \cdot \int \frac{\cos^2 x}{\sin x}dx - \frac{\cos x}{x} \cdot \int \cos x dx$$

$$= \frac{\sin x}{x}[\ln|\tan(x/2)| + \cos x] - \frac{\cos x}{x} \cdot \sin x.$$

Thus, $u(x) = \dfrac{\sin x \ln|\tan(x/2)|}{x}$, and the general solution of the given equation has the form

$$y = C_1 \cdot \frac{\sin x}{x} + C_2 \cdot \frac{\cos x}{x} + \frac{\sin x}{x} \cdot \ln|\tan(x/2)|.$$

Note. We want again to draw attention to the fact that a nonhomogeneous linear equation of the second order can be integrated by quadratures if one particular solution $y_1(x)$ of the corresponding homogeneous equation is known; the general solution of such an equation has

the form $y = C_1y_1 + C_2y_2 + u(x)$, where y is determined from y_1 by the formula

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)dx}}{y_1^2}\, dx,$$

and $u(x)$ is determined from y_1 and y_2 by the formula given above.

Method of selection of a particular solution (method of undetermined coefficients). This method can only be applied to linear equations with constant coefficients in the only case when its right-hand side has the following form:

$$f(x) = e^{\alpha x}[P_n(x) \cos \beta x + Q_m(x) \sin \beta x]$$

(or is the sum of the functions of such a form). Here α and β are constants, $P_n(x)$ and $Q_m(x)$ are polynomials in x of the nth and the mth degree respectively.

The particular solution of the nth-order equation

$$y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + \ldots + a_ny = f(x)$$

(where $f(x)$ is of the indicated form, and $a_1, a_2, \ldots, a_n$ are real constant coefficients) should be sought in the form

$$u(x) = x^r e^{\alpha x}[P_l(x) \cos \beta x + Q_l(x) \sin \beta x].$$

Here r is equal to the multiplicity of the root $\alpha + \beta i$ in the characteristic equation $k^n + a_1k^{n-1} + \ldots + a_n = 0$ (if the characteristic equation does not possess such a root, we must put $r = 0$); $P_l(x)$ and $Q_l(x)$ are complete polynomials in x of degree l with undetermined coefficients, l being equal to the greater of the numbers n and m ($l = n \geqslant m$, or $l = m \geqslant n$);

$$P_l(x) = A_0x^l + A_1x^{l-1} + \ldots + A_l; \; Q_l(x) = B_0x^l + B_1x^{l-1} + \ldots + B_l.$$

It should be emphasized that the polynomials $P_l(x)$ and $Q_l(x)$ must be *complete* (that is, contain all the degree of x from 0 to l), with various undetermined coefficients in x of the same degree in both polynomials and, besides, if at least one of the functions $\cos \beta x$ and $\sin \beta x$ appears in the expression for the function $f(x)$, then *both* functions must be introduced into $u(x)$.

Undetermined coefficients may be found from the system of linear algebraic equations obtained by identifying the coefficients of the similar terms on the right-hand and left-hand sides of the original equation after substituting into it $u(x)$ for y.

By comparing *all* the terms of the right-hand side of the equation with the terms of the left-hand side similar to them which appeared in it after the substitution of $u(x)$ it is possible to check the correctness of the selected form of the particular solution.

If the right-hand side of the original equation is equal to the sum of several different functions of the structure in question, then to find the particular solution of such an equation use should be made of the *theorem on superposition of solutions*: one should find the particular solutions corresponding to the separate terms on the right-hand side and take their sum, which is precisely the particular solution of the original equation (that is, of the equation with the sum of the corresponding functions on the right-hand side).

Note. The particular cases of the function $f(x)$ of the structure being discussed (whose presence on the right-hand side of the equation makes it possible to apply the method of selection of a particular solution) are the following functions:

(1) $f(x) = Ae^{\alpha x}$, A being constant $[\alpha + \beta i \equiv \alpha]$,
(2) $f(x) = A\cos\beta x + B\sin\beta x$, A and B being constants $[\alpha + \beta i \equiv \beta i]$,
(3) $f(x) = P_n(x)$ (a polynomial of the nth degree) $[\alpha + \beta i \equiv 0]$,
(4) $f(x) = P_n^n(x)e^{\alpha x}$ $[\alpha + \beta i \equiv \alpha]$,
(5) $f(x) = P_n(x)\cos\beta x + Q_m(x)\sin\beta x$ $[\alpha + \beta i \equiv \beta i]$,
(6) $f(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$, A and B being constants.

675. Find the particular solution of the equation $y'' - 2y' - 3y = e^{4x}$, satisfying the boundary conditions $y|_{x = \ln 2} = 1$; $y|_{x = 2\ln 2} = 1$.

Solution. The characteristic equation $k^2 - 2k - 3 = 0$ has the roots $k_1 = 3$, $k_2 = -1$. The general solution of the corresponding homogeneous equation is $y = C_1e^{3x} + C_2e^{-x}$. The particular solution of the original equation should be sought in the form $u = Ae^{4x}$ (since there is no sine or cosine on the right-hand side, the coefficient in the exponential function is a zero-degree polynomial, i.e. $l = n = 0$, and $r = 0$ since $\alpha = 4$ is not a root of the characteristic equation).

Thus we have

$$+\left|\begin{array}{r|l} -3 & u = Ae^{4x} \\ -2 & u' = 4Ae^{4x} \\ 1 & u'' = 16Ae^{4x} \end{array}\right.$$

$$u'' - 2u' - 3u = 5Ae^{4x} \equiv e^{4x}.$$

If follows that $A = 1/5$. Consequently, the general solution of the given equation is

$$y = C_1e^{3x} + C_2e^{-x} + \frac{1}{5}e^{4x}.$$

To find C_1 and C_2, we make use of the boundary conditions

$$\begin{cases} C_1e^{3\ln 2} + C_2e^{-\ln 2} + \dfrac{1}{5}e^{4\ln 2} = 1, \\ C_1e^{6\ln 2} + C_2e^{-2\ln 2} + \dfrac{1}{5}e^{8\ln 2} = 1, \end{cases} \quad \text{or} \quad \begin{cases} 8C_1 + \dfrac{1}{2}C_2 + \dfrac{16}{5} = 1, \\ 64C_1 + \dfrac{1}{4}C_2 + \dfrac{256}{5} = 1. \end{cases}$$

Hence $C_1 = -491/600$, $C_2 = 652/75$. Thus we have

$$y = \frac{1}{5}e^{4x} + \frac{652}{75}e^{-x} - \frac{491}{600}e^{3x}.$$

676. Integrate the equation $y'' + y' - 2y = \cos x - 3\sin x$, subject to the initial conditions $y(0) = 1$, $y'(0) = 2$.

Solution. **The characteristic equation $k^2 + k - 2 = 0$ has the roots $k_1 = 1$, $k_2 = -2$ and, therefore, the general solution of the homogeneous equation is $y = C_1e^{-2x} + C_2e^{x}$. The particular solution of the nonhomogeneous equation should be sought in the form**

$$u = A\cos x + B\sin x$$

(in the given case $\alpha = 0$, $\beta = 1$, $\alpha + \beta i = i$; since there is no such a root of the characteristic equation, it follows that $r = 0$; $m = n = 0$, and, consequently, $l = 0$ as well).

Thus we have

$$
+\left|\begin{array}{r|l} -2 & u = A\cos x + B\sin x \\ 1 & u' = -A\sin x + B\cos x \\ 1 & u'' = -A\cos x - B\sin x \end{array}\right.
$$

$$u'' + u' - 2u = (B - 3A)\cos x + (-3B - A)\sin x \equiv \cos x - 3\sin x.$$

We have obtained a system

$$\begin{cases} B - 3A = 1, \\ 3B + A = 3, \end{cases} \quad \text{i.e.} \quad A = 0,\ B = 1.$$

Consequently, the general solution of the given equation has the form

$$y = C_1 e^{-2x} + C_2 e^x + \sin x.$$

Next we shall find C_1 and C_2, making use of the initial conditions

$$\begin{cases} C_1 \cdot e^0 + C_2 \cdot e^0 + \sin 0 = 1, \\ -2C_1 e^0 + C_2 e^0 + \cos 0 = 2, \end{cases} \quad \text{or} \quad \begin{cases} C_1 + C_2 = 1, \\ -2C_1 + C_2 = 1. \end{cases}$$

Hence $C_1 = 0$, $C_2 = 1$, i.e. $y = e^x + \sin x$.

677. Integrate the equation $y'' - y' = \cosh 2x$, subject to the initial conditions $y(0) = y'(0) = 0$.

Solution. The characteristic equation $k^2 - k = 0$ has the roots $k_1 = 0$, $k_2 = 1$. The general solution of the homogeneous equation is $y = C_1 + C_2 e^x$. The particular solution of the nonhomogeneous equation should be sought in the form $u = A\cosh 2x + B\sinh 2x$. Differentiating and substituting into the original equation, we get

$$
+\left|\begin{array}{r|l} 0 & u = A\cosh 2x + B\sinh 2x \\ -1 & u' = 2A\sinh 2x + 2B\cosh 2x \\ 1 & u'' = 4A\cosh 2x + 4B\sinh 2x \end{array}\right.
$$

$$u'' - u' = (4A - 2B)\cosh 2x + (4B - 2A)\sinh 2x \equiv \cosh 2x.$$

Thus we have

$$\begin{cases} 4A - 2B = 1, \\ -2A + 4B = 0; \end{cases} \quad A = 1/3, \quad B = 1/6.$$

This means that the general solution of the original equation is

$$y = C_1 + C_2e^x + \frac{1}{3} \cosh 2x + \frac{1}{6} \sinh 2x.$$

To find C_1 and C_2, we use the initial conditions:

$$\begin{cases} C_1 + C_2 \cdot e^0 + \frac{1}{3} \cosh 0 + \frac{1}{6} \sinh 0 = 0, \\ C_2 \cdot e^0 + \frac{2}{3} \sinh 0 + \frac{1}{3} \cosh 0 = 0, \end{cases} \text{ or } \begin{cases} C_1 + C_2 = -\frac{1}{3}, \\ C_2 + \frac{1}{3} = 0. \end{cases}$$

Consequently, $C_1 = 0$, $C_2 = -1/3$. Thus we see that the desired particular solution has the form

$$y = -\frac{1}{3} e^x + \frac{1}{3} \cosh 2x + \frac{1}{6} \sinh 2x.$$

Note. In accordance with the general theory, we should have presented the right-hand side of the given equation in the form $(1/2)(e^{2x} + e^{-2x})$ and apply the superposition theorem, that is, seek separate solutions corresponding to the summands $(1/2)e^{2x}$ and $(1/2)e^{-2x}$ of the right-hand side. We should have got

$$\alpha = 2, \beta = 0; \alpha + \beta i = 2; r = 0; n = l = 0 \text{ for } (1/2)e^{2x};$$

thus, $u_1(x) = A_1e^{2x}$;

$$\alpha_1 = -2, \beta_1 = 0; \alpha_1 + \beta_1 i = -2; r = 0; n_1 = l_1 = 0 \text{ for } (1/2)e^{-2x};$$

thus, $u_2(x) = B_1e^{-2x}$.

Therefore, we should have sought the particular solution in the form

$$u(x) = u_1(x) + u_2(x) = A_1e^{2x} + B_1e^{-2x},$$

but

$$\begin{aligned} A_1e^{2x} + B_1e^{-2x} &= A_1 (\cosh 2x + \sinh 2x) + B_1(\cosh 2x - \sinh 2x) \\ &= (A_1 + B_1) \cosh 2x + (A_1 - B_1) \sinh 2x = A \cosh 2x + B \sinh 2x. \end{aligned}$$

We have sought the solution of the given equation precisely in this form.

It should be noted, that, in general, when applying the method of selection of a particular solution, the latter is always sought as a function of the same structure as that of the right-hand side of the given equation, but the function should be appropriately supplemented with additional summands and multipliers ensuring the possibility of identifying the terms appearing after the substitution into the left-hand side of the equation with all, similar to them, terms of the right-hand side.

678. Solve the equation $y'' - 2y' + 2y = x^2$.

Solution. The characteristic equation $k^2 - 2k + 2 = 0$ has the roots $k_{1,2} = 1 \pm i$, and, therefore, the general solution of the homogeneous equation is $y = e^x(C_1 \cos x + C_2 \sin x)$. The particular solution should be sought in the form $u = Ax^2 + Bx + C$ (in the present case, $\alpha = 0$, $\beta = 0$; $\alpha + \beta i = 0$; since the characteristic equation has no zero root, it follows that $r = 0$; $n = l = 2$). Thus,

$$+\left|\begin{array}{r|l} 2 & u = Ax^2 + Bx + C \\ -2 & u' = 2Ax + B \\ 1 & u'' = 2A \end{array}\right.$$

$$u'' - 2u' + 2u = 2Ax^2 + (2B - 4A)x + (2C - 2B + 2A) \equiv x^2.$$

Hence,

$$2A = 1,\ 2B - 4A = 0,\ 2C - 2B + 2A = 0,$$

i.e. $A = 1/2$, $B = 1$, $C = 1/2$.

Consequently, the general solution of the desired equation is

$$y = e^x(C_1\cos x + C_2\sin x) + \frac{1}{2}(x + 1)^2.$$

679. Solve the equation $y'' + y = xe^x + 3e^{-x}$.

Solution. The characteristic equation $k^2 + 1 = 0$ has the roots $k_{1,2} = \pm i$, and, therefore, the general solution of the homogeneous equation $y = C_1 \cos x + C_2 \sin x$. Applying the superposition principle, the particular solution of the original equation should be sought in the form $u = u_1 + u_2 = (Ax + B)e^x + Ce^{-x}$(we have $f_1(x) = xe^x$, $\alpha_1 = 1$, $\beta_1 = 0$; $\alpha_1 + \beta_1 i = i$ for u_1; since there is no such a root, it follows that $r_1 = 0$, $n = l = 1$; we have $f_2(x) = 2e^{-x}$; $\alpha_2 = -1$, $\beta_2 = 0$; $\alpha_2 + \beta_2 i = -1$; $r_2 = 0$; $n_1 = l_1 = 0$ for u_2).

Thus,

$$+\left|\begin{array}{r|l} 1 & u = (Ax + B)e^x + Ce^{-x} \\ 0 & u' = Ae^x + (Ax + B)e^x - Ce^{-x} \\ 1 & u'' = 2Ae^x + (Ax + B)e^x + Ce^{-x} \end{array}\right.$$

$$u'' + u = 2Axe^x + (2A + 2B)e^x + 2Ce^{-x} \equiv xe^x2e^{-x}.$$

Hence,

$$2A = 1,\ 2A + 2B = 0,\ 2C = 2, \text{ i.e. } A = 1/2,\ B = -1/2,\ C = 1.$$

Consequently, the general solution of the original equation is

$$y = C_1 \cos x + C_2 \sin x + \frac{1}{2}(x - 1)e^x + e^{-x}.$$

680. Solve the equation $y''' + y'' - 2y' = x - e^x$.

Solution. The characteristic equation $k^3 + k^2 - 2k = 0$ has the roots $k_1 = 0$, $k_2 = 1$, $k_3 = -2$ and, therefore, the general solution of the homogeneous equation is $y = C_1 + C_2e^x + C_3e^{-2x}$. We seek the particular solution, using the superposition principle, in the form $u = u_1 + u_2 = x(Ax + B) + Cxe^x$.

Thus we have

$$
+\left|\begin{array}{r|l}
0 & u = (Ax + B)x + Cxe^x \\
-2 & u' = 2Ax + B + Ce^x + Cxe^x \\
1 & u'' = 2A + 2Ce^x + Cxe^x \\
1 & u''' = 2Ce^x + Cxe^x
\end{array}\right.
$$

$$u''' + u'' - 2u' = -4Ax + (2A - 3B) + 3Ce^x \equiv x - e^x.$$

Hence, $-4A = 1$, $2A - 2B = 0$, $3C = -1$, i.e. $A = -1/4$, $B = -1/4$, $C = -1/3$. Consequently, the general solution of the original equation is

$$y = C_1 + C_2e^x + C_3e^{-2x} - \frac{1}{4}\,x(x + 1) - \frac{1}{3}\,xe^x.$$

681. Find the solution of the equation $y'' + y = 3\sin x$ satisfying the boundary conditions $y(0) + y'(0) = 0$, $y(\pi/2) + y'(\pi/2) = 0$.

Solution. The characteristic equation $k^2 + 1 = 0$ has the roots $k_{1,2} = \pm i$ and, therefore, the general solution of the homogeneous equation is $y = C_1\cos x + C_2\sin x$. The particular solution should be sought in the form $u = x(A\cos x + B\sin x)$ (in the given case, $\alpha = 0$, $\beta = i$, $\alpha + \beta i = i$; since i is a simple root of the characteristic equation, it follows that $r = 1$; $m = n = l = 0$).

Thus we have

$$
+\left|\begin{array}{r|l}
1 & u = (A\cos x + B\sin x)\cdot x \\
0 & u' = (-A\sin x + B\cos x) + (A\cos x + B\sin x) \\
1 & u'' = 2(-A\sin x + B\cos x) + (-A\cos x - B\sin x)\cdot x
\end{array}\right.
$$

$$u'' + u = -2A\sin x + 2B\cos x \equiv 3\sin x.$$

Hence, $-2A = 3$, $2B = 0$, i.e. $A = -3/2$, $B = 0$.
Consequently, the general solution of the original equation is

$$y = C_1\cos x + C_2\sin x - \frac{3}{2}\,x\cos x.$$

We shall find the constants C_1 and C_2 using the boundary conditions. We have

$$y' = -C_1\sin x + C_2\cos x + \frac{3}{2}\,x\sin x - \frac{3}{2}\cos x,$$

and, then,

$$y(0) = C_1\cos 0 + C_2\sin 0 - \frac{3}{2}\cdot 0\cdot\cos 0 = C_1,$$

$$y'(0) = -C_1 \sin 0 + C_2 \cos 0 + \frac{3}{2} \cdot 0 \cdot \sin 0 - \frac{3}{2} \cos 0 = C_2 - \frac{3}{2},$$

$$y\left(\frac{\pi}{2}\right) = C_1 \cos\frac{\pi}{2} + C_2 \sin\frac{\pi}{2} - \frac{3}{2} \cdot \frac{\pi}{2} \cdot \cos\frac{\pi}{2} = C_2.$$

$$y'\left(\frac{\pi}{2}\right) = -C_1 \sin\frac{\pi}{2} + C_2 \cos\frac{\pi}{2} + \frac{3}{2} \cdot \frac{\pi}{2} \sin\frac{\pi}{2} - \frac{3}{2} \cos\frac{\pi}{2} = -C_1 + \frac{3}{4}\pi.$$

Thus,

$$y(0) + y'(0) = C_1 + C_2 - 3/2 = 0,$$
$$y(\pi/2) + y'(\pi/2) = C_2 - C_1 + 3\pi/4 = 0,$$

whence we get the system of equations

$$\begin{cases} C_1 + C_2 = 3/2, \\ C_1 - C_2 = 3\pi/4. \end{cases}$$

Solving the system, we find $C_1 = 3(2 + \pi)/8$, $C_2 = 3(2 - \pi)/8$. Thus, the solution of the original equation, satisfying the boundary conditions, has the form

$$y = \frac{3}{8}[(\pi + 2)\cos x - (\pi - 2)\sin x] - \frac{3}{2} x \cos x.$$

682. Find the solution of the equation $y'' + y = \tan x$, satisfying the boundary conditions $y(0) = y(\pi/6) = 0$.

Solution. The characteristic equation $k^2 + 1 = 0$ has the roots $k_{1,2} = \pm i$, and, therefore, the general solution of the homogeneous equation is $y = C_1 \cos x + C_2 \sin x$. The particular solution of the original equation cannot be sought by the method of undetermined coefficients (the structure of the function $f(x)$ differs from the previous cases), and, therefore, we shall use the method of variation of arbitrary constants. We shall seek the solution of the equation in the form

$$y = C_1(x) \cos x + C_2(x) \sin x,$$

where the functions $C_1(x)$ and $C_2(x)$ should be found from the system of equations

$$\begin{cases} C_1'(x)y_1 + C_2'(x)y_2 = 0, \\ C_1'(x)y_1' + C_2'(x)y_2' = f(x), \end{cases} \quad \text{or} \quad \begin{cases} C_1'(x)\cos x + C_2'(x)\sin x = 0, \\ -C_1'(x)\sin x + C_2'(x)\cos x = \tan x. \end{cases}$$

Solving this system, we get $C_1'(x) = -\sin^2 x/\cos x$, $C_2'(x) = \sin x$, whence we get

$$C_1(x) = -\int \frac{\sin^2 x}{\cos x} dx + A = \sin x - \ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) + A;$$

$$C_2(x) = -\cos x + B.$$

(We could have used the formulas presented on p. 177 instead of solving this system.)

Thus, the general solution of the original equation is

$$y = A\cos x + B\sin x - \cos x \cdot \ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right),$$

where A and B are arbitrary constants which should be found with the aid of the boundary conditions:

$$\begin{cases} A\cos 0 + B\sin 0 - \cos 0 \cdot \ln\tan(\pi/4) = 0, \\ A\cos(\pi/6) + B\sin(\pi/6) - \cos(\pi/6)\ln\tan(\pi/3) = 0. \end{cases}$$

From this we have $A = 0$, $B = (\sqrt{3}/2)\ln 3$. Consequently, the solution satisfying the boundary conditions has the form

$$y = \frac{\sqrt{3}}{2}\ln 3\sin x - \cos x\ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

683. A homogeneous chain suspended freely from a hook is slipping from it due to its gravity (the friction can be neglected). Determine the time needed for the whole chain to slip from the hook, if at the initial moment a piece of 10 metres of the chain was hanging on one side and a piece of 8 metres, on the other side, and the speed of the chain is equal to zero.

Solution. Suppose the weight of one linear metre of the chain is P N. We shall designate by x the length (in metres) of the longer part of the chain hanging from the hook t seconds after the beginning of motion. A force $F = [x - (18 - x)]P$ N is applied to the centre of gravity of the chain. The mass of the chain is $18P/g$ kg, and its acceleration is $\ddot{x}$ m/s^2. Thus, we arrive at the equation of motion of the centre of gravity of the chain:

$$\frac{18}{g}P\ddot{x} = (2x - 18)P, \text{ or } \ddot{x} - \frac{g}{9}x = -g.$$

This equation must be integrated under the initial conditions: $x = 10$, $\dot{x} = 0$ for $t = 0$.

The roots of the characteristic equation $k_{1,2} = \pm\sqrt{g}/3$; the particular solution of the nonhomogeneous equation should be sought in the form $u = A$; after substitution into the equation, we get $A = 9$. Thus, the general solution of the equation has the form

$$x = C_1e^{t\sqrt{g}/3} + C_2e^{-t\sqrt{g}/3} + 9.$$

Making use of the initial conditions, we obtain

$$\begin{cases} C_1 + C_2 + 9 = 10, \\ \dfrac{\sqrt{g}}{3}(C_1 - C_2) = 0, \end{cases}$$

whence $C_1 = C_2 = 0.5$. Consequently,

$$x = (e^{t\sqrt{g}/3} + e^{-t\sqrt{g}/3})/2 + 9 = 9 + \cosh(t\sqrt{g}/3).$$

The time needed for the whole chain to slip down is determined from the condition $x = 18$ for $t = T$. Hence we have

$$18 = 9 + \cosh\left(\frac{T\sqrt{g}}{3}\right), \text{ or } \frac{e^{T\sqrt{g}/3} + e^{-T\sqrt{g}/3}}{2} = 9.$$

Solving the equation obtained for T, we find

$$T = \frac{3}{\sqrt{g}} \ln(9 + 4\sqrt{5}) \approx 2.76 \text{ s}.$$

Solve the following equations:

684. $y'' - 4y' + 3y = e^{5x}$; $y(0) = 3$, $y'(0) = 9$.
685. $y'' - 8y' + 16y = e^{4x}$; $y(0) = 0$, $y'(0) = 1$.
686. $y'' - 6y' + 25y = 2\sin x + 3\cos x$.
687. $y'' + y = \cos 3x$; $y(\pi/2) = 4$, $y'(\pi/2) = 1$.
688. $y'' - 6y' + 8y = 3x^2 + 2x + 1$.
689. $2y'' - y' = 1$; $y(0) = 0$, $y'(0) = 1$.
690. $y'' + 4y = \sin 2x + 1$; $y(0) = 1/4$, $y'(0) = 0$.
691. $y'' - 4y' = 2\sinh 2x$.
692. $y'' + 4y = \cos 2x$; $y(0) = y(\pi/4) = 0$.
693. $y'' + 3y' - 10y = xe^{-2x}$.
694. $y'' - (\alpha + \beta)y' + \alpha\beta y = ae^{\alpha x} + be^{\beta x}$.
695. $y'' - y = x\cos^2 x$.
696. $y'' - 9y' + 20y = x^2e^{4x}$.
697. $y'' - y = 2\sinh x$; $y(0) = 0$, $y'(0) = 1$.
698. $y'' - 4y = \cosh 2x$.
699. $y'' - 2y'\cos\varphi + y = 2\sin x \cos\varphi$.
700. $y'' - 2y' + 2y = e^x \sin x$.
701. $y'' + 9y = 2\sin x \sin 2x$; $y(0) = y(\pi/2) = 0$.

702. Show that the general solution of the differential equation $y'' - m^2y = 0$ can be represented in the form $y = C_1\cosh mx + C_2\sinh mx$.

703. Show that the general solution of the differential equation $y'' - 2\alpha y' + (\alpha^2 - \beta^2)y = 0$ can be represented in the form $y = e^{\alpha x}(C_1\cosh\beta x + C_2\sinh\beta x)$.

704. Determine the law of motion of a particle of mass m, moving along a straight line under the action of a restoring force which is directed towards the point of reference of the motion and is directly proportional to the distance of the particle

from the reference point, if there is no resistance of the medium, but an extermal force $F = A \sin \omega t$ is acting on the particle.

Use the method of variation of arbitrary constants in the following four problems.

705. $y'' + y = 1/\sqrt{\cos 2x}$.

706. $y'' + 5y' + 6y = 1/(1 + e^{2x})$.

707. $y'' + 4y = \cot 2x$.

708. $y'' \cos(x/2) + (1/4)y \cos(x/2) = 1$.

709. Solve Problem 683 taking into account the friction of the chain against the hook, if the force of friction is equal to the weight of 1 metre of the chain.

Hint. The equation of motion of the centre of gravity of the chain has the form

$$18 \frac{d^2x}{dt^2} = gx - (18 - x)g - g \cdot 1.$$

4.3.5. Euler's equation. The linear equation with variable coefficients of the form

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \ldots + a_{n-1} x y' + a_n y = f(x) \tag{1}$$

or of a more general form

$$(ax + b)^n y^{(n)} + a_1 (ax + b)^{n-1} y^{(n-1)} + \ldots + a_{n-1}(ax + b)y' + a_n y = f(x) \tag{2}$$

is known as *Euler's equation*. Here a_i are constant coefficients. By means of the substitutions $x = e^t$ for (1) and $ax + b = e^t$ for (2) these two equations can be transformed into linear equations with constant coefficients.

710. Solve the equation $x^2y'' - xy' + y = 0$.

Solution. Putting $x = e^t$, or $t = \ln x$, whence $dt/dx = 1/x = e^{-t}$, we get

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \dot{y}e^{-t},$$

$$y'' = \frac{d}{dt}[e^{-t}\dot{y}] \frac{dt}{dx} = (\dot{y}e^{-t})'_t \cdot e^{-t} = (\ddot{y} - \dot{y})e^{-2t}$$

(we denote by dots the differentiation with respect to t). Then, the original equation assumes the form

$$e^{2t} \cdot e^{-2t}(\ddot{y} - \dot{y}) - e^t \cdot e^{-t} \cdot \dot{y} + y = 0, \text{ or } \ddot{y} - 2\dot{y} + y = 0.$$

The characteristic equation $k^2 - 2k + 1 = 0$ has the roots $k_1 = k_2 = 1$. Consequently, the general solution is

$$y = (C_1 + C_2 t)e^t, \text{ or } y = (C_1 + C_2 \ln x)x.$$

711. Solve the equation $(4x - 1)^2 y'' - 2(4x - 1)y' + 8y = 0$.

Solution. We put $4x - 1 = e^t$. Then $dx = \frac{1}{4} e^t\, dt$, $\frac{dt}{dx} = 4e^{-t}$. Hence

$$y' = \frac{dy}{dt} \cdot \frac{dt}{dx} = 4e^{-t} \cdot \dot{y}, \quad y'' = 16e^{-2t}(\ddot{y} - \dot{y}).$$

The original equation assumes the form

$$16e^{2t} \cdot e^{-2t}(\ddot{y} - \dot{y}) - 4 \cdot 2e^{t} \cdot e^{-t} \cdot \dot{y} + 8y = 0, \text{ or } 2\ddot{y} - 3\dot{y} + y = 0.$$

The characteristic equation $2k^2 - 3k + 1 = 0$ has the roots $k_1 = 1$, $k_2 = 1/2$ Consequently, the general solution is

$$y = C_1e^t + C_2e^{t/2}, \text{ or } y = C_1(4x - 1) + C_2\sqrt{4x - 1}.$$

712. Solve the equation $y'' - xy' + y = \cos \ln x$.

Solution. We put $x = e^t$. Then $t = \ln x$, $\frac{dt}{dx} = \frac{1}{x} = e^{-t}$. Consequently, $y' = \dot{y} \cdot e^{-t}$, $y'' = (\ddot{y} - \dot{y})e^{-2t}$. The given equation will assume the form

$$\ddot{y} - 2\dot{y} + y = \cos t.$$

The general solution of the homogeneous equation is $y = (C_1 + C_2t)e^t$, and the particular solution of the nonhomogeneous equation should be sought in the form $u = A \cos t + B \sin t$. Then we have

$$\begin{array}{c|r|lcl} & 1 & u & = & A \cos t + B \sin t \\ + & -2 & u' & = & -A \sin t + B \cos t \\ & 1 & u'' & = & -A \cos t - B \sin t \\ \hline \end{array}$$

$$u'' - 2u' + u = -2B \cos t + 2A \sin t \equiv \cos t,$$

whence $B = -1/2$, $A = 0$. Consequently, the general solution of the original equation is

$$y = (C_1 + C_2t)e^t - \frac{1}{2} \sin t, \text{ or } y = (C_1 + C_2 \ln x)x - \frac{1}{2} \sin \ln x.$$

Solve the following equations:

713. $x^2y'' - xy' + 2y = 0$.
714. $x^2y'' - 3xy' + 3y = 3 \ln^2 x$.
715. $x^2y'' + xy' + y = \sin (2 \ln x)$.
716. $x^2y'' + 3xy' + y = 1/x$; $y(1) = 1$, $y'(1) = 0$.
717. $x^2y'' - 3xy' + 4y = x^3/2$; $y(1) = 1/2$, $y(4) = 0$.

4.4. Integration of Differential Equations with the Aid of Series

4.4.1. Application of series to solving differential equations. In certain cases, when integration of a differential equation in elementary functions is impossible, the solution of such an equation is sought in the form of a power series:

$$y = \sum_{n=0}^{\infty} C_n(x - x_0)^n.$$

The undetermined coefficients C_n ($n = 0, 1, 2, \ldots$) are found by means of a substitution of the series into the equation and equating the coefficients in the like powers of the difference $x - x_0$ in both parts of the equality obtained. If we manage to find all the coefficients of the series, then the series obtained determines the solution over the whole domain of its convergence.

In the cases when it is required to solve Cauchy's problem subject to the initial condition $y|_{x=x_0} = y_0$ for the equation $y' = f(x, y)$, the solution may be sought with the aid of Taylor's series

$$y = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}(x - x_0)^n,$$

where $y(x_0) = y_0$, $y'(x_0) = f(x_0, y_0)$, and the higher derivatives $y^{(n)}(x_0)$ can be found by means of successive differentiation of the original equation and by replacing $x, y, y', \ldots$ in the result of the differentiation by the values x_0, y_0, y_0' and all the rest consecutive derivatives obtained. Similar integration of higher-order equations can be performed with the aid of Taylor's series.

718. Integrate the equation $y'' - x^2y = 0$.

Solution. We shall seek the solution of this equation in the form of the series

$$y = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n + \ldots$$

Substituting y and y'' into the original equation, we find

$$[2 \cdot 1C_2 + 3 \cdot 2C_3x + 4 \cdot 3C_4x^2 + \ldots + (n + 2)(n + 1)C_{n+2}x^n + \ldots]$$
$$- x^2[C_0 + C_1x + C_2x^2 + \ldots + C_nx^n + \ldots] \equiv 0.$$

Collecting the terms with like powers of x, we get

$$2 \cdot 1C_2 + 3 \cdot 2C_3x^2 + \sum_{n=0}^{\infty} [(n + 4)(n + 3)C_{n+4} - C_n]x^{n+2} \equiv 0.$$

Equating to zero all the coefficients of the series obtained (to turn the equation into

an identity), we find

$$C_2 = C_3 = 0;\ C_{n+4} = \frac{C_n}{(n+3)(n+4)}\ (n = 0, 1, 2, \dots).$$

The last relation makes possible consecutive determination of all the coefficients of the desired expansion (C_0 and C_1 remain arbitrary and play the role of the arbitrary constants of integration):

$$C_{4k} = \frac{C_0}{3 \cdot 4 \cdot 7 \cdot 8 \dots (4k-1) \cdot 4k},\ C_{4k+1} = \frac{C_1}{4 \cdot 5 \cdot 8 \cdot 9 \dots 4k(4k+1)};$$

$$C_{4k+2} = C_{4k+3} = 0 \quad (k = 0, 1, 2, \dots).$$

Thus we have

$$y = C_0 \sum_{k=0}^{\infty} \frac{x^{4k}}{3 \cdot 4 \cdot 7 \cdot 8 \dots (4k-1)4k} + C_1 \sum_{k=0}^{\infty} \frac{x^{4k+1}}{4 \cdot 5 \cdot 8 \cdot 9 \dots 4k(4k+1)}.$$

The series obtained converge throughout the number axis and define two linearly independent particular solutions of the original equation.

Integrate the following equations expanding them into series in the powers of x and determine the domain of existence of the solution obtained:

719. $y' + xy = 0.$
720. $y' = x - 2y;\ y(0) = 0.$

Hint. In accordance with the initial condition put $C_0 = 0$.

721. $y'' + xy' + y = 0.$
722. $y'' - xy' - 2y = 0.$
723. $y'' + x^2y = 0;\ y(0) = 0;\ y'(0) = 1.$

Hint. In accordance with the initial conditions put $C_0 = 0$, $C_1 = 1$.

724. Using Taylor's series, perform an approximate integration of the equation $y' = x^2 + y^2$, $y(0) = 1$, taking the first six nonzero terms of the expansion.

Solution. From the equation and the initial conditions we find $y'(0) = 0^2 + 1^2 = 1$. Successive differentiation of the given equation yields

$$y'' = 2x + 2yy',\ y''' = 2 + 2y'^2 + 2yy'',\ y^{IV} = 6y'y'' + 2yy''',$$
$$y^{V} = 6y''^2 + 8y'y''' + 2yy^{IV}.$$

Setting $x = 0$ and using the values $y(0) = 1$, $y'(0) = 1$, we find consecutively: $y''(0) = 2$, $y'''(0) = 8$, $y^{IV}(0) = 28$, $y^{V}(0) = 144$. The solution sought has the form

$$y = 1 + \frac{x}{1!} + \frac{2x^2}{2!} + \frac{8x^3}{3!} + \frac{28x^4}{4!} + \frac{144x^5}{5!} + \ldots .$$

725. $y'' = x + y^2$, $y(0) = 0$, $y'(0) = 1$. Find the first four (different from zero) terms of the expansion.

Solution. Differentiating the equation $y'' = x + y^2$, we have

$$y''' = 1 + 2yy', \; y^{IV} = 2yy'' + 2y'^2, \; y^{V} = 2yy''' + 6y'y'',$$

$$y^{VI} = 2yy^{IV} + 8y'y^{IV} + 6y''^2.$$

At $x = 0$, we get

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 1, \quad y^{IV}(0) = 2, \quad y^{V}(0) = 0,$$
$$y^{VI}(0) = 16.$$

The solution is of the form

$$y = \frac{x}{1!} + \frac{x^3}{3!} + \frac{2x^4}{4!} + \frac{16x^6}{6!} + \ldots = x + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots .$$

726. $y' = x^2y + y^3$, $y(0) = 1$. Find the first four (nonzero) terms of the expansion.

727. $y' = x + 2y^2$, $y(0) = 0$. Find the first two (nonzero) terms of the expansion.

728. $y'' - xy^2 = 0$, $y(0) = 1$, $y'(0) = 1$. Find the first four (nonzero) terms of the expansion.

729. $y' = 2x - y$; $y(0) = 2$. Find the exact solution.

730. $y' = y^2 + x$; $y(0) = 1$. Find the first five terms of the expansion.

731. $y'' = (2x - 1)y - 1$; $y(0) = 0$, $y'(0) = 1$. Find the first five terms of the expansion.

4.4.2. Bessel's equations. The linear differential equation

$$x^2y'' + xy' + (x^2 - \lambda^2)y = 0 \qquad (\lambda = \text{const}) \tag{1}$$

with variable coefficients is known as *Bessel's equation* (the equation $x^2y'' + xy' + (m^2x^2 - \lambda^2)y = 0$ can be reduced to the same form by the substitution $mx = \xi$).

The solution of equation (1) will be sought in the form of the generalized power series, that is, the product of a certain power of x by the power series:

$$y = x^r(a_0 + a_1x + a_2x^2 + \dots) = \sum_{k=0}^{\infty} a_k x^{r+k}. \tag{2}$$

Substituting the generalized power series into equation (1) and equating to zero the coefficients in each power of x on the left-hand side of the equation, we get the system

$$\begin{array}{r|l}
x^r & (r^2 - \lambda^2) \cdot a_0 = 0, \\
x^{r+1} & [(r+1)^2 - \lambda^2] \cdot a_1 = 0, \\
x^{r+2} & [(r+2)^2 - \lambda^2] \cdot a_2 + a_0 = 0, \\
\dots & \dots\dots\dots\dots\dots\dots\dots\dots \\
x^{r+k} & [(r+k)^2 - \lambda^2] \cdot a_k + a_{k-2} = 0. \\
\dots & \dots\dots\dots\dots\dots\dots\dots\dots
\end{array}$$

Assuming that $a_0 \neq 0$, we find from the given system that $r_{1,2} = \pm\lambda$. Suppose $r_1 = \lambda$. Then, from the second equation of the system we find $a_1 = 0$, and from the equation $[(r+k)^2 - \lambda^2]a_k = -a_{k-2}$, assigning to k the values 3, 5, 7, ... , we infer that $a_3 = a_5 = a_7 = \dots = a_{2k+1} = 0$. For the coefficients with even numbers we get the expressions

$$a_2 = \frac{-a_0}{(2\lambda + 2) \cdot 2}, \; a_4 = \frac{-a_2}{(2\lambda + 4) \cdot 4} = \frac{a_0}{(\lambda + 1)(\lambda + 2) \cdot 1 \cdot 2 \cdot 2^4}, \dots ,$$

$$a_{2k} = \frac{-a_{2k-2}}{(2\lambda + 2) \cdot 2 \cdot k}$$

$$= (-1)^{k+1} \cdot \frac{a_0}{2 \cdot 4 \cdot 6 \dots 2k(2\lambda + 2)(2\lambda + 4) \dots (2\lambda + 2k)}.$$

Substituting the coefficients obtained into series (2), we get the solution

$$y_1(x) = a_0 \cdot x^\lambda \left[1 - \frac{x^2}{2(2\lambda + 2)} + \frac{x^4}{2 \cdot 4(2\lambda + 2)(2\lambda + 4)} - \frac{x^6}{2 \cdot 4 \cdot 6\,(2\lambda + 2)(2\lambda + 4)(2\lambda + 6)} + \dots \right]$$

$$= a_0 \cdot \sum_{k=0}^{\infty} \frac{(-1)^k x^{\lambda + 2k}}{4^k k!\,(\lambda + 1)(\lambda + 2) \dots (\lambda + k)},$$

where the coefficient a_0 remains arbitrary.

At $r_2 = -\lambda$ all the coefficients a_k are determined by analogy only when λ is not an integer. Then the solution can be obtained by substituting, in the previous solution $y_1(x)$, the value of $-\lambda$ for λ:

$$y_2(x) = a_0 x^{-\lambda} \left[1 - \frac{x^2}{2(-2\lambda + 2)} + \frac{x^4}{2 \cdot 4\,(-2\lambda + 2)(-2\lambda + 4)} \right.$$

$$-\frac{x^6}{2\cdot 4\cdot 6\,(-2\lambda+2)(-2\lambda+4)(-2\lambda+6)}+\ldots\Bigg]$$

$$=a_0\sum_{k=0}^{\infty}\frac{(-1)^k x^{-\lambda+2k}}{4^k k!\,(-\lambda+1)(-\lambda+2)\ldots(-\lambda+k)}.$$

The power series obtained converge for all the values of x, which fact can be easily ascertained by means of D'Alembert's test. The solutions $y_1(x)$ and $y_2(x)$ are linearly independent since their ratio is not constant.

The solution $y_1(x)$ multiplied by the constant $a_0 = \dfrac{1}{2^\lambda\Gamma(\lambda+1)}$ is known as the *Bessel function of order* λ *of the first kind* and is designated as $J_\lambda(x)$. The solution y_2 is designated $J_{-\lambda}(x)$.

Consequently, the general solution of equation (1) for λ not equal to an integer has the form

$$y(x) = C_1 J_\lambda(x) + C_2 J_{-\lambda}(x),$$

where C_1 and C_2 are arbitrary constant quantities.

The generally accepted method of choosing the constant a_0 requires the participation of the gamma-function $\Gamma(\lambda+1)$, which is defined by the improper integral (see p. 41):

$$\Gamma(\lambda) = \int_0^\infty e^{-x}x^{\lambda-1}dx\ (\lambda > 0).$$

It can be shown that at λ equal to half an odd number, the Bessel function is expressed in terms of elementary functions since in that case the gamma-function appearing in the definition of the Bessel function

$$J_\lambda(x) = \frac{1}{2^\lambda\cdot\Gamma(\lambda+1)}\cdot\sum_{k=0}^{\infty}\frac{(-1)^k\cdot x^{\lambda+2k}}{4^k\cdot k!\,(\lambda+1)(\lambda+2)\ldots(\lambda+k)}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(\lambda+k+1)}\cdot\left(\frac{x}{2}\right)^{\lambda+2k},$$

(the product $(\lambda+1)(\lambda+2)\ldots(\lambda+k)\Gamma(\lambda+1)$ is replaced, according to the property of the gamma-function, by $\Gamma(\lambda+k+1)$) assumes the following values:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}\cdot x^{-1/2}dx = 2\int_0^\infty e^{-t^2}dt = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

(use is made here of Poisson's integral):

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi};$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(1 + \frac{3}{2}\right) = \frac{3}{2} \cdot \Gamma\left(\frac{3}{2}\right) = \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi};$$

$$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(1 + \frac{5}{2}\right) = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}; \ldots .$$

For $\lambda = n$ (natural), Bessel's function J_λ can be briefly written as follows:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n + k + 1)} \cdot \left(\frac{x}{2}\right)^{2k+n} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n + k)!} \cdot \left(\frac{x}{2}\right)^{2k+n}.$$

For a negative and integral λ the particular solution is not expressed by the Bessel function of the first kind and must be sought in the form

$$K_n(x) = J_n(x) \cdot \ln x + x^{-n} \sum_{k=0}^{\infty} b_k x^k.$$

Substituting this expression into equation (1), we shall determine the coefficients b_k. The function $K_n(x)$, multiplied by some constant, is called *Bessel's function of the nth order of the second kind.*

732. Find Bessel's function at $\lambda = 0$.
Solution. Using the equality

$$J_\lambda(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\lambda + k + 1)} \left(\frac{x}{2}\right)^{2k+\lambda},$$

we obtain for $\lambda = 0$

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k + 1)} \left(\frac{x}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k}}{4^k \cdot k! \cdot k!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k}}{4^k \cdot (k!)^2} = 1 - \frac{x^2}{4} + \frac{x^4}{4^2 \cdot (1 \cdot 2)^2} - \frac{x^6}{4^3(1 \cdot 2 \cdot 3)^2} + \ldots .$$

733. Solve the equation $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$.

Solution. Since $\lambda = 1/2$, the general solution of the equation has the form

$$y = C_1J_{1/2} + C_2J_{-1/2},$$

where

$$J_{1/2}=\frac{1}{2^{1/2}\cdot\Gamma\left(\frac{3}{2}\right)}\cdot x^{1/2}\cdot\left[1-\frac{x^2}{2\cdot 3}+\frac{x^4}{2\cdot 4\cdot 3\cdot 5}-\frac{x^6}{2\cdot 4\cdot 6\cdot 3\cdot 5\cdot 7}+\dots\right]$$

$$=\frac{2}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{x}}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots\right)=\sqrt{\frac{2}{\pi}}\cdot\frac{\sin x}{\sqrt{x}}.$$

In the same way we obtain $J_{-1/2}=\sqrt{\frac{2}{\pi}}\cdot\frac{\cos x}{x}$.

Consequently, the general solution is

$$y=\sqrt{\frac{2}{\pi x}}\,(C_1\sin x+C_2\cos x).$$

734. Find $J_1(x)$

735. Solve the equation $x^2y''+xy'+\left(x^2-\frac{9}{4}\right)y=0.$

736. Solve the equation $x^2y''+xy'+\left(x^2-\frac{4}{9}\right)y=0.$

4.5. Systems of Differential Equations

4.5.1. System of differential equations in normal form. The system of differential equations of the form

$$\begin{cases}\dfrac{dx_1}{dt}=f_1(t,x_1,x_2,\dots,x_n),\\ \dfrac{dx_2}{dt}=f_2(t,x_1,x_2,\dots,x_n),\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ \dfrac{dx_n}{dt}=f_n(t,x_1,x_2,\dots,x_n),\end{cases}$$

where $x_1, x_2, \dots, x_n$ are the unknown functions of the independent variable t, is said to be a *system in normal form*.

If the right-hand sides of the system of differential equations in normal form are linear functions with respect to $x_1, x_2, \dots, x_n$, then the system of the differential equations is said to be *linear*.

Sometimes, it is possible to reduce a system of differential equations in normal form to one equation of the nth order containing one unknown function. This can be done by differentiating one of the equations of the system and eliminating all the unknowns except one (the so-called *method of elimination*).

In certain cases, combining the equations of the system, it is possible to obtain, after simple transformations, easily integrable equations (the so-called *method of integrable combinations*), which enables us to find the solution of the system.

737. Solve the system of differential equations

$$\frac{dx}{dt} = x + y, \quad \frac{dy}{dt} = x - y$$

subject to the initial conditions $x(0) = 2$, $y(0) = 0$.

Solution. Let us differentiate the first equation with respect to t: $\frac{d^2x}{dt^2} = \frac{dx}{dt} + \frac{dy}{dt}$; eliminating $\frac{dy}{dt}$ and y from the equation obtained, we get $\frac{d^2x}{dt^2} - 2x = 0$. The characteristic equation $k^2 - 2 = 0$ has the roots $k_{1,2} = \pm\sqrt{2}$. Consequently, the general solution for x can be written in the form

$$x = C_1 e^{t\sqrt{2}} + C_2 e^{-t\sqrt{2}}.$$

The general solution for y can be found from the first equation:

$$y = \frac{dx}{dt} - x = C_1(\sqrt{2} - 1)e^{t\sqrt{2}} - C_2(\sqrt{2} + 1)e^{-t\sqrt{2}}.$$

Let us use the initial conditions to find the arbitrary constants:

$$C_1 + C_2 = 2, \sqrt{2}\,(C_1 - C_2) - (C_1 + C_2) = 0.$$

It follows from this that $C_1 = (\sqrt{2} + 2)/2$, $C_2 = (2 - \sqrt{2})/2$. Thus, the desired particular solution has the form

$$x = \left(\frac{\sqrt{2}}{2} + 1\right)e^{t\sqrt{2}} + \left(1 - \frac{\sqrt{2}}{2}\right)e^{-t\sqrt{2}}, \quad y = \frac{\sqrt{2}}{2}e^{t\sqrt{2}} - \frac{\sqrt{2}}{2}e^{-t\sqrt{2}}.$$

738. Solve the system of differential equations

$$\frac{dx}{dt} = \frac{x}{2x + 3y}, \quad \frac{dy}{dt} = \frac{y}{2x + 3y}$$

subject to the initial conditions $x(0) = 1$, $y(0) = 2$.

Solution. Let us derive the first integrable combination. Dividing the first equation by the second, we get

$$\frac{dx}{dy} = \frac{x}{y}; \ \frac{dx}{x} = \frac{dy}{y}; \ \ln x = \ln y + \ln C_1, \text{ i.e. } x = C_1 y.$$

Now we derive the second integrable combination. Adding up the doubled first equation and the trebled second equation, we obtain

$$2 \cdot \frac{dx}{dt} + 3 \cdot \frac{dy}{dt} = 1; \ 2\,dx + 3\,dy = dt, \text{ i.e. } 2x + 3y = t + C_2.$$

The system of equations $x = C_1 y$, $2x + 3y = t + C_2$ yields the general solution of the system:

$$x = \frac{C_1(t + C_2)}{2C_1 + 3}, \ y = \frac{t + C_2}{2C_1 + 3}.$$

Using the initial conditions, we get

$$1 = \frac{C_1 C_2}{2C_1 + 3}, \ 2 = \frac{C_2}{2C_1 + 3}, \text{ i.e. } C_1 = \frac{1}{2}, \ C_2 = 8.$$

Substituting the values of C_1 and C_2 obtained into the general solution, we get the particular solutions satisfuing the initial conditions:

$$x = \frac{1}{8}t + 1, \quad y = \frac{1}{4}t + 2.$$

739. Solve the system of differential equations

$$\frac{dx}{dt} = 2y, \ \frac{dy}{dt} = 2z, \ \frac{dz}{dt} = 2x.$$

Solution. We differentiate the first equation with respect to t: $\dfrac{d^2x}{dt^2} = 2 \cdot \dfrac{dy}{dt}$. Eliminating $\dfrac{dy}{dt}$ from the equation obtained, we get $\dfrac{d^2x}{dt^2} = 4z$. Now we differentiate the second-order equation we have obtained with respect to t: $\dfrac{d^3x}{dt^3} = 4 \cdot \dfrac{dz}{dt}$. Eliminating $\dfrac{dz}{dt}$, we get

$$\frac{d^3x}{dt^3} - 8x = 0,$$

that is, we have arrived at an equation in one unknown function. Solving this homogeneous linear equation of the third order, we find

$$x = C_1 e^{2t} + e^{-t}(C_2 \cos t\sqrt{3} + C_3 \sin t\sqrt{3}).$$

We obtain the general solution for y from the first equation of the system:

$$y = \frac{1}{2}\cdot\frac{dx}{dt} = \frac{1}{2}[2C_1 e^{2t} - e^{-t}(C_2 \cos t\sqrt{3} + C_3 \sin t\sqrt{3}) + e^{-t}\sqrt{3}(C_3 \cos t\sqrt{3} - C_2 \sin t\sqrt{3})],$$

or

$$y = C_1 e^{2t} + \frac{1}{2}e^{-t}[(C_3\sqrt{3} + C_2)\cos t\sqrt{3} - (C_2\sqrt{3} + C_3)\sin t\sqrt{3}].$$

The second equation of the system yields the solution for z:

$$z = \frac{1}{2}\cdot\frac{dy}{dt} = C_1 e^{2t} - \frac{1}{2}e^{-t}[(C_3\sqrt{3} + C_2)\cos t\sqrt{3} - (C_2\sqrt{3} - C_3)\sin t\sqrt{3}].$$

Solve the following systems of differential equations:

740. $\dfrac{dx}{dt} = 2x + y, \quad \dfrac{dy}{dt} = x + 2y; \quad x(0) = 1,\ y(0) = 3.$

741. $\dfrac{dx}{dt} = 4x + 6y, \quad \dfrac{dy}{dt} = 2x + 3y + t.$

742. $\dfrac{dx}{dt} = e^{3t} - y, \quad \dfrac{dy}{dt} = 2e^{3t} - x.$

743. $y' = e^x - z, \quad z' = e^{-x} + y.$

744. $\dfrac{dx}{dt} = y + t, \quad \dfrac{dy}{dt} = x + e^t; \quad x(0) = 1,\ y(0) = 0.$

745. $\dfrac{dx}{dt} = \dfrac{x}{x + y}, \quad \dfrac{dy}{dt} = \dfrac{y}{x + y}; \quad x(0) = 2,\ y(0) = 4.$

746. $\dfrac{dx}{dt} = 2x + y + \cos t, \quad \dfrac{dy}{dt} = -x + 2\sin t.$

747. $\dfrac{dx}{dt} + \dfrac{dy}{dt} = 2(x + y), \quad \dfrac{dy}{dt} = 3x + y.$

748. $\dfrac{dx}{dt} + \dfrac{2x}{t} = 1, \quad \dfrac{dy}{dt} = x + y + \dfrac{2x}{t} - 1.$

749. $\dfrac{dx}{dt} = x^2 + xy, \quad \dfrac{dy}{dt} = xy + y^2.$

Hint. Consider two integrable combinations: (1) sum up the equations; (2) divide termwise the first equation by the second.

750. $\dfrac{dx}{dt} + \dfrac{2dy}{dt} - 2(x - y) = 3e^t, \quad \dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = 4e^{2t}.$

751. $\dfrac{d^2x}{dt^2} + m^2y = 0, \quad \dfrac{d^2y}{dt^2} - m^2x = 0.$

752. $\dfrac{dx}{dt} = \dfrac{x}{x^2 + y^2 + z^2}, \quad \dfrac{dy}{dt} = \dfrac{y}{x^2 + y^2 + z^2}, \quad \dfrac{dz}{dt} = \dfrac{z}{x^2 + y^2 + z^2}.$

4.5.2. Using matrices to solve homogeneous linear systems of differential equations with constant coefficients (modified Euler's method). Suppose we are given a system of n linear differential equations in n unknown functions whose coefficients are constant:

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n, \\ \dfrac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n. \end{cases}$$

This system can be written as one matrix differential equation

$$\frac{dX}{dt} = AX,$$

Here

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \frac{dX}{dt} = \begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \\ \vdots \\ \dfrac{dx_n}{dt} \end{pmatrix}$$

We seek the solution of the system in the form

$$x_1 = p_1e^{\lambda t}, x_2 = p_2e^{\lambda t}, \dots, x_n = p_ne^{\lambda t}.$$

Substituting the values of $x_1, x_2, \dots, x_n$ into the system of differential equations, we obtain a system of linear algebraic equations with respect to $p_1, p_2, \dots, p_n$:

$$\begin{cases} (a_{11} - \lambda)p_1 + a_{12}p_2 + \dots + a_{1n}p_n = 0, \\ a_{21}p_1 + (a_{22} - \lambda)p_2 + \dots + a_{2n}p_n = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}p_1 + a_{n2}p_2 + \dots + (a_{nn} - \lambda)p_n = 0. \end{cases}$$

The system must have a nonzero solution, therefore, to determine λ we get an nth-degree equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0.$$

The last equation is a characteristic equation of the matrix A and at the same time the characteristic equation of the system.

Suppose that the characteristic equation has n distinct roots $\lambda_1, \lambda_2, \dots, \lambda_n$, which are the characteristic numbers of the matrix A. Each characteristic number is associated with an eigenvector. Assume that the characteristic number λ_k is associated with the eigenvector $(p_{1k}; p_{2k}; \dots; p_{nk})$, where $k = 1, 2, \dots, n$. Then the system of differential equations possesses n solutions:

the 1st solution corresponding to the root $\lambda = \lambda_1$:

$$x_{11} = p_{11}e^{\lambda_1 t},\quad x_{21} = p_{21}e^{\lambda_1 t}, \dots, x_{n1} = p_{n1}e^{\lambda_1 t};$$

the 2nd solution corresponding to the root $\lambda = \lambda_2$:

$$x_{12} = p_{12}e^{\lambda_2 t},\quad x_{22} = p_{22}e^{\lambda_2 t}, \dots, x_{n2} = p_{n2}e^{\lambda_2 t};$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

the nth solution corresponding to the root $\lambda = \lambda_n$:

$$x_{1n} = p_{1n}e^{\lambda_n t},\quad x_{2n} = p_{2n}e^{\lambda_n t}, \dots, x_{nn} = p_{nn}e^{\lambda_n t}.$$

We have obtained a fundamental system of solutions. The general solution of the system is

$$\begin{aligned} x_1 &= C_1x_{11} + C_2x_{12} + \dots + C_nx_{1n}, \\ x_2 &= C_1x_{21} + C_2x_{22} + \dots + C_nx_{2n}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n &= C_1x_{n1} + C_2x_{n2} + \dots + C_nx_{nn}. \end{aligned}$$

The cases of complex and multiple roots will be considered by way of examples.

753. Find the general solution of the system of equations

$$\begin{cases} \dfrac{dx_1}{dt} = 7x_1 + 3x_2, \\ \dfrac{dx_2}{dt} = 6x_1 + 4x_2. \end{cases}$$

Solution. Let us derive the characteristic equation for the matrix of the system

$$\begin{vmatrix} 7-\lambda & 3 \\ 6 & 4-\lambda \end{vmatrix} = 0, \quad \text{or} \quad \lambda^2 - 11\lambda + 10 = 0.$$

Its roots $\lambda_1 = 1$, $\lambda_2 = 10$ are the characteristic numbers of the matrix.

At $\lambda = 1$, the equations specifying the eigenvector have the form $(7-1)p_1 + 3p_2 = 0$ and $6p_1 + (4-1)p_2 = 0$ and can be reduced to one equation $2p_1 + p_2 = 0$. The last equation specifies the vector $(1; -2)$.

At $\lambda = 10$, we get the equation $(7-10)p_1 + 3p_2 = 0$, $6p_1 + (4-10)p_2 = 0$, or $p_1 - p_2 = 0$. This equation specifies the vector $(1; 1)$.

We get a fundamental system of solutions:

at $\lambda = 1$: $x_{11} = e^t$, $x_{21} = -2e^t$.

at $\lambda = 10$: $x_{12} = e^{10t}$, $x_{22} = e^{10t}$.

The general solution of the system has the form

$$x_1 = C_1e^t + C_2e^{10t},\ x_2 = -2C_1e^t + C_2e^{10t}.$$

754. Find the general solution of the system of equations

$$\begin{cases} \dfrac{dx}{dt} = 6x - 12y - z, \\ \dfrac{dy}{dt} = x - 3y - z, \\ \dfrac{dz}{dt} = -4x + 12y + 3z. \end{cases}$$

Solution. We derive the characteristic equation of the matrix of the system:

$$\begin{vmatrix} 6-\lambda & -12 & -1 \\ 1 & -3-\lambda & -1 \\ -4 & 12 & 3-\lambda \end{vmatrix} = 0.$$

Expanding the determinant, we find

$$(6-\lambda)(\lambda^2 - 9) - 48 - 12 + 12 + 4\lambda + 72 - 12\lambda + 36 - 12 = 0,$$

or, finally,

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

This equation has the roots $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

We determine the eigenvectors of the matrix A.

At $\lambda = 1$, we get the system of equations

$$\begin{cases} 5p_1 - 12p_2 - p_3 = 0, \\ p_1 - 4p_2 - p_3 = 0, \\ -4p_1 + 12p_2 + 2p_3 = 0, \end{cases}$$

one of which is the consequence of the other two.

Let us consider, for example, the first two equations:

$$5p_1 - 12p_2 - p_3 = 0, \quad p_1 - 4p_2 - p_3 = 0.$$

From this we find

$$p_1 = \begin{vmatrix} -12 & -1 \\ -4 & -1 \end{vmatrix} \cdot k = 8k, \; p_2 = -\begin{vmatrix} 5 & -1 \\ 1 & -1 \end{vmatrix} \cdot k = 4k, \; p_3 = \begin{vmatrix} 5 & -12 \\ 1 & -4 \end{vmatrix} \cdot k = -8k.$$

Assuming $k = 1/4$, we get the eigenvector $(2;\ 1;\ -2)$.

At $\lambda = 2$, we have the system

$$\begin{cases} 4p_1 - 12p_2 - p_3 = 0, \\ p_1 - 5p_2 - p_3 = 0, \\ -4p_1 + 12p_2 + p_3 = 0. \end{cases}$$

Using again the first two equations (the third equation being their consequence), we find

$$p_1 = \begin{vmatrix} -12 & -1 \\ -5 & -1 \end{vmatrix} \cdot k = 7k, \; p_2 = -\begin{vmatrix} 4 & -1 \\ 1 & -1 \end{vmatrix} \cdot k = 3k, \; p_3 = \begin{vmatrix} 4 & -12 \\ 1 & -5 \end{vmatrix} \cdot k = -8k.$$

Putting $k = 1$, we find the eigenvector $(7;\ 3;\ -8)$.

At $\lambda = 3$, we have the system

$$\begin{cases} 3p_1 - 12p_2 - p_3 = 0, \\ p_1 - 6p_2 - p_3 = 0, \\ -4p_1 + 12p_2 = 0. \end{cases}$$

The last equation yields $p_1 = 3p_2$. We substitute this value of p_1 into the first equation and find $p_3 = -3p_2$. Assuming $p_2 = 1$, we get $p_1 = 3$, $p_3 = -3$, that is, the eigenvector $(3;\ 1;\ -3)$.

The fundamental system of solutions:

for $\lambda = 1$: $x_{11} = 2e^t$, $x_{21} = e^t$, $x_{31} = -2e^t$,

for $\lambda = 2$: $x_{12} = 7e^{2t}$, $x_{22} = 3e^{2t}$, $x_{32} = -8e^{2t}$,

for $\lambda = 3$: $x_{13} = 3e^{3t}$, $x_{23} = e^{3t}$, $x_{33} = -3e^{3t}$.

The general solution is written as

$$x_1 = 2C_1e^t + 7C_2e^{2t} + 3C_3e^{3t},$$
$$x_2 = C_1e^t + 3C_2e^{2t} + C_3e^{3t},$$
$$x_3 = -2C_1e^t - 8C_2e^{2t} - 3C_3e^{3t}.$$

755. Find the general solution of the system of equations

$$\begin{cases} \dfrac{dx_1}{dt} = 4x_1 - 3x_2, \\[2ex] \dfrac{dx_2}{dt} = 3x_1 + 4x_2. \end{cases}$$

Solution. We derive the characteristic equation of the matrix of the system:

$$\begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = 0; \quad (4-\lambda)^2 = -9, \quad \lambda - 4 = \pm 3i, \quad \lambda = 4 \pm 3i.$$

Next we determine the eigenvectors.

At $\lambda_1 = 4 + 3i$, we get the system of equations

$$\begin{cases} 3ip_1 - 3p_2 = 0, \\ 3p_1 + 3ip_2 = 0. \end{cases}$$

Thus we have $p_2 = ip_1$. Assuming $p_1 = 1$, we get $p_2 = i$, that is, the eigenvector $(1; i)$.

At $\lambda_2 = 4 - 3i$, we get the system of equations

$$\begin{cases} -3ip_1 - 3p_2 = 0, \\ 3p_1 - 3ip_2 = 0. \end{cases}$$

From this we find the eigenvector $(1; -i)$.

The fundamental system of solutions:

for $\lambda_1 = 4 + 3i$:

$$x_{11} = e^{(4+3i)t} = e^{4t}(\cos 3t + i \sin 3t),$$
$$x_{21} = ie^{(4+3i)t} = e^{4t}(-\sin 3t + i \cos 3t);$$

for $\lambda_2 = 4 - 3i$:

$$x_{12} = e^{(4-3i)t} = e^{4t}(\cos 3t - i \sin 3t),$$
$$x_{22} = e^{4t}(-\sin 3t - i \cos 3t).$$

Thus we arrive at the general solution

$$x_1 = C_1 e^{4t}(\cos 3t + i \sin 3t) + C_2 e^{4t}(\cos 3t - i \sin 3t),$$
$$x_2 = C_1 e^{4t}(-\sin 3t + i \cos 3t) + C_2 e^{4t}(-\sin 3t - i \cos 3t),$$

i.e.

$$x_1 = e^{4t}[(C_1 + C_2)\cos 3t + (C_1 - C_2) i \sin 3t],$$
$$x_2 = e^{4t}[-(C_1 + C_2)\sin 3t + (C_1 - C_2) i \cos 3t].$$

Putting $C_1 + C_2 = C_1^*$, $(C_1 - C_2)i = C_2^*$, we obtain

$$x_1 = e^{4t}(C_1^* \cos 3t + C_2^* \sin 3t),$$
$$x_2 = e^{4t}(-C_1^* \sin 3t + C_2^* \cos 3t).$$

There is another way of finding the general solution. Isolating the real and imaginary parts in the solutions corresponding to one of the complex characteristic numbers (we do not consider the conjugate characteristic number),

$$e^{(4+3i)t} = e^{4t}\cos 3t + ie^{4}\sin 3t,$$
$$ie^{(4+3i)t} = -e^{4t}\sin 3t + ie^{4t}\cos 3t,$$

we get two linearly independent particular solutions: $x_{11} = e^{4t}\cos 3t$, $x_{21} = -e^{4t}\sin 3t$, $x_{12} = e^{4t}\sin 3t$, $x_{22} = e^{4t}\cos 3t$.

The general solution is

$$x_1 = C_1x_{11} + C_2x_{12},\quad x_2 = C_1x_{21} + C_2x_{22},$$

i.e.

$$x_1 = e^{4t}(C_1\cos 3t + C_2\sin 3t),\quad x_2 = e^{4t}(-C_1\sin 3t + C_2\sin 3t).$$

756. Find the general solution of the system of equations

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 - x_3, \\ \dfrac{dx_2}{dt} = x_1, \\ \dfrac{dx_3}{dt} = x_1 - x_2. \end{cases}$$

Solution. We derive the characteristic equation

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & -\lambda & 0 \\ 1 & -1 & -\lambda \end{vmatrix} = 0,\quad \text{or}\quad (1-\lambda)(1+\lambda^2) = 0.$$

The characteristic numbers are $\lambda_1 = 1$, $\lambda_2 = i$, $\lambda_3 = -i$.

At $\lambda = 1$, to determine the eigenvector, we get the system of equations

$$\begin{cases} -p_3 = 0, \\ p_1 - p_2 = 0, \\ p_1 - p_2 - p_3 = 0. \end{cases}$$

The system defines the eigenvector (1; 1; 0).

At $\lambda = i$, we get the system of equations

$$\begin{cases} (1-i)p_1 - p_3 = 0, \\ p_1 - ip_2 = 0, \\ p_1 - p_2 - ip_3 = 0. \end{cases}$$

The system defines the eigenvector $(1; -i; 1-i)$.

The eigenvector corresponding to the characteristic number $\lambda = -i$ will not be considered.

The value $\lambda = 1$ is associated with the solutions

$$x_{11} = e^t,\quad x_{21} = e^t,\quad x_{31} = 0.$$

The value $\lambda = i$ is associated with the solutions

$$e^{it} = \cos t + i\sin t,\quad -ie^{it} = -\sin t + i\cos t,$$

$$(1-i)e^{it} = (\cos t + \sin t) + i(\sin t - \cos t).$$

Isolating the real parts, we get the solutions

$$x_{12} = \cos t, \quad x_{22} = -\sin t, \quad x_{32} = \cos t + \sin t.$$

Isolating the imaginary parts, we find the solutions

$$x_{13} = \sin t, \quad x_{23} = \cos t, \quad x_{33} = \sin t - \cos t.$$

The general solution is

$$\begin{aligned} x_1 &= C_1e^t + C_2\cos t + C_3\sin t,\\ x_2 &= C_1e^t + C_2\sin t + C_3\cos t,\\ x_3 &= C_2(\cos t + \sin t) + C_3(\sin t - \cos t). \end{aligned}$$

757. Find the general solution of the system of equations

$$\begin{cases} \dfrac{dx_1}{dt} = 5x_1 - x_2,\\[2mm] \dfrac{dx_2}{dt} = x_1 + 3x_2. \end{cases}$$

Solution. We solve the characteristic equation:

$$\begin{vmatrix} 5-\lambda & -1\\ 1 & 3-\lambda \end{vmatrix} = 0; \quad (5-\lambda)(3-\lambda) + 1 = 0; \quad \lambda^2 - 8\lambda + 16 = 0;$$
$$\lambda_1 = \lambda_2 = 4.$$

If λ_1 is a root of the characteristic equation of multiplicity m, then the root is associated with the solution

$$x_1 = p_1(t)e^{\lambda_1 t}, \quad x_2 = p_2(t)e^{\lambda_1 t}, \dots, x_n = p_n(t)e^{\lambda_1 t},$$

where $p_1(t), p_2(t), \dots, p_n(t)$ are polynomials of the degree not higher than $m-1$.

Thus, the double root $\lambda = 4$ is associated with the solution

$$x_1 = e^{4t}(a_1t + a_2), \quad x_2 = e^{4t}(b_1t + b_2).$$

Differentiating x_1 and x_2, we get

$$\frac{dx_1}{dt} = a_1e^{4t} + 4(a_1t + a_2)e^{4t}, \quad \frac{dx_2}{dt} = b_1e^{4t} + 4(b_1t + b_2)e^{4t}.$$

We substitute the values of $x_1, x_2, \dfrac{dx_1}{dt}, \dfrac{dx_2}{dt}$ into the system of equations. After cancelling by e^{4t} we have

$$\begin{aligned} a_1 + 4(a_1t + a_2) &= 5(a_1t + a_2) - (b_1t + b_2),\\ b_1 + 4(b_1t + b_2) &= a_1t + a_2 + 3(b_1t + b_2). \end{aligned}$$

Equating the coefficients in t and constant terms, we obtain the systems of equations

$$\begin{cases} 4a_1 = 5a_1 - b_1,\\ 4b_1 = a_1 + 3b_1, \end{cases} \quad \begin{cases} a_1 + 4a_2 = 5a_2 - b_2,\\ b_1 + 4b_2 = a_2 + 3b_2. \end{cases}$$

It follows that $a_1 = b_1$; $a_2 - b_2 = a_1 = b_1$. Putting $a_1 = C_1$, $a_2 = C_2$ (C_1 and C_2 being arbitrary constants), we find $b_1 = C_1$, $b_2 = C_2 - C_1$. Consequently,

$$x_1 = e^{4t}(C_1 t + C_2), \quad x_2 = e^{4t}(C_1 t + C_2 - C_1).$$

It is easier to solve this system by the method of elimination. Indeed, solving the first equation for x_2 by its expression and differentiating, we then substitute the values of x_2 and $\frac{dx_2}{dt}$ into the second equation. As a result, we obtain a homogeneous linear equation of the second order with respect to x_1. We recommend the reader to solve the given system by the elimination method.

Find the general solutions of the following systems:

758. $\begin{cases} \dfrac{dx_1}{dt} = -ax_2, \\ \dfrac{dx_2}{dt} = -ax_1. \end{cases}$ **759.** $\begin{cases} \dfrac{dx_1}{dt} = 8x_2 - x_1, \\ \dfrac{dx_2}{dt} = x_1 + x_2. \end{cases}$

760. $\begin{cases} \dfrac{dx_1}{dt} = -x_1 + x_2 + x_3, \\ \dfrac{dx_2}{dt} = x_1 - x_2 + x_3, \\ \dfrac{dx_3}{dt} = x_1 + x_2 + x_3. \end{cases}$ **761.** $\begin{cases} \dfrac{dx_1}{dt} = x_1 - x_2 + x_3, \\ \dfrac{dx_2}{dt} = x_1 + x_2 - x_3, \\ \dfrac{dx_3}{dt} = 2x_1 - x_2. \end{cases}$

762. $\begin{cases} \dfrac{dx_1}{dt} = 12x_1 - 5x_2, \\ \dfrac{dx_2}{dt} = 5x_1 + 12x_2. \end{cases}$ **763.** $\begin{cases} \dfrac{dx_1}{dt} = x_1 - 2x_2, \\ \dfrac{dx_2}{dx} = x_1 - x_2. \end{cases}$

764. $\begin{cases} \dfrac{dx_1}{dt} = -15x_1 - 6x_2 + 16x_3, \\ \dfrac{dx_2}{dt} = -15x_1 - 7x_2 + 18x_3, \\ \dfrac{dx_3}{dt} = -19x_1 - 8x_2 + 21x_3. \end{cases}$ **765.** $\begin{cases} \dfrac{dx}{dt} = (a + 1)x - y, \\ \dfrac{dy}{dt} = x + (a - 1)y. \end{cases}$

766. $\begin{cases} \dfrac{dx}{dt} = x - 4y, \\ \dfrac{dy}{dt} = x + y. \end{cases}$ **767.** $\begin{cases} \dfrac{dx}{dt} = 3x + y, \\ \dfrac{dy}{dt} = -4x - y. \end{cases}$

Chapter 5

Elements of the Probability Theory

5.1. Random Events, Their Frequency and Probability

The events that may occur and may not occur when the set of conditions connected with the possibility of their occurrence is realized are called *random* or *stochastic*.

Random events are denoted by the letters A, B, C, ... Each realization of the indicated set of conditions is called a *trial*. The number of trials can increase indefinitely. The ratio of the number m of the occurrences of some random event A in a given series of trials to the total number n of the trials of that series is called the *frequency* of occurrence of the event A in the given series of trials (or simply the frequency of the event A) and is denoted as $\bar{P}(A)$. Thus, $\bar{P}(A) = m/n$.

The frequency of a random event is always contained between zero and unity: $0 \leqslant \bar{P}(A) \leqslant 1$.

A feature of mass-scale random events is the *stability* of their frequency: the values of the frequency of a given random event observed in various series of homogeneous trials (with a sufficiently large number of trials for a series) vary from series to series in rather small limits.

Precisely this fact enables us to use mathematical methods when studying random events, ascribing to every mass-scale random event its *probability*, which is the number (generally not known beforehand) around which the observed frequency of the event varies.

The probability of the random event A is designated as $P(A)$. Just like its frequency, the probability of a random event is contained between zero and unity: $0 \leqslant P(A) \leqslant 1$.

The probability of a *certain* (*sure*) event (i.e. an event which is sure to occur in each trial) is $P(A) = 1$.

The probability of an *impossible* event (i.e. an event which cannot occur whatever the trial) is $P(A) = 0$.

In some simple cases the probability of a random event can be predetermined. This can be done, for instance, when the possible outcomes of homogeneous trials can be represented as n only possible, mutually exclusive and equally probable outcomes ("events") (that is, there can be no other outcomes except these n events, neither two of them can take place simultaneously, and there is no ground to believe that some one of them is more probable than the others). If m out of these n only possible, mutually exclusive and equally probable events are connected with the occurrence of the event A (or, as it is said in the theory of probability, they are favourable to A), then the probability of the event A is assumed to be the ratio between m and n: $P(A) = m/n$.

768. A box contains 10 balls labelled with the numbers from 1 to 10. We draw one ball. What is the probability that the number of the drawn ball does not exceed 10?

Solution. Since the number of any ball in the box does not exceed 10, it follows that the number of events favourable to the event A is equal to the number of all the possible cases, i.e. $m = n = 10$ and $P(A) = 1$. In this case, the event A is certain.

769. An urn contains 15 balls of which 5 are white and 10 black. What is the probability of drawing a blue ball?

Solution. There are no blue balls in the urn, i.e. $m = 0$ and $n = 15$. Consequently, $P(A) = 0/15 = 0$. In the given case, the event A is impossible.

770. There are 12 balls in the urn, of which 3 are white, 4 black and 5 red. What is the probability of drawing a black ball?

Solution. Here $m = 4$, $n = 12$ and $P(A) = 4/12 = 1/3$.

771. There are 10 balls in the urn, of which 6 are white and 4 black. We draw two balls. What is the probability of both balls being white?

Solution. The number of all events here is $n = C_{10}^2 = (10 \cdot 9)/(1 \cdot 2) = 45$. The number of cases favourable to the event A is determined by the equation $m = C_6^2$, i.e. $m = (6 \cdot 5)/(1 \cdot 2) = 15$. Thus we have $P(A) = 15/45 = 1/3$.

772. There are 2000 tickets in the lottery. One of the tickets wins 100 rubles, four other tickets win 50 rubles each, ten tickets win 20 rubles each, 20 tickets win 10 rubles each, 165 tickets win 5 rubles each, and 400 tickets win 1 ruble each. The rest of the tickets are not winning. What is the probability of one ticket winning not less than 10 rubles?

Solution. Here $m = 1 + 4 + 10 + 20 = 35$, $n = 2000$, i.e. $P(A) = m/n = = 35/2000 = 0.0175$.

773. An urn contains 20 balls labelled with the numbers from 1 to 20. What is the probability of drawing the ball with the number 37?

774. We toss a coin twice. What is the probability of the coin falling heads up both times?

775. One box contains balls labelled with the numbers from 1 to 5, and the other box, balls with the numbers from 6 to 10. We draw one ball from each box. What is the probability that the sum of the numbers of the drawn balls (1) is not less than 7; (2) is equal to 11; (3) does not exceed 11?

776. There are 1000 tickets in the lottery, of which 500 are winning tickets and 500 are non-winning. We buy two tickets. What is the probability of both tickets being winning?

777. Six students from a group of 30 have passed the test with excellent marks, ten students with good marks, and nine students have got satisfactory marks. What is the probability that three students chosen randomly from this group have got failing marks for the test?

5.2. Probability Addition and Multiplication Rules

The *union* (or *sum*) of several random events is an event consisting in the occurrence of at least one of the given events. The union of the events $A_1, A_2, \dots, A_n$ is denoted as $A_1 + A_2 + \dots + A_n$.

If the events being summed up are pairwise mutually exclusive (neither two of them can take place simultaneously), then *the probability of summing up several events is equal to the sum of the probabilities of these events* (**probability addition rule**):

$$P(A_1 + A_2 + \dots + A_n) = P(A_1) + P(A_2) + \dots + P(A_n).$$

The event consisting in the non-occurrence of the random event A is said to be an event *opposite*, or *contrary*, to the event A and is denoted $\bar{A}$. The union of the events A and $\bar{A}$ yields a "certain" event, and since the events A and $\bar{A}$ are mutually exclusive, we have

$$P(A) + P(\bar{A}) = 1 \quad \text{or} \quad P(\bar{A}) = 1 - P(A).$$

If only one of the mutually exclusive events $A_1, A_2, \dots, A_n$ can occur as a result of some trial, then the events $A_1, A_2, \dots, A_n$ form a so-called *complete group of events*. Since the union of the events forming a complete group is a "certain" event, the equality

$$P(A_1) + P(A_2) + \dots + P(A_n) = 1$$

holds true for such events.

The *intersection* (or *product*) of two random events A_1 and A_2 is a compound event consisting in a simultaneous or a consecutive occurrence of the two events. The intersection of the events A_1 and A_2 is denoted A_1A_2.

The conditional probability of the event A_2 relative to the event A_1 (the notation is $P(A_2/A_1)$) is understood as the probability of the occurrence of the event A_2 defined relative to the hypothesis that the event A_1 has occurred.

The probability that two events, A_1 and A_2, occur simultaneously is equal to the product of the probability of one of them by the conditional probability of the other relative to the first one (**probability multiplication rule**):

$$P(A_1 \cdot A_2) = P(A_1) \cdot P(A_2/A_1) = P(A_2) \cdot P(A_1/A_2).$$

Two random events A_1 and A_2 are said to be independent if the conditional probability of one of them relative to the other is equal to the absolute (unconditional) probability of the first event: $P(A_2/A_1) = P(A_2)$. In this case the following equalities hold true:

$$P(A_2/\bar{A}_1) = P(A_2/A_1) = P(A_2); \quad P(A_1/A_2) = P(A_1/\bar{A}_2) = P(A_1).$$

The probability of the intersection of independent events is equal to the product of their probabilities:

$$P(A_1A_2) = P(A_1) \cdot P(A_2).$$

The intersection of n events $A_1, A_2, \dots, A_n$ (defined by analogy) is denoted as $A_1A_2 \dots A_n$.

The conditional probability of the event A_k, defined on the assumption that the events $A_1, A_2, \dots A_{k-1}$ have occurred, is designated as $P(A_k/A_1A_2 \dots, A_{k-1})$.

By the probability multiplication rule, the probability of intersection of n events is specified by the formula

$$P(A_1A_2 \dots A_n) = P(A_1)\cdot P(A_2/A_1)\cdot P(A_3/A_1A_2) \dots P(A_n/A_1A_2 \dots A_{n-1}).$$

The n events $A_1, A_2, \dots, A_n$ are said to be *collectively independent* if the probability of occurrence of each of them is not affected by the occurrence of any other events taken in an arbitrary combination.

The probability of intersection of n collectively independent events is equal to the product of their probabilities:

$$P(A_1A_2 \dots A_n) = P(A_1)\cdot P(A_2) \dots P(A_n).$$

778. An urn contains 10 white balls, 15 black, 20 blue and 25 red balls. We draw one ball. Find the probability of the drawn ball being white; black; blue; red; white or black; blue or red; white, black or blue.

Solution. We have $n = 10 + 15 + 20 + 25 = 70$, $P(\text{white}) = 10/70 = 1/7$, $P(\text{black}) = 15/70 = 3/14$, $P(\text{blue}) = 20/70 = 2/7$, $P(\text{red}) = 25/70 = 5/14$. Applying the probability addition rule, we get

$$P(\text{white} + \text{black}) = P(\text{white}) + P(\text{black}) = 1/7 + 3/14 = 5/14;$$

$$P(\text{blue} + \text{red}) = P(\text{blue}) + P(\text{red}) = 2/7 + 5/14 = 9/14;$$

$$P(\text{white} + \text{black} + \text{blue}) = 1 - P(\text{red}) = 1 - 5/14 = 9/14.$$

779. The first box contains 2 white and 10 black balls; the second box contains 8 white and 4 black balls. We draw one ball from each box. What is the probability of both balls being white?

Solution. The given case involves intersection of two events, A and B, where the event A is the drawing of a white ball from the first box and the event B is the drawing of a white ball from the second box, A and B being independent events. We have $P(A) = 2/12 = 1/6$, $P(B) = 8/12 = 2/3$. Applying the probability multiplication rule, we find

$$P(AB) = P(A)\cdot P(B) = (1/6)\cdot(2/3) = 1/9.$$

780. Under the conditions of the previous problem determine the probability of one drawn ball being white and the other black.

Solution. Assume that

event A is the drawing of a white ball from the first box;

event B is the drawing of a white ball from the second box;

event C is the drawing of a black ball from the first box; $(C = \bar{A})$;

event D is the drawing of a black ball from the second box; $(D = \bar{B})$.

Then $P(A) = 1/6$, $P(B) = 2/3$, $P(C) = P(\bar{A}) = 1 - 1/6 = 5/6$, $P(D) = P(\bar{B}) = 1 - 2/3 = 1/3$.

Let us determine the probability of the ball drawn from the first box being white and that drawn from the second box being black:

$$P(AD) = P(A)\cdot P(D) = (1/6)\cdot(1/3) = 1/18.$$

Let us determine the probability that the ball drawn from the first box is black and that drawn from the second box is white:

$$P(BC) = P(B)\cdot P(C) = (2/3)\cdot(5/6) = 5/9.$$

Let us now determine the probability that the ball drawn from one box (no matter the first or the second) is white and the ball drawn from the other box is black. We apply the probability addition rule:

$$P = P(AD) + P(BC) = 1/18 + 5/9 = 11/18.$$

781. A box contains 6 white and 8 black balls. We draw two balls (without replacing the drawn ball into the box). Find the probability of both balls being white.

Solution. Suppose the event A is the drawing of a white ball in the first trial and the event B is the drawing of a white ball in the second trial. By the probability multiplication rule for the case of dependent events we have $P(AB) = P(A)\cdot P(B/A)$. But $P(A) = 6/(6 + 8) = 6/14 = 3/7$ (the probability of a white ball drawn first); $P(B/A) = (6 - 1)/(6 + 8 - 1) = 5/13$ (the probability of a white ball being drawn second on the assumption that the first white ball has already been drawn). Consequently, $P(AB) = (3/7)\cdot(5/13) = 15/91$.

782. Three marksmen do target practice. For the first marksman the probability of hitting the target is 0.75, for the second, 0.8, for the third, 0.9. Determine the probability of the three marksmen hitting the target.

Solution. We have

$$P(A) = 0.75, \quad P(B) = 0.8, \quad P(C) = 0.9;$$

$$P(ABC) = P(A)\cdot P(B)\cdot P(C) = 0.75\cdot 0.8\cdot 0.9 = 0.54.$$

783. Under the conditions of the previous problem determine the probability of at least one marksman hitting the target.

Solution. Here $P(\bar{A}) = 1 - 0.75 = 0.25$ (the probability of the first marksman missing the target); $P(\bar{B}) = 1 - 0.8 = 0.2$ (the probability of the second marksman missing the target); $P(\bar{C}) = 1 - 0.9 = 0.1$ (the probability of the third marksman missing the target); then $P(\bar{A}\bar{B}\bar{C})$, the probability of all three marksmen missing the target, is determined as follows:

$$P(\bar{A}\bar{B}\bar{C}) = P(\bar{A})\cdot P(\bar{B})\cdot P(\bar{C}) = 0.25\cdot 0.2\cdot 0.1 = 0.005.$$

But the event contrary to the event $\bar{A}\bar{B}\bar{C}$ consists in the target being hit by at least one marksman. Consequently, the sought-for probability is $P = 1 - P(\bar{A}\bar{B}\bar{C})$, i.e. $P = 1 - 0.005 = 0.995$.

784. The probability of a machine-tool failing during one workday is α (α is a small positive number whose second degree may be neglected). What is the probability that the tool will work without failing for five days? Solve the problem for $\alpha = 0.01$.

Solution. Since $1 - \alpha$ is the probability of the machine-tool working for a day without failing, it follows, from the probability multiplication rule, that $(1 - \alpha)^5$ is the probability that the tool will work without failing for 5 days.

Making use of the binomial expansion and neglecting the terms containing α^2, α^3, α^4 and α^5, we obtain an approximate equality $(1 - \alpha)^5 \approx 1 - 5\alpha$, i.e. $P \approx 1 - 5\alpha$. Assuming $\alpha = 0.01$, we get $P \approx 0.95$.

785. A box contains a white and b black balls. What is the probability that one of the drawn balls is white and the other black? (The drawing is without replacement.)

Solution. Assume that
event A is the drawing of a white ball in the first trial;
event B is the drawing of a black ball in the second trial;
event C is the drawing of a black ball in the first trial;
event D is the drawing of a white ball in the second trial.

Let us calculate the probability of a white ball being drawn first and a black ball being the second:

$$P_1 = P(A)\cdot P(B/A) = \frac{a}{a+b}\cdot\frac{b}{a+b-1} = \frac{ab}{(a+b)(a+b-1)}.$$

Let us now find the probability of a black ball being the first and a white ball being the second:

$$P_2 = P(C)\cdot P(D/C) = \frac{b}{a+b}\cdot\frac{a}{a+b-1} = \frac{ab}{(a+b)(a+b-1)}.$$

Thus, the probability of one of the drawn balls being white and the other black is determined by the addition rule: $P = P_1 + P_2$, i.e.

$$P = \frac{2ab}{(a+b)(a+b-1)}.$$

786. A box contains a white, b black and c blue balls. We draw one ball. Calculate the probability that the drawn ball is (1) white; (2) black; (3) blue; (4) white or black; (5) white or blue; (6) black or blue.

787. The first box contains a white and b black balls; the second box contains c white and d black balls. We draw one ball from each box. What is the probability of both balls being black?

788. The probability of the first marksman hitting the target is p_1, that of the second, p_2. The marksmen fire simultaneously. What is the probability that one of them will hit the target and the other miss it?

789. The probability that any July day the temperature in a Southern town N is lower than 5 °C is α (α is a small number whose square may be neglected). What is the probability that the first three days in July the temperature will not be lower than 5 °C?

790. The first box contains 1 white, 2 red and 3 blue balls; the second box contains 2 white, 6 red and 4 blue balls. We draw one ball from each box. What is the probability that we have not drawn blue balls?

791. The probability of the machine-tool failing during a workday is 0.03. What is the probability of the machine-tool working for four days running without failing?

792. There are 12 boys and 18 girls in a class. A random choice of two pupils must be made. What is the probability that we shall select (1) two boys; (2) two girls; (3) a boy and a girl?

793. An urn contains 9 white balls and 1 black ball. We draw three balls at once. What is the probability of all the balls being white?

794. Three shells are fired at a target. The probability of each shell hitting the target is 0.5. Determine the probability of only one shell finding the target.

5.3. Bernoulli's Formula. The Most Probable Number of Occurrences of an Event

If n independent trials are performed in each of which the probability of occurrence of the event A is the same and is equal to p, then the probability of the event A occurring m times in these n trials is expressed by *Bernoulli's formula*

$$P_{m,n} = C_n^m p^m q^{n-m},$$

where $q = 1 - p$. Thus we have

$$P_{0,n} = q^n, \quad P_{1,n} = npq^{n-1}, \quad P_{2,n} = \frac{n(n-1)}{1\cdot 2} p^2 \cdot q^{n-2}, \ \ldots, \ P_{n,n} = p^n.$$

The number m_0 is known as the *most probable number* of occurrences of the event A in n trials if at $m = m_0$ the value of $P_{m,n}$ is not less than all the other values of $P_{m,n}$, i.e. $P_{m_0,n} \geqslant P_{m_i,n}$ at $m_i \neq m_0$.

If $p \neq 0$ and $p \neq 1$, then the number m_0 can be determined from the double inequality

$$np - q \leqslant m_0 \leqslant np + p.$$

The difference between the end point values in this double inequality is 1. If $np + p$ is not an integer, then the double inequality specifies only one most probable value m_0. Now if $np + p$ is an integer, then there are two most probable values, $m_0' = np - q$ and $m_0'' = np + p$.

795. There are 20 white and 10 black balls in the urn. Four balls are drawn successively with replacement, the urn being shaken before every new drawing. What is the probability of two of the four drawn balls being white?

Solution. The probability of a white ball being drawn, $p = 20/30 = 2/3$, may be assumed to be the same in the four trials; $q = 1 - p = 1/3$. Using Bernoulli's formula, we get

$$P_{2,4} = C_4^2 p^2 q^2 = \frac{4\cdot 3}{1\cdot 2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 = \frac{8}{27}.$$

796. The probability of occurrence of the event A is 0.4. What is the probability of the event A occurring not more than three times in 10 trials?

Solution. Here $p = 0.4$, $q = 0.6$. We have:

probability of occurrence of event A 0 times: $P_{0,10} = q^{10}$;

probability of occurrence of event A 1 time: $P_{1,\,10} = 10pq^9$;
probability of occurrence of event A 2 times: $P_{2,\,10} = 45p^2q^8$;
probability of occurrence of event A 3 times: $P_{3,\,10} = 120p^3q^7$.

The probability of the event A occurring not more than three times is equal to

$$P = P_{0,\,10} + P_{1,\,10} + P_{2,\,10} + P_{3,\,10}$$

that is, .

$$P = q^{10} + 10pq^9 + 45p^2q^8 + 120p^3q^7, \text{ or } P = q^7(q^3 + 10q^2p + 45qp^2 + 120p^3).$$

Putting $p = 0.4$, $q = 0.6$, we get

$$P = 0.6^7(0.216 + 1.44 + 4.32 + 7.68) \approx 0.38$$

797. Determine the probability of the fact that in a family having five children there are three girls and two boys. The probabilities of a girl and a boy being born are assumed to be equal.

Solution. The probability of a girl being born is $p = 0.5$, then $q = 1 - p = 0.5$ (the probability of a boy being born). Hence, the sought-for probability is

$$P_{3,5} = C_5^3 p^3q^2 = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} \cdot (0.5)^3 \cdot (0.5)^2 = \frac{5}{16}.$$

798. Under the conditions of the previous problem find the probability of the fact that there will not be more than three girls among the children born to the family.

Solution. We have

$$P_{0,5} = \left(\frac{1}{2}\right)^5 = \frac{1}{32}; \quad P_{1,5} = 5 \cdot \left(\frac{1}{2}\right)^5 = \frac{5}{32};$$

$$P_{2,5} = 10 \cdot \left(\frac{1}{2}\right)^5 = \frac{5}{16}; \quad P_{3,5} = 10 \cdot \left(\frac{1}{2}\right)^5 = \frac{5}{16};$$

$$P = P_{0,5} + P_{1,5} + P_{2,5} + P_{3,5} = \frac{13}{16}.$$

799. We toss a coin eight times. What is the probability that we shall have heads six times?

800. We toss a coin six times. What is the probability that we shall have heads not more than three times?

801. There are 20 boys and 10 girls in a class. Each of the three questions put by the teacher was answered by one pupil. What is the probability that there were two boys and one girl among the pupils who answered the questions?

802. Each of the four boxes contains 5 white and 15 black balls. We draw one ball from each box. What is the probability of drawing two white and two black balls?

803. There are 10 white and 40 black balls in the urn. We successively draw 14 balls, register the colour of each drawn ball and replace it into the urn. Determine the most probable number of occurrences of a white ball.

Solution. Here $n = 14$, $p = 10/50 = 1/5$, $q = 1 - p = 4/5$. Using the double

inequality $np - q \leqslant m_0 \leqslant np + p$ at the values of n, p and q indicated, we get

$$14/5 - 4/5 \leqslant m_0 \leqslant 14/5 + 1/5, \quad \text{i.e.} \quad 2 \leqslant m_0 \leqslant 3.$$

Thus, the problem admits of two solutions: $m_0' = 2$, $m_0'' = 3$.

804. The probability of a marksman hitting the target is 0.7. He fires 25 shots. Determine the most probable number of times he hits the target.

Solution. Here $n = 25$, $p = 0.7$, $q = 0.3$. Consequently,

$$25 \cdot 0.7 - 0.3 \leqslant m_0 \leqslant 25 \cdot 0.7 + 0.7, \quad \text{i.e.} \quad 17.2 \leqslant m_0 \leqslant 18.2.$$

Since m is an integer, we have $m_0 = 18$.

805. As a result of long observations it was established that the probability of rain falling on October 1 in a given town is 1/7. Determine the most probable number of rainy days on October 1 in that town for 40 years.

Solution. We have $n = 40$, $p = 1/7$, $q = 6/7$. Thus we have

$$40 \cdot \frac{1}{7} - \frac{6}{7} \leqslant m_0 \leqslant 40 \cdot \frac{1}{7} + \frac{1}{7}, \quad 4\,\frac{6}{7} \leqslant m_0 \leqslant 5\,\frac{6}{7}, \quad \text{i.e.} \quad m_0 = 5.$$

806. There are 20 boxes containing homogeneous machine parts. The probability of the machine parts turning out to be unified in one randomly chosen box is 0.75. Find the most probable number of boxes in which all the parts are unified.

807. An urn contains 100 white and 80 black balls. We draw n balls (with replacement). The most probable number of drawing a white ball is 11. Find n.

Solution. It follows from the double inequality $np - q \leqslant m_0 \leqslant np + p$, that

$$(m_0 - p)/p \leqslant n \leqslant (m_0 + q)/p.$$

Here $m_0 = 11$, $p = 100/180 = 5/9$, $q = 4/9$; consequently,

$$\frac{11 - 5/9}{5/9} \leqslant n \leqslant \frac{11 + 4/9}{5/9}, \quad \text{i.e.} \quad 18.8 \leqslant n \leqslant 20.6$$

Thus, the problem admits of two solutions: $n_1 = 19$, $n_2 = 20$.

808. Can the numerical values of m_0 and p appearing in the previous problem be altered so that the problem will have no solution?

809. One worker can manufacture 120 articles during a shift, the other, 140 articles, the probabilities of the articles being of a high quality are 0.94 and 0.8 respectively. Determine the most probable number of high-quality articles manufactured by each worker.

810. There are 100 urns containing white and black balls. The probability of drawing a white ball from each urn is 0.6. Find the most probable number of urns in which all the balls are white.

5.4. Total Probability Formula. Bayes's Formula

If the event A is known to occur together with one of the events $H_1, H_2, \ldots, H_n$ forming a complete group of mutually exclusive events, then the event A can be

represented as a union of the events AH_1, AH_2, ..., AH_n, i.e. $A = AH_1 + AH_2 + \ldots + AH_n$. The probability of the event A can be found from the formula

$$P(A) = P(H_1) \cdot P(A/H_1) + P(H_2) \cdot P(A/H_2) + \ldots + P(H_n) \cdot P(A/H_n),$$

or

$$P(A) = \sum_{i=1}^{n} P(H_i) \cdot P(A/H_i).$$

This formula is known as the *total* (or *composite*) *probability formula.*

Provided the event A has occurred, the conditional probability of the event H_i can be determined by *Bayes' formula*

$$P(H_i/A) = \frac{P(AH_i)}{P(A)} = \frac{P(H_i)) \cdot P(A/H_i)}{\sum_{i=1}^{n} P(A/H_i) \cdot P(H_i)}$$

$$(i = 1, 2, \ldots, n).$$

The probabilities $P(H_i/A)$ calculated by Bayes' formula are often called *probabilities of hypotheses.*

811. There are four urns. The first urn contains 1 white and 1 black ball, the second, 2 white and 3 black balls, the third, 3 white and 5 black balls, and the fourth, 4 white and 7 black balls. The event H_i is the selection of the ith urn ($i = 1, 2, 3, 4$). The probability of selecting the ith urn is $i/10$, i.e. $P(H_1) = 1/10$, $P(H_2) = 1/5$, $P(H_3) = 3/10$, $P(H_4) = 2/5$. We randomly select one of the urns and draw a ball from it. Find the probability of the ball being white.

Solution. It follows from the hypothesis that $P(A/H_1) = 1/2$ (conditional probability of drawing a white ball from the first box); similarly, $P(A/H_2) = 2/5$, $P(A/H_3) = 3/8$, $P(A/H_4) = 4/11$. The probability of drawing a white ball can be found by the formula of total probability:

$$P(A) = P(H_1) \cdot P(A/H_1) + P(H_2) \cdot P(A/H_2) + P(H_3) \cdot P(A/H_3)$$

$$+ P(H_4) \cdot P(A/H_4) = \frac{1}{10} \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{2}{5} + \frac{3}{10} \cdot \frac{3}{8} + \frac{2}{5} \cdot \frac{4}{11} + \frac{1707}{4400}.$$

812. We have three boxes identical in appearance. The first box contains 20 white balls, the second box contains 10 white and 10 black balls, and the third box contains 20 black balls. We draw a white ball from a randomly selected box. Calculate the probability of the ball being drawn from the first box.

Solution. Suppose H_1, H_2, H_3 are the hypotheses consisting in selecting the first, the second and the third box respectively; the event A is the drawing of a white ball. Then $P(H_1) = P(H_2) = P(H_3) = 1/3$ (the selection of any of the boxes is equally probable); $P(A/H_1) = 1$ (the probability of drawing a white ball from the first box), $P(A/H_2) = 10/20 = 1/2$ (the probability of drawing a white ball from the

second box), $P(A/H_3) = 0$ (the probability of drawing a white ball from the third box).

The desired probability $P(H_1/A)$ can be found from Bayes' formula

$$P(H_1/A) = \frac{1 \cdot (1/3)}{1 \cdot (1/3) + (1/2) \cdot (1/3) + 0 \cdot (1/3)} = \frac{2}{3}.$$

813. A box contains N articles some of which may be defective. A randomly selected article has turned out to be effective. Determine the probability of the fact that all the articles in the box are effective; $N - 1$ articles are effective and one article is defective; $N - 2$ articles are effective and two articles are defective; . . . ; all N articles in the box are defective.

Solution. A priori probabilities: H_0, all the articles in the box are effective; H_1, one article is defective; H_2, two articles are defective; ...; H_N, all the articles are defective. The event A is the appearance of an effective article. It is required to find $P(H_0/A)$, $P(H_1/A)$, $P(H_2/A)$, ..., $P(H_N/A)$.

Assume that all a priori hypotheses are equally probable:

$$P(H_0) = P(H_1) = P(H_2) = \dots = P(H_N) + \frac{1}{N+1},$$

that is,

$$P(A/H_0) = 1, \quad P(A/H_1) = \frac{N-1}{N}, \quad P(A/H_2) = \frac{N-2}{N}, \dots,$$

$$P(A/H_{N-1}) = \frac{1}{N}, \quad P(A/H_N) = 0.$$

This yields

$$P(H_0/A) = \frac{1 \cdot \frac{1}{N+1}}{1 \cdot \frac{1}{N+1} + \frac{N-1}{N} \cdot \frac{1}{N+1} + \dots + \frac{1}{N} \cdot \frac{1}{N+1} + 0 \cdot \frac{1}{N+1}}$$

$$= \frac{1}{\frac{1}{N} + \frac{2}{N} + \dots + \frac{N-1}{N} + 1} = \frac{N}{1 + 2 + \dots + N - 1 + N} = \frac{2}{N+1}.$$

By analogy we obtain

$$P(H_1/A) = \frac{2}{N+1} \cdot \frac{N-1}{N},$$

$$P(H_2/A) = \frac{2}{N+1} \cdot \frac{N-2}{N}, \quad \dots, \quad P(H_N/A) = \frac{2}{N+1} \cdot 0 = 0.$$

814. The first urn contains 5 white and 10 black balls, the second, 3 white and 7 black balls. We draw one ball from the second urn and place it into the first one, then we randomly draw one ball from the first urn. Determine the probability of the drawn ball being white.

Solution. After a ball was drawn from the second urn and placed into the first, two collections of balls have turned out in the first urn: (1) 5 white and 10 black balls it contained prior to replacement; (2) one ball replaced from the second urn. The probability of appearance of a white ball belonging to the first collection is $P(A/H_1) = 5/15 = 1/3$, and from the second collection, $P(A/H_2) = 3/10$. The probability that the randomly drawn ball belongs to the first collection is $P(H_1) = 15/16$, and to the second, $P(H_2) = 1/16$.

Using the total probability formula, we obtain

$$P(A)=P(H_1)\cdot P(A/H_1)+P(H_2)\cdot P(A/H_2)$$

$$= \frac{15}{16}\cdot\frac{1}{3}+\frac{1}{16}\cdot\frac{3}{10}=\frac{53}{160}.$$

815. The first urn contains 1 white and 2 black balls, the second, 100 white and 100 black balls. One ball was drawn from the second urn and placed into the first, and then one ball was randomly drawn from the first urn. What is the probability that the drawn ball had previously belonged to the second urn if it is known to be white?

5.5. Random Variable and the Law of Its Distribution

If every elementary event A belonging to a certain set of events ω can be associated with a definite quantity $X = X(A)$, then a *random* variable is said to be given which can be regarded as a function of the event A with the domain of definition ω.

A random variable can assume one or another value from a certain number set, but it is not known beforehand which value it will assume. It is customary to designate random variables by capital letters $X, Y, \ldots$, and the values they assume, by the corresponding lower-case letters $x, y, \ldots$.

If the values that the given random variable X can assume form a discrete* (finite or infinite) number series $x_1, x_2, \ldots, x_n, \ldots$, then the random variable itself is also called *discrete*.

Now if the values that the given random variable can assume fill up the whole finite or infinite interval (a, b) of a number axis Ox, then the random variable is said to be *continuous*.

To each value x_n of a random variable of a discrete type there corresponds a definite probability p_n; to each interval (a, b) belonging to the range of a random variable of the continuous type there also corresponds a definite probability $P(a < X < b)$ that the value assumed by the random variable will fall in that interval.

*A finite or infinite number series is said to be *discrete* if every number x_n of this series can be put into correspondence with the interval (a_n, b_n) containing no other numbers of that series in its interior.

The relationship establishing in one way or another the connection between the possible values of a random variable and their probabilities is called the *law of distribution* of a random variable.

The law of distribution of a discrete random variable is usually given as a distribution series:

x_i	x_1	x_2	x_3	...	x_n
p_i	p_1	p_2	p_3	...	p_n

In this case $\sum_{i=1}^{n} p_i = 1$, where the summation covers the whole (finite or infinite) set of the possible values of the given random variable X.

It is convenient to represent the law of distribution of a continuous random variable with the aid of the so-called *probability density function* $f(x)$. The probability $P(a < X < b)$ of the fact that the value assumed by the random variable X will fall in the interval (a, b) is defined by the equality

$$P(a < X < b) = \int_a^b f(x)dx.$$

The graph of the function $f(x)$ is called a *distribution curve*. In terms of geometry, the probability that the random variable will fall in the interval (a, b) is equal to the area of the corresponding curvilinear trapezoid bounded by the distribution curve, the Ox axis and the straight lines $x = a$, $x = b$ (Fig. 35).

The probability density function $f(x)$ possesses the following properties:

1°. $f(x) \geqslant 0$.

2°. $\int_{-\infty}^{+\infty} f(x)dx = 1$

(if all the values of the random variable X belong to the interval (a, b), the last property can be written as $\int_a^b f(x)dx = 1$).

Let us now consider the function $F(x) = P(X < x)$. This function is called the *probability distribution function* of the random variable X. The function $F(x)$ exists both for discrete and for continuous random variables. If $f(x)$ is the probability density function of the continuous random variable X, then

$$F(x) = \int_{-\infty}^{x} f(x)dx.$$

It follows from the last equality that

$$f(x) = F'(x).$$

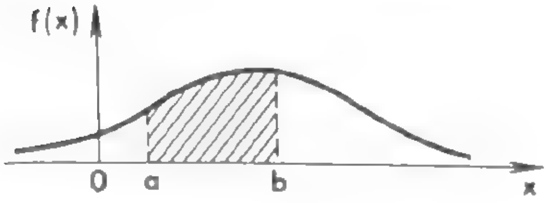


Fig. 35

The function $f(x)$ is sometimes called a *probability distribution differential function*, and the function $F(x)$, a *probability distribution integral function*.

Note the most significant properties of a probability distribution function:

1°. $F(x)$ is a non-decreasing function.

2°. $F(-\infty) = 0$.

3°. $F(+\infty) = 1$.

816. Given the probabilities of the values of the random variable X: the value 10 has the probability 0.3; the value 2 has the probability 0.4; the value 8, the probability 0.1; the value 4, the probability 0.2. Construct the distribution series for the random variable X.

Solution. Arranging the values of the random variable in the order of its increase, we get a distribution series:

x_i	2	4	8	10
p_i	0.4	0.2	0.1	0.3

We take the points (2; 0.4), (4; 0.2), etc. on the xOp plane. Connecting the consecutive points by line segments, we obtain the so-called polygon of distribution of the random variable X (Fig. 36).

817. The random variable X obeys the distribution law with density $f(x)$, with the following equality holding true:

$$f(x)=\begin{cases} 0, & \text{if} \quad x<0; \\ a(3x-x^2), & \text{if} \quad 0\leqslant x\leqslant 3; \\ 0, & \text{if} \quad x>3. \end{cases}$$

It is required: (1) to find the coefficient a; (2) to construct the graph of density distribution $y = f(x)$; (3) to find the probability of X falling in the interval (1, 2).

Solution. (1) Since all the values of the given random variable belong to the interval [0, 3], we have $\int\limits_0^3 a(3x - x^2)dx = 1$, whence

$$a\left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3 = 1, \quad \text{or} \quad a\left(\frac{27}{2} - 9\right) = 1, \quad \text{i.e.} \quad a = \frac{2}{9}.$$

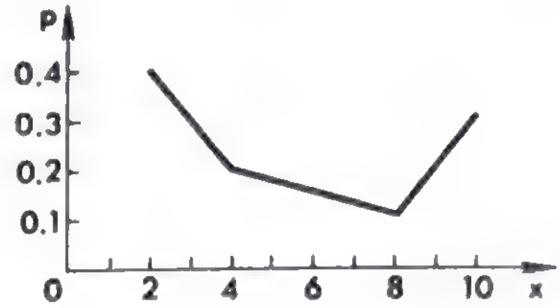


Fig. 36

The graph of the function $f(x)$ in the integral (0, 3) is the parabola $y = \frac{2}{3}x \frac{2}{9}x^2$, and outside the interval the graph is the abscissa axis itself. (Fig. 37). (Fig. 37).

(3) The probability of the random variable X falling in the interval (1, 2) can be found from the equality

$$P(1 < X < 2) = \int_1^2 \left(\frac{2}{3}x - \frac{2}{9}x^2\right) dx$$

$$= \left[\frac{x^2}{3} - \frac{2x^2}{27}\right]_1^2 = \frac{4}{3} - \frac{16}{27} - \frac{1}{3} + \frac{2}{27} = \frac{13}{27}.$$

818. Given the distribution series for the random variable X:

x_i	10	20	30	40	50
p_i	0.2	0.3	0.35	0.1	0.005

Construct the probability distribution function for that variable.

Solution.

If $x \leqslant 10$, then $F(x) = P(X < x) = 0$;
if $10 < x \leqslant 20$, then $F(x) = P(X < x) = 0.2$;
if $20 < x \leqslant 30$, then $F(x) = P(X < x) = 0.2 + 0.3 = 0.5$;
if $30 < x \leqslant 40$, then $F(x) = P(X < x) = 0.2 + 0.3 + 0.35 = 0.85$;

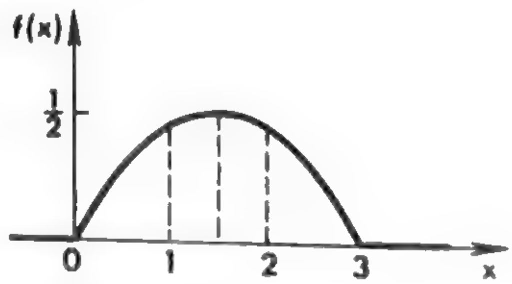


Fig. 37

if $40 < x \leqslant 50$, then $F(x) = P(X < x) = 0.2 + 0.3 + 0.35 + 0.1 = 0.95$;
if $x > 50$, then $F(x) = P(X < x) = 0.2 + 0.3 + 0.35 + 0.1 + 0.05 = 1$.

819. The random variable X is defined by the distribution function (integral function)

$$F(x) = \begin{cases} 0; & \text{if } x < 1; \\ (x-1)/2, & \text{if } 1 \leqslant x \leqslant 3; \\ 1, & \text{if } x > 3. \end{cases}$$

Calculate the probabilities of the random variable X falling in the intervals (1.5; 2.5) and (2.5; 3.5).

Solution. We have

$$P_1 = F(2.5) - F(1.5) = (2.5 - 1)/2 - (1.5 - 1)/2 = 0.75 - 0.25 = 0.5,$$
$$P_2 = F(3.5) - F(2.5) = 1 - (2.5 - 1)/2 = 1 - 0.75 = 0.25.$$

820. The random variable X is defined by the distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < 2; \\ (x-2)^2, & \text{if } 2 \leqslant x \leqslant 3; \\ 1, & \text{if } x > 3. \end{cases}$$

Calculate the probabilities of the random variable X falling in the intervals (1; 2.5) and (2.5; 3.5).

Solution. We have

$$P_1 = F(2.5) - F(1) = (2.5 - 2)^2 - 0 = 0.25,$$
$$P_2 = F(3.5) - F(2.5) = 1 - (2.5 - 2)^2 = 1 - 0.25 = 0.75.$$

821. The random variable X is defined by the distribution function indicated in the preceding problem. Find the probability density function (probability distribution differential function) of the random variable.

Solution. The density of probability distribution is equal to the derivative of the distribution function, i.e. $f(x) = F'(x)$, therefore

$$f(x) = \begin{cases} 0, & \text{if } x < 2; \\ 2(x-2) & \text{if } 2 \leqslant x \leqslant 3; \\ 0, & \text{if } x > 3. \end{cases}$$

822. A marksman fires three shots. The probability of hitting the target each time is 0.3. Construct the distribution series for the number of hits.

Hint. Use Bernoulli's formula.

823. An urn contains four balls labelled with the numbers from 1 to 4. We draw two balls. The random variable X is the sum of the numbers of the balls. Construct the distribution series for the random variable X.

824. The random variable X obeys the distribution law with the density

$$f(x) = \begin{cases} a/\sqrt{a^2 - x^2}, & \text{if } |x| < a; \\ 0, & \text{if } |x| \geqslant a. \end{cases}$$

It is required: (1) to find the coefficient a; (2) to find the probability of the random variable X falling in the interval $(a/2, a)$; (3) to construct the graph of the probability density function.

825. Show that the function $f(x) = 1/(x^2 + \pi^2)$ is the probability density of some random variable X and calculate the probability of the random variable X falling in the interval (π, ∞).

826. Given the probability density function of the random variable X:

$$f(x)=\begin{cases} 0, & \text{if} \quad x < 0; \\ a\sin x, & \text{if} \quad 0 \leqslant x \leqslant \pi; \\ 0, & \text{if} \quad x > \pi. \end{cases}$$

Determine a and $F(x)$.

827. An urn contains 5 white and 25 black balls. We have drawn one ball. The random variable X is the number of white balls we have drawn. Construct the distribution function $F(x)$.

5.6. The Mean Value and the Variance of a Random Variable

The *mean value* (*mathematical expectation*) *of a discrete random variable* is the sum of the products of the values of the random variable by the probabilities of these values.

If the random variable X is characterized by the finite distribution series

x_i	x_1	x_2	x_3	...	x_n
p_i	p_1	p_2	p_3	...	p_n

then the mean value $M(X)$ can be determined from the formula

$$M(X) = x_1p_1 + x_2p_2 + \ldots + x_np_n, \quad \text{or} \quad M(X) = \sum_{i=1}^{n} x_ip_i. \tag{1}$$

Since $p_1 + p_2 + \ldots + p_n = 1$, it follows that

$$M(x) = \frac{x_1p_1 + x_2p_2 + \ldots + x_np_n}{p_1 + p_2 + \ldots + p_n}.$$

Thus, $M(X)$ is the weighted arithmetic mean of the values $x_1, x_2, \ldots, x_n$ of the random variable for the weights $p_1, p_2, \ldots, p_n$.

If $n = \infty$, then

$$M(X) = \sum_{i=1}^{\infty} x_ip_i$$

(provided the sum of the series is finite).

The concept of the mean can be extended to a continuous random variable. Suppose $f(x)$ is the probability density function of the random variable X. Then the *mean value of the continuous random variable X* is specified by the equality

$$M(X) = \int_{-\infty}^{+\infty} xf(x)dx$$

(provided the value of the integral is finite).

In terms of geometry, the mean value of a continuous as well as a discrete random variable is equal to the abscissa of the centre of gravity of the area bounded by the curve (polygon) of distribution and the abscissa axis. Therefore, if the distribution curve (polygon) is symmetric about some straight line parallel to the axis of ordinates, then the mean coincides with the abscissa of the intersection point of that axis of symmetry and the abscissa axis.

The point of the Ox axis with the abscissa equal to the mean of the random variable is often called the *centre of distribution* of that random variable.

The *variance of a random variable* is the expectation of the square of the deviation of the random variable from its mean value:

$$D(X) = M[X - M(X)]^2.$$

The variance of a random variable is the spread of its values about its expectation.

If we introduce the notation $M(X) = m$, then the formulas for calculating the variance of the discrete random variable X will be written in the form

$$\left.\begin{aligned} D(X) &= \sum_{i=1}^{\infty} p_i(x_i - m)^2, \\ D(X) &= \sum_{i=1}^{\infty} p_i(x_i - m)^2 \quad (\text{for } n = \infty), \end{aligned}\right\} \tag{2}$$

and for the continuous random variable X, in the form

$$D(X) = \int_{-\infty}^{+\infty} (x - m)^2 f(x)dx. \tag{3}$$

The formula

$$D(X) = M[(X - a)^2] - [M(X) - a]^2 \tag{4}$$

or

$$D(X) = M[(x - a)^2] - (m - a)^2$$

is valid for the variance of a random variable, where a is an arbitrary number. This formula is often used to calculate the variance of a random variable since the calculations performed by this formula are simpler than by formulas (2) and (3).

The *standard deviation* of the random variable X is the quantity $\sigma_x = \sqrt{D(X)}$.

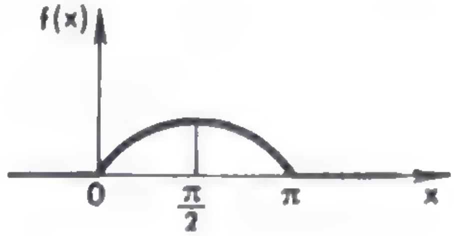


Fig. 38

828. Given the function

$$f(x) = \begin{cases} 0\,; & \text{if} \quad x<0; \\ (1/2)\sin x, & \text{if} \quad 0 \leqslant x \leqslant \pi; \\ 0\,, & \text{if} \quad x > \pi. \end{cases}$$

Show that $f(x)$ can serve as the probability density function of some random variable X. Find the mathematical expectation and the variance of the random variable X.

Solution. We have

$$\int_{-\infty}^{+\infty} f(x)dx = \int_0^{\pi} f(x)dx = \frac{1}{2}\int_0^{\pi} \sin x dx = -\frac{1}{2}\cos x \Big|_0^{\pi} = 1.$$

In addition, $f(x) \geqslant 0$. Consequently, $f(x)$ can serve as the probability density function of some random variable. Since the straight line $x = \pi/2$ is the symmetry axis of the corresponding arc of the curve $y = (1/2)\sin x$ (Fig. 38), the expectation of the random variable X is $\pi/2$, i.e $M(X) = \pi/2$.

Let us find the variance. We put $a = 0$, $M(X) = \pi/2$ in formula (4) and calculate the integral determining $M(X^2)$; we have

$$M(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx = \frac{1}{2}\int_0^{\pi} x^2 \sin x\, dx$$

$$= \frac{1}{2}[-x^2\cos x + 2x\sin x + 2\cos x]_0^{\pi} = \frac{1}{2}(\pi^2 - 4).$$

Therefore

$$D(X) = \frac{1}{2}(\pi^2 - 4) - \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4} - 2,$$

$$\sigma_x = \sqrt{\frac{\pi^2}{4} - 2} \approx 0.69.$$

829. The random variable X is characterized by the distribution series.

x_i	0	1	2	3	4
p_i	0.2	0.4	0.3	0.08	0.02

Determine its mean and variance.

Solution. We find the mean by formula (1):

$$M(X) = 0 \cdot 0.2 + 1 \cdot 0.4 + 2 \cdot 0.3 + 3 \cdot 0.08 + 4 \cdot 0.02 = 1.32$$

We find the variance by formula (4) putting $a = 2$; hence $M(X) - a = 1.32 - 2 = -0.68$. We compile a table:

x_i	0	1	2	3	4
$x_i - a$	-2	-1	0	1	2
$(x_i - a)^2$	4	1	0	1	4
p_i	0.2	0.4	0.3	0.08	0.02
$p_i(x_i - a)^2$	0.8	0.4	0	0.08	0.08

Now we find

$$M[(X - a)^2] = \sum_{i=0}^{4} p_i(x_i - a)^2 = 1.36;$$

$$D(X) = 1.36 - (-0.68)^2 = 1.36 - 0.4634 = 0.8966;$$

$$\sigma_x = \sqrt{0.8966} = 0.95.$$

830. An urn contains 6 white and 4 black balls. We draw a ball from it five times in succession, each time replacing the ball and shaking the urn. Assuming the number of white balls we have drawn to be the random variable X, formulate the distribution law for this variable, determine its mean and variance.

831. Given the function

$$f(x) = \begin{cases} 0, & \text{if} \quad x < 0; \\ \lambda(4x - x^3), & \text{if} \quad 0 \leqslant x \leqslant 2; \\ 0, & \text{if} \quad x > 2. \end{cases}$$

At what value of λ can the function $f(x)$ be taken as the probability density function of the random variable X? Determine this value of λ, find the mean and the standard deviation of the corresponding random variable X.

5.7. The Mode and the Median

The *mode of the discrete random variable* X is its most frequent value.

The *mode of the continuous random variable* X is the point at which the probability density function has its greatest value.

The mode is designated as $\bar{M}$.

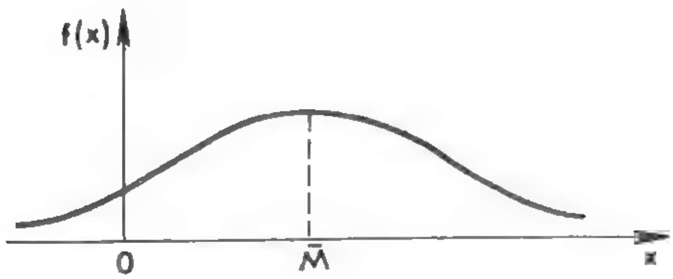


Fig. 39

The *median* of the continuous random variable X is its value μ for which it is equally probable that the random variable turns out to be less than or greater than μ, i.e.

$$P(X < \mu) = P(X > \mu) = 0.5.$$

In terms of geometry, the mode is the abscissa of the point of the distribution curve (polygon) whose ordinate is maximal. In its turn, the ordinate drawn at the point with the abscissa $x = \mu$ divides in half the area bounded by the distribution curve. If the straight line $x = a$ is the symmetry axis of the distribution curve $y = f(x)$, then $\bar{M} = \mu = M(X) = a$ (Fig. 39).

832. Given the probability density function of the random variable $f(x) = ae^{2x - x^2}$ $(a > 0)$. Find the mode of this random variable.

Solution. To find the maximum of the function $y = f(x)$, we find the derivatives of the first and second orders:

$$f'(x) = 2a(1 - x)e^{2x - x^2}, \quad f''(x) = -2ae^{2x - x^2} + 4a(1 - x)^2e^{2x - x^2}.$$

From the equation $f'(x) = 0$ we get $x = 1$. Since $f''(1) = -2ae < 0$, it follows that for $x = 1$ the function $f(x)$ possesses a maximum, i.e. $\bar{M} = 1$. We have not determined the value of the constant quantity a since the maximum of the function $f(x) = ae^{2x - x^2}$ does not depend on the numerical value of a.

833. Given the probability density function of the random variable X:

$$f(x) = \begin{cases} 0, & \text{if} \quad x < 0; \\ x - x^3/4, & \text{if} \quad 0 \leqslant x \leqslant 2; \\ 0, & \text{if} \quad x > 2. \end{cases}$$

Find the median of this random variable.

Solution. We find the median μ from the condition $P(X < \mu) = 0.5$. In the given case

$$P(X < \mu) = \int_0^{\mu} \left(x - \frac{1}{4}x^3\right) dx = \frac{\mu^2}{2} - \frac{\mu^4}{16}.$$

Thus we arrive at the equation $\mu^2/2 = \mu^4/16 = 0.5$, or $\mu^4 - 8\mu^2 + 8 = 0$, whence $\mu = \pm\sqrt{4 \pm \sqrt{8}}$. From the four roots of the equation we should choose the root contained between 0 and 2. Hence, $\mu = \sqrt{4 - \sqrt{8}} \approx 1.09$.

834. Given the distribution series for a discrete random variable

x_i	10	20	30	40	50	60
p_i	0.24	0.36	0.20	0.15	0.03	0.02

Find the mode.

835. Given the probability density function of the continuous random variable

$$f(x) = \begin{cases} 0, & \text{if} \quad x < 2; \\ a(x-2)(4-x), & \text{if} \quad 2 \leqslant x \leqslant 4; \\ 0, & \text{if} \quad x > 4. \end{cases}$$

Determine the value of a, the mode and the median.

5.8. Uniform Distribution

The distribution of random variables whose all values lie in an interval $[a, b]$ and possess a constant probability density on that interval is known as *uniform* distribution (Fig. 40). Thus,

$$f(x) = \begin{cases} 0, & \text{if} \quad x < a; \\ h, & \text{if} \quad a \leqslant x \leqslant b; \\ 0, & \text{if} \quad x > b. \end{cases}$$

Since $h(b - a) = 1$, we have $h = 1/(b - a)$ and, consequently,

$$f(x) = \begin{cases} 0, & \text{if} \quad x < a; \\ 1/(b-a), & \text{if} \quad a \leqslant x \leqslant b; \\ 0, & \text{if} \quad x > b. \end{cases}$$

836. Determine the mean of a random variable with uniform distribution.

Solution. We have

$$M(X) = \int_a^b xf(x)dx = \int_a^b \frac{x}{b-a}\,dx = \frac{1}{b-a} \cdot \frac{1}{2}x^2 \Big|_a^b = \frac{1}{b-a} \cdot \frac{b^2 - a^2}{2},$$

i.e. $M(X) = (a + b)/2$, which should be expected because of the symmetry of distribution.

837. Calculate the variance and the standard deviation for a random variable with uniform distribution.

Solution. We use the formula $D(X) = M(X^2) - [M(X)]^2$, taking into account the value $M(X) = (a + b)/2$ found in the preceding problem. Thus, it remains to calculate $M(X^2)$; we have

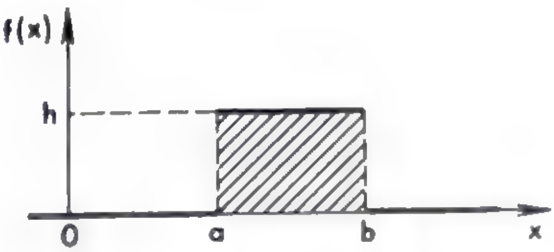


Fig. 40

$$M(X^2) = \int_a^b \frac{x^2}{b-a}\,dx = \frac{1}{3(b-a)}x^3 \Bigg|_a^b = \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + 2ab + a^2}{3}.$$

It follows that

$$D(X) = \frac{b^2 + ab + a^2}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}.$$

Consequently,

$$\sigma_x = \sqrt{D(X)} = \frac{b-a}{2\sqrt{3}}.$$

838. All the values of the uniformly distributed random variable belong to the interval [2, 8]. Find the probability of the random variable falling in the interval (3, 5).

839. Streetcars run at 5-minute intervals. A passenger comes to the stop at some moment of time. What is the probability of the passenger coming not earlier than one minute after the preceding car has left and not later than two minutes before the arrival of the next car?

5.9. Binomial Distribution. Poisson's Distribution

If the probability of occurrence of a random event in each trial is equal to p, then, as is known, the probability of the event occurring m times in n trials is specified by Bernoulli's formula

$$P_{m,n} = C_n^m p^m q^{n-m} \quad (\text{where } q = 1 - p).$$

The distribution of the random variable X which can assume $n + 1$ values $(0, 1, \ldots, n)$ described by Bernoulli's formula is known as a *binomial distribution*.

The distribution of the random variable X which can assume any integral non-negative values $(0, 1, 2, \ldots, n)$ is described by the formula

$$P(X = m) = \frac{a^m}{m!}e^{-a},$$

is known as *Poisson's distribution*.

The following random variables have a Poisson's distribution:

(a) Suppose n points are randomly distributed over the interval $(0, N)$ of the Ox axis and the events consisting in the falling of one point in any preassigned segment of constant (say, unit) length are equally probable.

If $N \to \infty$, $n \to \infty$ and $a = \lim\limits_{N \to \infty} \frac{n}{N}$, then the random variable X equal to the the number of points falling in the preassigned segment of unit length (which can assume the values 0, 1, . . ., m, . . .) has Poisson's distribution.

(b) If n is an average number of calls received by a given telephone exchange during one hour, then the number of calls received during one minute is approximately distributed by Poisson's law, with $a = n/60$.

The mathematical expectation and the variance of random variables are determined by the following formulas:

for the binomial distribution: $M(X) = np$; $D(X) = npq$;

for the Poisson distribution: $M(X) = a$; $D(X) = a$.

840. An automatic telephone exchange receives an average of 300 calls an hour. What is the probability that it will receive only two calls during the given minute?

Solution. The telephone exchange receives an average of $300/60 = 5$ calls a minute, i.e. $a = 5$. It is required to find P_2. Applying Poisson's formula, we find

$$P_2 = \frac{5^2}{2!} e^{-5} = \frac{25}{2e^5} \approx 0.09.$$

841. A book of 1000 pages contains 100 misprints. What is the probability that there are not less than four misprints on a randomly chosen page?

Solution. The average number of misprints per randomly chosen page is $a = 100/1000 = 0.1$. In this case we should apply Poisson's formula:

$$P_m = \frac{(0.1)^m}{m!} e^{-0.1}.$$

Here P_m is the probability of having m misprints on one page.

If $m = 0$, then $P_0 = e^{-0.1}$; if $m = 1$, then $P_1 = 0.1 \cdot e^{-0.1}$; if $m = 2$, then $P_2 = 0.005 \cdot e^{-0.1}$; if $m = 3$, then $P_3 = 0.000167 \cdot e^{-0.1}$. The sum $P_0 + P_1 + P_2 + P_3$ is the probability of not more than three misprints appearing on the page. This sum is equal to $1.105167 \cdot e^{-0.1}$. Now the probability that there are not less than four misprints on a randomly chosen page is equal to

$$1 - 1.105167 \cdot e^{-0.1} = 1 - 1.105167 \cdot 0.904837 = 1 - 0.999996 = 0.000004.$$

842. There are 0.4% of weed seeds among the rye seeds. What is the probability of discovering 5 weed seeds in a random sample of 5000 seeds?

843. Determine the mean and the variance of the frequency m/n of occurrence of a random event in n trials if the probability of occurrence of the event in one trial is p.

844. Show that the binomial distribution approaches Poisson's distribution in the limit if $n \to \infty$, $p \to 0$ but $np = a$.

Hint. Make use of the equality

$$C_n^m p^m q^{n-m} = \frac{(np)^m\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{m-1}{n}\right)}{m!}(1-p)^{\frac{np}{p}-m}$$

and pass to the limit.

845. The random variable X is binomially distributed $P(X = m) = C_n^m p^m q^{n-m}$. Determine its mean value.

Solution. We have

$$M(X) = \sum_{m=0}^{n} mC_n^m p^m q^{n-m} = \sum_{m=1}^{n} mC_n^m p^m q^{n-m} = np\sum_{m=1}^{n}\frac{m}{n}C_n^m p^{m-1}p^{n-m}$$

but

$$\frac{m}{n}C_n^m = \frac{m}{n}\cdot\frac{n(n-1)(n-2)\ldots(n-m+1)}{1\cdot 2\cdot 3\ldots(m-1)m} = C_{n-1}^{m-1}.$$

Consequently,

$$M(X) = np\sum_{m=1}^{n} C_{n-1}^{m-1}p^{m-1}q^{(n-1)-(m-1)} = np(p+q)^{n-1},$$

i.e. $M(X) = np$.

846. Determine the variance of the binomial random variable X.

Solution. First we find the mean of the random variable X^2:

$$M(X^2) = \sum_{m=1}^{n} m^2 C_p^m p^m q^{n-m} = np\sum_{m=1}^{n} m\cdot\frac{m}{n}C_n^m p^{m-1}q^{(n-1)-(m-1)}$$

$$= np\sum_{m=1}^{n} mC_{n-1}^{m-1}p^{m-1}q^{(n-1)-(m-1)}$$

$$= np\left(\sum_{m=1}^{n}(m-1)C_{n-1}^{m-1}p^{m-1}q^{(n-1)-(m-1)}\right.$$

$$\left.+\sum_{m=1}^{n} C_{n-1}^{m-1}p^{m-1}q^{(n-1)-(m-1)}\right).$$

The first sum in the parentheses is the mean value of the random variable X distributed binomially $P(X = m-1) = C_{n-1}^{m-1}p^{m-1}q^{(n-1)-(m-1)}$ and, therefore, it is equal to $(n-1)p$ (see the preceding problem). The second sum is equal to $(p+q)^{n-1} = 1$.

Thus we have $M(X^2) = np(n-1)p + np$. But $D(X) = M(X^2) - [M(X)]^2$, therefore

$$D(X) = n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq.$$

847. Find the mean of the random variable X having Poisson's distribution $P = (X = m) = \dfrac{a^m e^{-a}}{m!}$.

Solution. We have

$$M(X) = \sum_{m=0}^{\infty} m \frac{a^m e^{-a}}{m!} = \sum_{m=1}^{\infty} m \frac{a^m \cdot e^{-a}}{m!} = ae^{-a} \sum_{m=1}^{\infty} \frac{a^{m-1}}{(m-1)!}.$$

But $\sum_{m=1}^{\infty} \frac{a^{m-1}}{(m-1)!} = e^a$, consequently, $M(X) = a$.

848. Find the variance of the random variable X having Poisson's distribution.

Solution. We first find

$$M(X^2) = \sum_{m=1}^{\infty} m^2 \frac{a^m e^{-a}}{m!} = \sum_{m=1}^{\infty} m \frac{a^m e^{-a}}{(m-1)!}$$

$$= \sum_{m=1}^{\infty} (m-1) \frac{a^m e^{-a}}{(m-1)!} + \sum_{m=1}^{\infty} \frac{a^m e^{-a}}{(m-1)!}$$

$$= a \sum_{m=1}^{\infty} (m-1) \frac{a^{m-1} e^{-a}}{(m-1)!} + ae^{-a} \sum_{m=1}^{\infty} \frac{a^{m-1}}{(m-1)!}.$$

The first sum is the mean of the Poisson random variable X and the second sum is equal to e^a. Hence we have $M(X^2) = a^2 + a$.

Consequently, $D(X) = a^2 + a - a^2 = a$.

849. The probability of the marksman hitting the target is 2/3. The marksman fired 15 shots. The random variable X is the number of hits. Find the mean value and the variance of the random variable X.

Solution. We must use here the mean values and the variances of the binominal distribution:

$$M(X) = np = 15 \cdot 2/3 = 10, \quad D(X) = npq = 15 \cdot 2/3 \cdot 1/3 = 10/3.$$

5.10. Normal Distribution. Laplace's Function

The *normal distribution* is characterized by the density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}.$$

It is easy to see that the function $f(x)$ satisfies two conditions a density function is subject to: (1) $f(x) > 0$; (2) $\int_{-\infty}^{+\infty} f(x)dx = 1$.

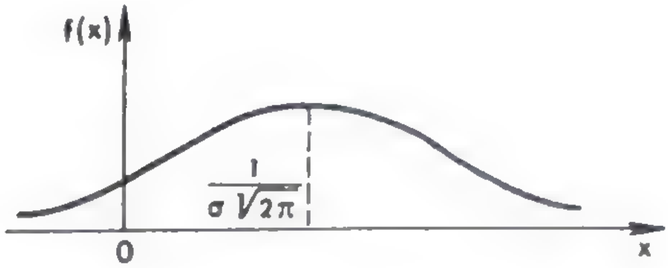


Fig. 41

The curve $y = f(x)$ is shaped as shown in Fig. 41. It is symmetric about the straight line $x = m$, the maximum ordinate of the curve (at $x = m$) is equal to $1/(\sigma\sqrt{2\pi})$ and the abscissa axis is an asymptote to that curve. Since $\int_{-\infty}^{+\infty} xf(x)dx = m$, the parameter m is the mean of the random variable X. On the other hand, $\int_{-\infty}^{+\infty} (x - m)^2 f(x)dx = \sigma^2$, whence $D(x) = \sigma^2$, that is, σ is the standard deviation of the variable X.

We introduce the notation

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

The function $\Phi(x)$ is called the *Laplace function* or the *error integral*. This function is also known as an *error function* and is designated as erf x. Sometimes other forms of Laplace's function are used, for instance the form $\overline{\Phi}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt$ (*normalized Laplace function*), which is related to the error function $\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ as $\overline{\Phi}(x) = 0.5\Phi(x/\sqrt{2})$, or $\overline{\Phi}(x\sqrt{2}) = 0.5\Phi(x)$.

A special table is used to calculate the values of Laplace's function (see Table II on p. 479).

The probability that the normally distributed random variable X will fall in the interval (a, b) is determined from the values of the Laplace function by the formula

$$P(a < X < b) = 0.5 \left[\Phi\left(\frac{b - m}{\sigma\sqrt{2}}\right) - \Phi\left(\frac{a - m}{\sigma\sqrt{2}}\right) \right].$$

Here are some properties of Laplace's function.

1°. $\Phi(0) = 0$, since $\int_0^0 e^{-t^2} dt = 0$;

2°. $\Phi(+\infty) = 1$, since $\Phi(+\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1$;

3°. $\Phi(x)$ is an odd function.

The following formula is also valid:

$$P(|X - m| < \varepsilon) = \Phi\left(\frac{\varepsilon}{\delta\sqrt{2}}\right).$$

This formula aids in seeking the probability that the normally distributed random variable falls in the interval symmetric about the point m.

850. The normal random variable X is distributed with the mathematical expectation $m = 40$ and the variance $D = 200$. Calculate the probability of the random variable falling in the interval (30, 80).

Solution. Here $a = 30$, $b = 80$, $m = 40$, $\sigma = \sqrt{200} = 10\sqrt{2}$; using Table II (see Appendix), we find

$$P(30 < X < 80) = 0.5\left[\Phi\left(\frac{80 - 40}{10\sqrt{2}\cdot\sqrt{2}}\right) - \Phi\left(\frac{30 - 40}{10\sqrt{2}\cdot\sqrt{2}}\right)\right]$$

$$= 0.5[\Phi(2) + \Phi(0.5)] = 0.5[0.995 + 0.521] = 0.758.$$

851. The deviation of the length of the articles being manufactured from the standard is considered to be a normal random variable. Suppose the standard length is $m = 40$ cm and the standard deviation is $\sigma = 0.4$ cm. What accuracy of the length of the article can be ensured with the probability 0.8?

Solution. It is required to find the positive number ε such that $P(|X - 40| < \varepsilon) = 0.8$. Since

$$P(|X - 40| < \varepsilon) = \Phi\left(\frac{\varepsilon}{0.4\sqrt{2}}\right) = \Phi(1.77\varepsilon),$$

the problem reduces to solving the inequality $\Phi(1.77\varepsilon) > 0.8$. We establish, with the aid of Table II, that $1.77\varepsilon > 0.91$. It remains to find the minimum value of ε satisfying this inequality, whence it follows that $\varepsilon = 0.52$.

852. Shells are fired from the point O along the straight line Ox. The average firing distance is m. Assuming the firing distance X to be normally distributed with the standard deviation $\sigma = 80$ m, find the percentage of the shots with an overshoot of 120 to 160 m.

853. The normal random variable X is distributed with the mathematical expectation m and the standard deviation σ. Calculate with an accuracy to within 0.01 the probability of X falling in the intervals $(m, m + \sigma)$, $(m + \sigma, m + 2\sigma)$, $(m + 2\sigma, m + 3\sigma)$.

854. Given a random variable X with expectation m and standard deviation σ, obeying the normal distribution law. Show that the probability of the variable falling in the interval (a, b) will not change if each of the numbers a, b, m and σ is increased λ times ($\lambda > 0$).

5.11. Moments, Skewness and Excess of a Random Variable

The *initial moment* of the sth-order discrete random variable X defined by the distribution series

x_i	x_1	x_2	...	x_n	...
p_i	p_1	p_2	...	p_n	...

is the sum of the series

$$\alpha_s = x_1^s p_1 + x_2^s p_2 + \ldots + x_n^s p_n + \ldots .$$

For the continuous random variable X with the probability density function $f(x)$ the initial moment of the sth order is the integral

$$\alpha_s = \int_{-\infty}^{+\infty} x^s f(x)dx.$$

It is easy to see that the first-order initial moment of the random variable X is the mathematical expectation of that random variable: $\alpha_1 = M(X)$.

The *central moment* of the s-order of the discrete random variable X is the sum of the series

$$\mu_s = (x_1 - m_x)^s \cdot p_1 + (x_2 - m_x)^s \cdot p_2 + \ldots + (x_n - m_x)^s \cdot p_n + \ldots,$$

where m_x is the mean.

For a continuous random variable the central moment of order s is the integral

$$\mu_s = \int_{-\infty}^{+\infty} (x - m_x)^s f(x)dx.$$

For any random variable the central moment of the first order is zero, i.e. $\mu_1 = 0$.

The central moment of the second order of any random variable is the variance of the random variable, i.e. $\mu_2 = D(X)$.

The central and initial moments of the first, second, third and fourth orders are related as

$$\begin{aligned}
\mu_1 &= 0,\\
\mu_2 &= \alpha_2 - \alpha_1^2,\\
\mu_3 &= \alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3,\\
\mu_4 &= \alpha_4 - 4\alpha_1\alpha_3 + 6\alpha_1^2\alpha_2 - 3\alpha_1^4.
\end{aligned}$$

If the distribution is symmetric about the mathematical expectation, then all the odd central moments are zero, i.e. $\mu_1 = \mu_3 = \mu_5 = \ldots = 0$.

The ratio between the central moment of the third order and the cube of the standard deviation is an *asymmetry* (*skewness*)

$$S_k = \mu_3/\sigma_x^3.$$

The distribution curve is called a *histogram*.

If the distribution is symmetric about the mathematical expectation, then $S_k = 0$.

Figures 42 and 43 show the histograms for $S_k > 0$ and $S_k < 0$.

The *excess* of the random variable X is the quantity E_x specified by the equation

$$E_x = \mu_4/\sigma_x^4 - 3.$$

For the normal distribution law $E_x = 0$.

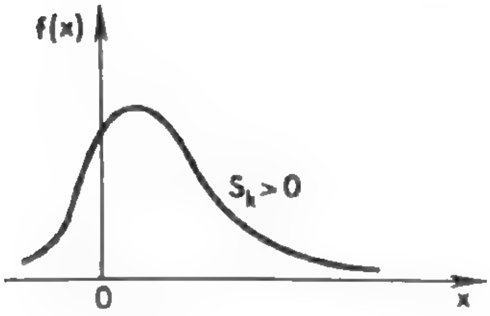


Fig. 42

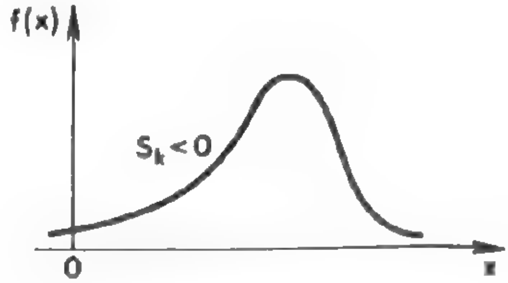


Fig. 43

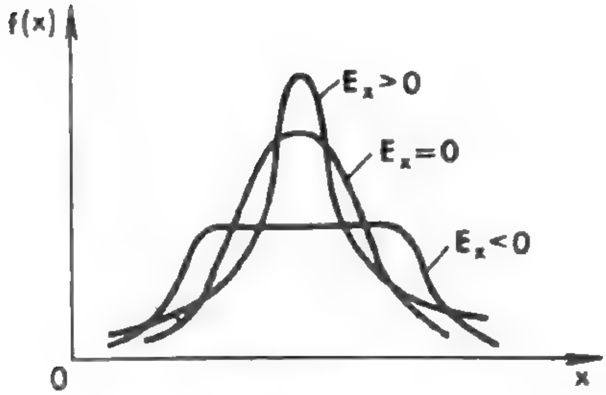


Fig. 44

Curves more picked as compared to the normal (so-called *Gaussian*) curve possess a positive excess; for curves less picked, $E_x < 0$ (Fig. 44).

855. Given the distribution series for the random variable X:

x_i	1	3	5	7	9
p_i	0.1	0.4	0.2	0.2	0.1

Find the initial and central moments of the first four orders of that random variable, determine the skewness and the excess.

Solution. The first-order initial moments is

$$\alpha_1 = 1 \cdot 0.1 + 3 \cdot 0.4 + 5 \cdot 0.2 + 7 \cdot 0.2 + 9 \cdot 0.1 = 4.6.$$

The first-order initial moments is the mathematical expectation, therefore $M(X) = 4.6$.

Let us find the initial moment of the second order:

$$\alpha_2 = 1 \cdot 0.1 + 9 \cdot 0.4 + 25 \cdot 0.2 + 49 \cdot 0.2 + 81 \cdot 0.1 = 26.6.$$

The initial moment of the third order is

$$\alpha_3 = 1 \cdot 0.1 + 27 \cdot 0.4 + 125 \cdot 0.2 + 343 \cdot 0.2 + 729 \cdot 0.1 = 177.4.$$

The initial moment of the fourth order is

$$\alpha_4 = 1 \cdot 0.1 + 81 \cdot 0.4 + 625 \cdot 0.2 + 2401 \cdot 0.2 + 6561 \cdot 0.1 = 1293.8.$$

Let us now find the central moments. As is known, $\mu_1 = 0$. The second-order central moment will be found by the formula

$$\mu_2 = \alpha_2 - \alpha_1^2 = 26.6 - 4.6^2 = 26.6 - 21.16 = 5.44.$$

This central moment is the variance of the random variable, i.e. $D(X) = 5.44$.

Hence it is easy to determine the standard deviation:

$$\sigma_x = \sqrt{D(X)} = \sqrt{5.44} = 2.33.$$

The third-order central moment can be found by the formula

$$\mu_3 = \alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3 = 177.4 - 3 \cdot 4.6 \cdot 26.6 + 2 \cdot 4.6^3$$
$$= 177.4 - 367.08 + 194.672 = 4.992.$$

Now it is easy to determine the skewness:

$$S_k = \frac{\mu_3}{\sigma_x^3} = \frac{4.992}{5.44 \cdot 2.33} = \frac{4.992}{12.675} = 0.394.$$

To determine the fourth-order central moment, use will be made of the formula

$$\mu_4 = \alpha_4 - 4\alpha_1\alpha_3 + 6\alpha_1^2\alpha_2 - 3\alpha_1^4 = 1293.8 - 4 \cdot 4.6 \cdot 177.4 + 6 \cdot 4.6^2 \cdot 26.6 -$$
$$3 \cdot 4.6^4 = 1293.8 - 3264.16 + 3377.136 - 1343.227 = 64.55.$$

Now we can find the excess:

$$E_x = \frac{\mu_4}{\sigma_x^4} - 3 = \frac{64.55}{5.44^2} - 3 = \frac{64.55}{29.59} - 3 = 2.18 - 3 = -0.82.$$

856. Given the function

$$f(x) = \begin{cases} 0, & \text{if} \quad x < 0; \\ ax^2, & \text{if} \quad 0 \leqslant x < 1; \\ a(2 - x)^2, & \text{if} \quad 1 \leqslant x < 2; \\ 0, & \text{if} \quad x \geqslant 2 \end{cases}$$

(Fig. 45). At what value of a is $f(x)$ the probability density function of the random variable X? Determine the initial and central moments of the first four orders, the skewness and the excess.

Solution. To find a we have the equation

$$a \int_0^1 x^2 dx + a \int_1^2 (2 - x)^2 dx = 1,$$

whence

$$a \cdot \frac{x^3}{3}\Big|_0^1 - a \cdot \frac{(2 - x)^3}{3}\Big|_0^2 = 1; \quad \frac{a}{3} + \frac{a}{3} = 1, \quad \text{i.e.} \quad a = \frac{3}{2}.$$

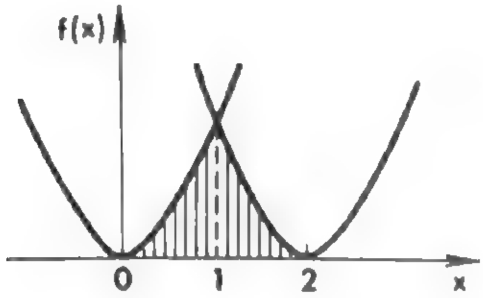


Fig. 45

We find the initial moments:

$$\alpha_1 = \frac{3}{2}\int_0^1 x^3dx + \frac{3}{2}\int_1^2 x(2-x)^2dx = \frac{3}{2}\cdot\frac{1}{4} + \frac{3}{2}\left(6 - \frac{28}{3} + \frac{15}{4}\right) = 1;$$

$$\alpha_2 = \frac{3}{2}\int_0^1 x^4dx + \frac{3}{2}\int_1^2 x^2(2-x)^2dx$$
$$= \frac{3}{10} + \frac{3}{2}\left[\frac{4x^3}{3} - x^4 + \frac{x^3}{5}\right]_1^2 = \frac{3}{10} + 14 - \frac{45}{2} + \frac{93}{10} = 1.1;$$

$$\alpha_3 = \frac{3}{2}\int_0^1 x^5dx + \frac{3}{2}\int_1^2 x^3(2-x)^2dx$$
$$= \frac{3}{12} + \frac{3}{2}\left[x^4 - \frac{4}{5}x^5 + \frac{x^6}{6}\right]_1^2 = \frac{1}{4} + \frac{45}{2} - \frac{186}{5} + \frac{63}{4} = 1.3;$$

$$\alpha_4 = \frac{3}{2}\int_0^1 x^6dx + \frac{3}{2}\int_1^2 x^4(2-x)^2dx$$
$$= \frac{3}{14} + \frac{3}{2}\left[\frac{4x^5}{5} - \frac{2x^6}{3} + \frac{x^7}{7}\right]_1^2 = \frac{3}{14} + \frac{186}{5} - 63 + \frac{381}{14} = 1\frac{22}{35}.$$

Now we find the central moments:

$$\mu_1 = 0;$$
$$\mu_2 = \alpha_2 - \alpha_1^2 = 1.1 - 1 = 0.1;$$
$$\mu_3 = \alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3 = 1.3 - 3\cdot 1.1 + 2 = 0$$

(indeed, the curve possesses a vertical symmetry axis);

$$\mu_4 = \alpha_4 - 4\alpha_1\alpha_3 + 6\alpha_1^2\alpha_2 - 3\alpha_1^4 = 1\frac{22}{35} - 4\cdot 1.3 + 6\cdot 1.1 - 3 = \frac{1}{35}.$$

From this we get

$$D(X) = \mu_2 = 0.1 \quad \text{(variance)};$$
$$\sigma_x = \sqrt{D(X)} = \sqrt{0.1} = 0.316 \quad \text{(standard deviation)}.$$

We find the skewness: $S_k = \mu_3/\sigma_x^3 = 0$.

Now we find the excess: $E_x = \frac{\mu_4}{\sigma_x^4} - 3 = \frac{1/35}{0.01} - 3 = -\frac{1}{7}$.

857. Given the distribution series of a random variable:

x_i	2	4	6	8
p_i	0.4	0.3	0.2	0.1

Find the initial and central moments of the first four orders of that random variable and also the skewness and the excess.

858. The probability density function of the random variable X is defined as follows:

$$f(x) = \begin{cases} 0, & \text{if} \quad x < 0; \\ x, & \text{if} \quad 0 \leqslant x < 1; \\ 2 - x, & \text{if} \quad 1 \leqslant x < 2; \\ 0, & \text{if} \quad x \geqslant 2. \end{cases}$$

Find the initial and central moments of the first four orders, the skewness and the excess.

859. The random variable X is distributed with the density $f(x) = \lambda e^{-|x|}$. Determine the value of λ and the excess of the random variable X.

5.12. Law of Large Numbers

5.12.1. Chebyshev's theorem. The random variable X_n is said to *converge in probability* to a if for sufficiently large n there holds the inequality

$$P(|X_n - a| < \varepsilon) > 1 - \delta,$$

where ε is an arbitrary small positive number, and the value of δ depends on the choice of ε and n. In terms of the given definition **Chebyshev's theorem** may be formulated as follows: *for a sufficiently large number of independent trials the arithmetic mean of the observed values of the random variable converged in probability to its mathematical expectation*, that is,

$$P\left(\left|\left(\sum_{i=1}^{n} x_i\right)\Big/n - M(X)\right| < \varepsilon\right) > 1 - \delta.$$

In this inequality we can assume $0 < \delta < \frac{D(X)}{n\varepsilon^2}$, where $D(X)$ is the variance of the random variable X.

Chebyshev's theorem is one of the laws of large numbers which underlie all practical applications of the theory of probability.

5.12.2. Bernoulli's theorem. Another and more simple law of large numbers, established earlier than all the other laws, is **Bernoulli's theorem** which states that *as*

the number of trials increases indefinitely, the frequency of the random event converges in probability to the probability of the event, that is

$$P\left(\left|\frac{m}{n}-p\right|<\varepsilon\right)>1-\delta \tag{1}$$

$\left(\text{we can assume } 0<\delta<\dfrac{pq}{n\varepsilon^2}\right)$, if the probability of the event does not vary from trial to trial and is equal to p $(q = 1 - p)$.

860. A coin is tossed 1000 times. Estimate from above the fact that the probability of deviation of the frequency of the coin falling heads up from the probability of its falling heads up is less than 0.1.

Solution. Here $n = 1000$, $p = q = 1/2$, $\varepsilon = 0.1$. Using inequality (1), we obtain

$$P\left(\left|\frac{m}{1000}-\frac{1}{2}\right|<0.1\right)>1-\frac{1/2\cdot 1/2}{1000\cdot 0.01}=\frac{39}{40}.$$

Since the inequality $\left|\dfrac{m}{1000}-\dfrac{1}{2}\right|<0.1$ is equivalent to the double inequality $400 < m < 600$, we may say that the probability of heads falling in the interval (400, 600) exceeds 39/40.

861. An urn contains 1000 white and 2000 black balls. We draw (with replacement) 300 balls. Estimate from above the probability that the number m of the drawn white balls satisfies the double inequality $80 < m < 120$.

Solution. The given double inequality can be rewritten as

$$-20 < m - 100 < 20, \quad \text{or} \quad -\frac{1}{15}<\frac{m}{300}-\frac{1}{3}<\frac{1}{15}.$$

It is required to evaluate the probability of the inequality $\left|\dfrac{m}{300}-\dfrac{1}{3}\right|<\dfrac{1}{15}$; consequently, $\varepsilon = \dfrac{1}{15}$ and

$$P\left(\left|\frac{m}{300}-\frac{1}{3}\right|<\frac{1}{15}\right)>1-\frac{1/3\cdot 2/3}{300\cdot 1/225}=\frac{5}{6}.$$

862. Suppose that 100 independent trials have yielded the values of the random variable X: $x_1, x_2, \ldots, x_{100}$. Assume that the mathematical expectation is $M(X) = 10$ and the variance $D(X) = 1$. Estimate from above the probability that the absolute value of the difference between the arithmetic mean of the observed values of the random variable $\left(\sum\limits_{i=1}^{100} x_i\right)\Big/100$ and the mathematical expectation is less than 1/2.

Solution. We make use of the inequality

$$P\left(\left| \left(\sum_{i=1}^{n} x_i \right) \Big/ n - M(X) \right| < \varepsilon \right) > 1 - \frac{D(X)}{n\varepsilon^2}.$$

By the hypothesis $n = 100$, $M(X) = 10$, $D(X) = 1$, $\varepsilon = 1/2$, we get

$$P\left(\left| \left(\sum_{i=1}^{100} x_i \right) \Big/ 100 - 10 \right| < \frac{1}{2} \right) > 1 - \frac{1}{100 \cdot (1/4)} = \frac{24}{25}.$$

Thus, the desired probability exceeds 0.96.

863. Each of the two urns contains 10 balls labelled with the numbers from 1 to 10. The experiment consists in drawing (with replacement) a ball from each urn. The random variable X is the sum of the numbers of the balls drawn from the two urns. A hundred trials have been performed. Estimate from above the probability of the sum $\sum\limits_{i=1}^{100} x_i$ falling in the interval (800, 1400).

Solution. Let us find the law of distribution of the random variable X. This random variable (the sum of the numbers of the balls drawn from the urns) can assume the values $x_1 = 2$; $x_2 = 3$; . . .; $x_{19} = 20$.

Let us find the probability of X assuming the value $x_k = k + 1$. If $k \leqslant 10$, then the sum of the numbers of the drawn balls can be equal to $k + 1$ in the following k equiprobable cases:

the first ball is labelled with the number 1, the second, with k;

the first ball is labelled with the number 2, the second, with $k - 1$;

....

the first ball is labelled with the number k, the second, with 1.

The probability of each of these combinations being $(1/10) \cdot (1/10) = 1/100$, the probability p_k of two balls having the sum of the numbers $k + 1$ (provided $k \leqslant 10$) is $k/100$. Thus, $p_k = k/100$ (provided $k \leqslant 10$).

If $k > 10$, the sum of the numbers of the drawn balls can be equal to $k + 1$ in the following equiprobable cases (whose number is $20 - k$):

the first ball is labelled with the number $k - 9$, the second, with 10,

the first ball is labelled with the number $k - 8$, the second, with 9.

....

the first ball is labelled with the number 10, the second, with $k - 9$.

Since the probability of each of these combinations is equal to 1/100 as before, we have $p_k = (20 - k)/100$ for $k > 10$.

To determine $M(X)$ and $D(X)$, we compile a table:

k	x_k	p_k	$p_k x_k$	$x_k - M(X)$	$[x_k - M(X)]^2$	$p_k[x_k - M(X)]^2$
1	2	0.01	0.02	−9	81	0.81
2	3	0.02	0.06	−8	64	1.28
3	4	0.03	0.12	−7	49	1.47
4	5	0.04	0.20	−6	36	1.44
5	6	0.05	0.30	−5	25	1.25
6	7	0.06	0.42	−4	16	0.96
7	8	0.07	0.56	−3	9	0.63
8	9	0.08	0.72	−2	4	0.32
9	10	0.09	0.90	−1	1	0.09
10	11	0.10	1.10	0	0	0
11	12	0.09	1.08	1	1	0.09
12	13	0.08	1.04	2	4	0.32
13	14	0.07	0.98	3	9	0.63
14	15	0.06	0.90	4	16	0.96
15	16	0.05	0.80	5	25	1.25
16	17	0.04	0.68	6	36	1.44
17	18	0.03	0.54	7	49	1.47
18	19	0.02	0.38	8	64	1.28
19	20	0.01	0.20	9	81	0.81
Σ		1.00	11.00	—	—	16.50

Thus we have

$$M(X) = \sum_{k=1}^{19} p_k x_k = 11, \quad D(X) = 16.5.$$

It is evident that

$$\left(800 < \sum_{i=1}^{100} x_i < 1400\right) \Leftrightarrow \left(-300 < \sum_{i=1}^{100} x_i - 1100 < 300\right)$$

$$\Leftrightarrow -3 < \left(\sum_{i=1}^{100} x_i\right)\Big/100 - 11 < 3 \Leftrightarrow \left| \left(\sum_{i=1}^{100} x_i\right)\Big/100 - 11 \right| < 3.$$

Thus, $\varepsilon = 3$. Consequently,

$$P\left(\left| \left(\sum_{i=1}^{100} x_i\right)\Big/100 - 11 \right| < 3\right) > 1 - \frac{16.5}{900} \approx 0.982.$$

864. A die is thrown 10000 times. Estimate the fact that the probability of deviation of the frequency of rolling a six from the probability of a six coming up is less than 0.01.

865. An urn contains 100 white and 100 black balls. Fifty balls have been drawn with replacement. Estimate from above the probability that the number of white balls drawn satisfies the double inequality $15 < m < 35$.

866. Two hundred independent trials yielded the values $x_1, x_2, \ldots, x_{200}$ for the random variable, with $M(X) = D(X) = 2$. Estimate from above the probability that the absolute value of the difference between the arithmetic mean of the random variable $\left(\sum_{i=1}^{200} x_i\right) \Big/ 200$ and the mathematical expectation is less than 1/5.

5.13. De Moivre-Laplace Theorem

If n trials are performed in each of which the probability of the event A is equal to p, then the frequency m/n of the occurrence of the event is a binomial random variable whose mean and variance are equal to p and $\sqrt{pq/n}$ respectively. The random variable $\tau_n = \dfrac{m/n - p}{\sqrt{pq/n}}$, whose mean is zero and the variance is unity, is called the *normalized* (*standardized*) frequency of the random event (its distribution is also binomial).

The De Moivre-Laplace theorem states that with an infinite increase in the number n of trials, the binomial distribution of the normalized frequency turns, in the limit, into the normal distribution with the same mean value (zero) and the same variance (unity). Because of this, at large values of n for the probabilities of the inequalities which must be satisfied by the frequency (or the occurrences) of the random event, use can be made of an approximate estimation by means of the probability integral (Laplace's function); namely, the following approximations are valid:

$$P\left(a < \frac{m/n - p}{\sqrt{pq/n}} < b\right) = P\left(a < \frac{m - np}{\sqrt{npq}} < b\right) \approx \frac{1}{2}\left[\Phi\left(\frac{b}{\sqrt{2}}\right) - \Phi\left(\frac{a}{\sqrt{2}}\right)\right].$$

867. What is the probability that in n trials the event A will occur from α to β times? The probability of occurrence of the event A is p.

Solution. It is evident that

$$\left(a < \frac{x - np}{\sqrt{npq}} < b\right) \Leftrightarrow (np + a\sqrt{npq} < x < np + b\sqrt{npq}).$$

Putting $np + a\sqrt{npq} = \alpha$, $np + b\sqrt{npq} = \beta$, we get $a = (\alpha - pn)/\sqrt{npq}$, $b = (\beta - np)/\sqrt{npq}$. Applying De Moivre-Laplace theorem, we obtain

$$P(\alpha < X < \beta) = \frac{1}{2}\left[\Phi\left(\frac{\beta - np}{\sqrt{2npq}}\right) - \Phi\left(\frac{\alpha - np}{\sqrt{2npq}}\right)\right].$$

868. The probability of the event A in each trial is equal to 0.7. What must be the sufficient number of trials to expect, with the probability 0.9, the frequency of occurrence of the event A to deviate from the probability of success by not more than 0.05?

Solution. It follows from the hypothesis that $\dfrac{X}{2} - 0.7 < 0.05$. Hence $0.65n < X < 0.75n$. In the formula obtained in the solution of problem 867 we set $\alpha = 0.65n$, $\beta = 0.75n$. Then we have

$$P(0.65n < X < 0.75n)$$
$$= \frac{1}{2}\left[\Phi\left(\frac{0.75n - 0.7n}{\sqrt{2n(1/2)\cdot(1/2)}}\right) - \Phi\left(\frac{0.65n - 0.7n}{\sqrt{2n(1/2)\cdot(1/2)}}\right)\right] = \Phi\left(\frac{\sqrt{2n}}{20}\right).$$

Using Table II (see Appendix), we find from the equation $\Phi(\sqrt{2n}/20) = 0.9$ that $\sqrt{2n}/20 = 1.17$, i.e. $n = 273$.

869. What is the probability that we shall have "heads" from 40 to 60 times if we toss the coin 100 times?

Hint. Use the result of problem 867 at $\alpha = 40$, $\beta = 60$, $n = 100$, $p = q = 1/2$.

870. An urn contains 80 white and 20 black balls. What is the number of balls that must be drawn from the urn (with replacement) to expect, with the probability 0.95, the frequency of a white ball to deviate from its probability by less than 0.1?

5.14. Systems of Random Variables

The result of the experiment is often described not by one random variable X but by several random variables $X_1, X_2, \ldots, X_n$. In that case, it is customary to say that the indicated random variables form a system $(X_1, X_2, \ldots, X_n)$.

A system of two random variables (X, Y) may be represented by a random point on a plane.

The event consisting in a random point $(X; Y)$ falling in the domain D is usually designated as $(X; Y) \subset D$.

The distribution of a system of two discrete random variables can be defined by means of the table

$X \backslash Y$	y_1	y_2	...	y_n
x_1	p_{11}	p_{12}	...	p_{1n}
x_2	p_{21}	p_{22}	...	p_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_m	p_{m1}	p_{m2}	...	p_{mn}

where $x_1 < x_2 < \ldots < x_m$, $y_1 < y_2 < \ldots < y_n$, p_{ij} is the probability of the event consisting in a simultaneous satisfaction of two equalities $X = x_i$, $Y = y_j$. In this case, $\sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij} = 1$. The table may contain an infinite number of rows and columns.

We shall define the law of distribution of the system of the continuous random variables (X, Y) by means of the probability density function $f(x, y)$.

The probability of the random point $(X; Y)$ falling in the domain D is specified by the equality

$$P[(X; Y) \subset D] = \iint_D f(x, y)dxdy.$$

The probability density function possesses the following properties:

1°. $f(x, y) \geqslant 0$.

2°. $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)dxdy = 1$.

If all the random points $(X; Y)$ belong to the finite domain D, the last condition assumes the form

$$\iint_D f(x, y)dxdy = 1.$$

The mean values of the discrete random variable X and Y entering into the system are specified by the formulas

$$m_x = M(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i p_{ij}, \quad m_y = M(Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} y_j p_{ij},$$

and the mean values of the continuous random variables, by the formulas

$$m_x = M(X) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x \cdot f(x, y)dxdy, \quad m_y = M(Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} y \cdot f(x, y)dxdy.$$

The point $(m_x; m_y)$ is called the *centre of dispersion* of the system of the random variables (X, Y).

The mean values m_x and m_y can be found in a simpler way if the random variables X and Y are independent. In that case use can be made of the distribution formula given in 5.6.

The variances of the discrete random variables X and Y are determined from the formulas

$$D(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}(x_i - m_x)^2, \quad D(Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}(y_i - m_y)^2.$$

The variance of the continuous random variables X and Y entering into the system are determined by the formulas

$$D(x) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - m_x)^2 \cdot f(x, y)dxdy,$$

$$D(Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (y - m_y)^2 \cdot f(x, y)dxdy.$$

The standard deviations of the random variables X and Y are found by the formulas

$$\sigma_x = \sqrt{D(X)}, \quad \sigma_y = \sqrt{D(Y)}.$$

The variances can also be found by the formulas

$$D(X) = M(X^2) - [M(X)]^2, \quad D(Y) = M(Y^2) - [M(Y)]^2.$$

An important role in the theory of systems of random variables is played by the so-called *correlation moment* (*covariance*)

$$C_{xy} = M[(X - m_x)(Y - m_y)].$$

For discrete random variables the covariance can be found by the formula

$$C_{xy} = \sum_m \sum_n (x_n - m_x)(y_m - m_y)p_{nm},$$

and for continuous random variables, by the formula

$$C_{xy} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - m_x)(y - m_y)f(x, y)dxdy.$$

The covariance can also be found by the formula

$$C_{xy} = M(XY) - M(X)M(Y).$$

Here

$$M(XY) = \sum_m \sum_n x_n y_m p_{mn}$$

for the discrete random variables X and Y, and

$$M(XY) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xyf(x, y)dxdy$$

for continuous random variables.

The random variables X and Y are said to be *independent* if the probability of one of them, taken to be the value lying in any interval of its range, is independent of the value assumed by the other variable. In this case

$$M(XY) = M(X) \cdot M(Y); \quad C_{xy} = 0.$$

To characterize the correlation between the variables X and Y, use is made of the so-called *correlation coefficient*

$$r_{xy} = \frac{C_{xy}}{\sigma_x \sigma_y},$$

which is a dimensionless quantity.

If the random variables X and Y are independent, then $r_{xy} = 0$. Now if the random variables X and Y are connected by the exact linear relationship $Y = aX + b$, then $r_{xy} = \operatorname{sign} a$, i.e. $r_{xy} = 1$ for $a > 0$ and $r_{xy} = -1$ for $a < 0$.

In general, the correlation coefficient satisfies the condition $-1 \leqslant r_{xy} \leqslant 1$.

871. Two boxes contain six balls each; the first box contains one ball labelled 1, two balls labelled 2, three balls labelled 3; the second box contains two balls labelled 1, three balls labelled 2 and one ball labelled 3. Suppose X is the number of the ball drawn from the first box, Y is the number of the ball drawn from the second box. We have drawn a ball from each box. Compile a table describing the distribution of the system of the random variables (X, Y).

Solution.

The random point (1; 1) is of multiplicity $1 \times 2 = 2$;
the random point (1; 2) is of multiplicity $1 \times 3 = 3$;
the random point (1; 3) is of multiplicity $1 \times 1 = 1$;
the random point (2; 1) is of multiplicity $2 \times 2 = 4$;
the random point (2; 2) is of multiplicity $2 \times 3 = 6$;
the random point (2; 3) is of multiplicity $2 \times 1 = 2$;
the random point (3; 1) is of multiplicity $3 \times 2 = 6$;
the random point (3; 2) is of multiplicity $3 \times 3 = 9$;
the random point (3; 3) is of multiplicity $3 \times 1 = 3$.

There are $6 \times 6 = 36$ random points all in all (the n-tuple point is taken as n points).

Since the relation between the multiplicity of a point and the whole number of points is equal to the probability of occurrence of that point, the table of distribution of the system of random variables has the form

X \ Y	1	2	3
1	1/18	1/12	1/36
2	1/9	1/6	1/18
3	1/6	1/4	1/12

The sum of all the probabilities indicated in the table is equal to unity.

872. Find the means of the random variables X and Y from the hypothesis of the preceding problem.

Solution. We have

$$m_x = 1 \cdot \frac{1}{18} + 2 \cdot \frac{1}{9} + 3 \cdot \frac{1}{6} + 1 \cdot \frac{1}{12} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{4} + 1 \cdot \frac{1}{36} + 2 \cdot \frac{1}{18} + 3 \cdot \frac{1}{12} = \frac{7}{3};$$

$$m_y = 1 \cdot \frac{1}{18} + 2 \cdot \frac{1}{12} + 3 \cdot \frac{1}{36} + 1 \cdot \frac{1}{9} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{18} + 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{12} = \frac{11}{6}.$$

The point (7/3; 11/6) is the centre of dispersion of the given system (X, Y).

The random variables X and Y being independent (see the hypothesis of problem 871), the mean values m_x and m_y can be calculated in a much simpler way, by using

the distribution series:

x_i	1	2	3	y_j	1	2	3
p_i	1/6	1/3	1/2	p_j	1/3	1/2	1/6

From this we find

$$m_x = \sum p_i x_i = \frac{1}{6} + \frac{2}{3} + \frac{3}{2} = \frac{7}{3}, \quad m_y = \sum p_j y_j = \frac{1}{3} + 1 + \frac{1}{2} = \frac{11}{6}.$$

873. Find the variances of the random variables X and Y from the hypothesis of problem 871.

Solution. We pass from the system of variables (X, Y) to the system of centred variables $(\mathring{X}, \mathring{Y})$, where $\mathring{X} = X - m_x = X - 7/3$, $\mathring{Y} = Y - m_y = Y - 11/6$. We compile a table

$\mathring{X}$ \ $\mathring{Y}$	$-5/6$	$1/6$	$7/6$
$-4/3$	1/18	1/12	1/36
$-1/3$	1/9	1/6	1/18
$2/3$	1/6	1/4	1/12

We have

$$D(X) = \left(-\frac{4}{3}\right)^2 \cdot \frac{1}{18} + \left(-\frac{1}{3}\right)^2 \cdot \frac{1}{9} + \left(\frac{2}{3}\right)^2 \cdot \frac{1}{6} + \left(-\frac{4}{3}\right)^2 \cdot \frac{1}{12}$$
$$+ \left(-\frac{1}{3}\right)^2 \cdot \frac{1}{6} + \left(\frac{2}{3}\right)^2 \cdot \frac{1}{4} + \left(-\frac{4}{3}\right)^2 \cdot \frac{1}{36} + \left(-\frac{1}{3}\right)^2 \cdot \frac{1}{18}$$
$$+ \left(\frac{2}{3}\right)^2 \cdot \frac{1}{12} = \frac{5}{9};$$

$$D(Y) = \left(-\frac{5}{6}\right)^2 \cdot \frac{1}{18} + \left(-\frac{5}{6}\right)^2 \cdot \frac{1}{9} + \left(-\frac{5}{6}\right)^2 \cdot \frac{1}{6} + \left(\frac{1}{6}\right)^2 \cdot \frac{1}{12}$$
$$+ \left(\frac{1}{6}\right)^2 \cdot \frac{1}{6} + \left(\frac{1}{6}\right)^2 \cdot \frac{1}{4} + \left(\frac{7}{6}\right)^2 \cdot \frac{1}{36} + \left(\frac{7}{6}\right)^2 \cdot \frac{1}{18} + \left(\frac{7}{6}\right)^2 \cdot \frac{1}{12} = \frac{17}{36}.$$

Hence $\sigma_x = \sqrt{5}/3$, $\sigma_y = \sqrt{17}/6$.

Note that $D(X)$ and $D(Y)$ can be found by the formulas $D(X) = M(X^2) - [M(X)]^2$, $D(Y) = M(Y^2) - [M(Y)]^2$.

874. Find the correlation coefficient from the hypothesis of problem 871.

Solution. We use the table of distribution of the system $(\mathring{X}, \mathring{Y})$ of the centred random variables.

We determine the covariance:

$$C_{xy} = \left(-\frac{4}{3}\right)\cdot\left(-\frac{5}{6}\right)\cdot\frac{1}{18} + \left(-\frac{4}{3}\right)\cdot\frac{1}{6}\cdot\frac{1}{12} + \left(-\frac{4}{3}\right)\cdot\frac{7}{6}\cdot\frac{1}{36}$$
$$+ \left(-\frac{1}{3}\right)\cdot\left(-\frac{5}{6}\right)\cdot\frac{1}{9} + \left(-\frac{1}{3}\right)\cdot\frac{1}{6}\cdot\frac{1}{6} + \left(-\frac{1}{3}\right)\cdot\frac{7}{6}\cdot\frac{1}{18}$$
$$+ \frac{2}{3}\cdot\left(-\frac{5}{6}\right)\cdot\frac{1}{6} + \frac{2}{3}\cdot\frac{1}{6}\cdot\frac{1}{4} + \frac{2}{3}\cdot\frac{7}{6}\cdot\frac{1}{12}$$
$$= -\frac{4}{3}\cdot\left(-\frac{5}{108} + \frac{1}{72} + \frac{7}{216}\right) - \frac{1}{3}\cdot\left(-\frac{5}{54} + \frac{1}{36} + \frac{7}{108}\right)$$
$$+ \frac{2}{3}\cdot\left(-\frac{5}{36} + \frac{1}{24} + \frac{7}{72}\right) = -\frac{4}{3}\cdot 0 - \frac{1}{3}\cdot 0 + \frac{2}{3}\cdot 0 = 0.$$

Since $C_{xy} = 0$, the correlation coefficient $r_{xy} = 0$ too.

The same result can be obtained without determining the covariance C_{xy}. Indeed, setting $Y = 1$, we see that the value $X = 1$ is repeated twice, the value $X = 2$, four times, and the value $X = 3$, six times. Hence, at $Y = 1$ we get the following distribution series of the random variable X:

x_i	1	2	3
p_i	1/6	1/3	1/2

If $Y = 2$, then the value $X = 1$ is repeated three times, the value $X = 2$, six times, and the value $X = 3$, nine times. Consequently, at $Y = 2$ we get the following distribution series of the random variable X:

x_i	1	2	3
p_i	1/6	1/3	1/2

Finally, if $Y = 3$, the value $X = 1$ is repeated once, the value $X = 2$, twice, and the value $X = 3$, three times. At $Y = 3$, the distribution series of the random variable X has the form

x_i	1	2	3
p_i	1/6	1/3	1/2

Thus, at different values of Y we get one and the same distribution series of the random variable X. Since this series does not depend on the values of the random variable Y, the random variables X and Y are independent. It follows that the correlation coefficient is zero.

875. The system of the random variables (X, Y) obeys the law of distribution with the density

$$f(x, y) = \begin{cases} a(x^2 + y^2), & \text{if} \quad x^2 + y^2 \leqslant r^2, \\ 0, & \text{if} \quad x^2 + y^2 > r^2. \end{cases}$$

Find the coefficient a.

Solution. The coefficient a can be determined from the equation

$$a \iint_D (x^2 + y^2)dxdy = 1,$$

where D is a disc bounded by the circle $x^2 + y^2 = r^2$. Passing to polar coordinates, we obtain

$$a \int_0^{2\pi} \int_0^r \rho^3 d\rho d\theta = 1, \quad \frac{r^4}{4} \cdot 2\pi a = 1, \quad \text{i.e.} \quad a = \frac{2}{\pi r^4}.$$

876. The system of random variables (X, Y) obeys the law of distribution with the density

$$f(x, y) = \begin{cases} a(x + y) & \text{in the domain } D; \\ 0 & \text{outside that domain.} \end{cases}$$

The domain D is a square bounded by the straight lines $x = 0$, $x = 3$, $y = 0$, $y = 3$. It is required: (1) to determine the coefficient a; (2) to calculate the probability of the random point $(X; Y)$ falling in the square Q bounded by the straight lines $x = 1$, $x = 2$, $y = 1$, $y = 2$; (3) to find the means m_x and m_y; (4) to find the standard deviations σ_x and σ_y.

Solution. (1) The coefficient a can be found from the equation

$$a \int_0^3 \int_0^3 (x + y)dxdy = 1,$$

whence

$$a \int_0^3 \int_0^3 (x + y)dxdy = a \int_0^3 \left[xy + \frac{y^2}{2}\right]_0^3 dx = a \int_0^3 \left(3x + \frac{9}{2}\right)dx$$

$$= a\left[\frac{3}{2}x^2 + \frac{9}{2}x\right]_0^3 = a\left(\frac{27}{2} + \frac{27}{2}\right) = 27a, \quad 27a = 1, \quad \text{i.e.} \quad a = \frac{1}{27}.$$

(2) $P[(X;\ Y) \subset Q] = \dfrac{1}{27} \int_1^2 \int_1^2 (x + y)\,dx\,dy$

$$= \frac{1}{27} \int_1^2 \left[xy + \frac{y^2}{2}\right]_1^2 dx = \frac{1}{27} \int_1^2 \left(2x + 2 - x - \frac{1}{2}\right)dx$$

$$= \frac{1}{27} \int_1^2 \left(x + \frac{3}{2}\right)dx = \frac{1}{27}\left[\frac{x^2}{2} + \frac{3x}{2}\right]_1^2 = \frac{1}{27}\left(2 + 3 - \frac{1}{2} - \frac{3}{2}\right) = \frac{1}{9}.$$

(3) Now we find the means m_x and m_y; we have

$$m_x = \frac{1}{27}\int_0^3 \int_0^3 x(x + y)\,dx\,dy = \frac{1}{27}\int_0^3 \left[x^2y + \frac{xy^2}{2}\right]_0^3 dx$$

$$= \frac{1}{27}\int_0^3 \left(3x^2 + \frac{9}{2}x\right)dx = \frac{1}{27}\left[x^3 + \frac{9}{2}x^2\right]_0^3 = \frac{1}{27}\left(27 + \frac{81}{4}\right) = 7/4.$$

Consequently, $m_y = 7/4$ as well.

(4) Finally we find the standard deviations σ_x and σ_y:

$$\sigma_x^2 = \iint_D (x - m_x)^2 \cdot f(x, y)\,dx\,dy = \frac{1}{27}\int_0^3 \int_0^3 \left(x - \frac{7}{4}\right)^2 \cdot (x + y)\,dy\,dx$$

$$= \frac{1}{27}\int_0^3 \int_0^3 \left(x - \frac{7}{4}\right)^2 \cdot \left(x - \frac{7}{4} + y + \frac{7}{4}\right) dy\,dx$$

$$= \frac{1}{27}\int_0^3 \int_0^3 \left(x - \frac{7}{4}\right)^3 dy\,dx + \frac{1}{27}\int_0^3 \int_0^3 \left(x - \frac{7}{4}\right)^2 \cdot \left(y + \frac{7}{4}\right) dy\,dx$$

$$= \frac{1}{27} \cdot \int_0^3 \left(x - \frac{7}{4}\right)^3 \cdot y\Big|_0^3 dx + \frac{1}{27 \cdot 2} \cdot \int_0^3 \left(x - \frac{7}{4}\right)^2 \cdot \left(y + \frac{7}{4}\right)^2\Big|_0^3 dx$$

$$= \frac{1}{9} \cdot \frac{\left(x - \frac{7}{4}\right)^4}{4}\Bigg|_0^3 + \frac{1}{27 \cdot 2} \cdot \frac{1}{3}\left(x - \frac{7}{4}\right)^3 \cdot \left(\frac{361}{16} - \frac{49}{16}\right)\Bigg|_0^3 = \frac{11}{16}.$$

Thus $\sigma_x = \sigma_y = \sqrt{11}/4$.

877. The system of the random variables (X, Y) obeys the law of distribution with the density

$$f(x, y) = \begin{cases} a \sin (x + y) & \text{in the domain } D; \\ 0 & \text{outside that domain.} \end{cases}$$

The domain D is specified by the inequalities $0 \leqslant x \leqslant \pi/2$, $0 \leqslant y \leqslant \pi/2$. Find:

(1) the coefficient a; (2) the means m_x and m_y; (3) the standard deviations σ_x and σ_y; (4) the correlation coefficient r_{xy}.

Solution. (1) The coefficient a can be found from the equation

$$a\int_0^{\pi/2}\int_0^{\pi/2}\sin(x+y)\,dy\,dx = 1.$$

Hence

$$a\int_0^{\pi/2}\int_0^{\pi/2}\sin(x+y)\,dy\,dx = -a\int_0^{\pi/2}\cos(x+y)\Big|_0^{\pi/2}dx$$

$$= a\int_0^{\pi/2}(\sin x+\cos x)\,dx = a(\sin x-\cos x)\Big|_0^{\pi/2} = 2a.$$

Thus, $a = 1/2$, i.e. $f(x, y) = (1/2)\sin(x+y)$ in the domain D.

$$(2)\ m_x = \frac{1}{2}\int_0^{\pi/2}\int_0^{\pi/2}x\sin(x+y)\,dy\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2}x\,dx\int_0^{\pi/2}\sin(x+y)\,dy = -\frac{1}{2}\int_0^{\pi/2}\cos(x+y)\Big|_0^{\pi/2}x\,dx$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[\cos\left(x+\frac{\pi}{2}\right)-\cos x\right]x\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2}x(\sin x+\cos x)\,dx = \frac{1}{2}x(\sin x-\cos x)\Big|_0^{\pi/2}$$

$$-\frac{1}{2}\int_0^{\pi/2}(\sin x-\cos x)dx = \frac{\pi}{4}+\frac{1}{2}(\cos x+\sin x)\Big|_0^{\pi/2} = \frac{\pi}{2}.$$

The means $m_y = \pi/4$ as well.

$$(3)\ \sigma_x^2 = M(X^2)-[M(X)]^2 = \frac{1}{2}\int_0^{\pi/2}\int_0^{\pi/2}x^2\sin(x+y)\,dy\,dx-\frac{\pi^2}{16}$$

$$= -\frac{1}{2}\int_0^{\pi/2}x^2\cos(x+y)\Big|_0^{\pi/2}dx-\frac{\pi^2}{16} = \frac{1}{2}\int_0^{\pi/2}x^2(\sin x+\cos x)\,dx-\frac{\pi^2}{16}$$

$$= \frac{1}{2}x^2(\sin x-\cos x)\Big|_0^{\pi/2}-\int_0^{\pi/2}x(\sin x-\cos x)\,dx-\frac{\pi^2}{16}$$

$$= \frac{\pi^2}{8}+x(\sin x+\cos x)\Big|_0^{\pi/2}-\int_0^{\pi/2}(\sin x+\cos x)\,dx-\frac{\pi^2}{16}$$

$$= \frac{\pi^2}{8}+\frac{\pi}{2}+(\sin x-\cos x)\Big|_0^{\pi/2}-\frac{\pi^2}{16} = \frac{\pi^2}{16}+\frac{\pi}{2}-2.$$

Consequently, $\sigma_x^2 = \sigma_y^2 = \sigma_x\sigma_y = (\pi^2 + 8\pi - 32)/16$.

(4) Now we define the covariance:

$$C_{xy} = M(XY) - M(X)\cdot M(Y) = \frac{1}{2}\int_0^{\pi/2}\int_0^{\pi/2} xy\sin(x+y)\,dy\,dx - \frac{\pi}{4}\cdot\frac{\pi}{4}$$

$$= \frac{1}{2}\int_0^{\pi/2} x\,dx\int_0^{\pi/2} y\sin(x+y)\,dy - \frac{\pi^2}{16}$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[y\cos(x+y)\Big|_0^{\pi/2} - \int_0^{\pi/2}\cos(x+y)\,dy\right]x\,dx - \frac{\pi^2}{16}$$

$$= -\frac{1}{2}\int_0^{\pi/2} x\cdot\left[\frac{\pi}{2}\cdot\cos\left(x+\frac{\pi}{2}\right) - \sin\left(x+\frac{\pi}{2}\right) + \sin x\right]dx - \frac{\pi^2}{16}$$

$$= -\frac{1}{2}\cdot\int_0^{\pi/2} x\left(-\frac{\pi}{2}\sin x - \cos x + \sin x\right)dx - \frac{\pi^2}{16}$$

$$= \frac{1}{2}\int_0^{\pi/2} x\left(\frac{\pi}{2}\sin x + \cos x - \sin x\right)dx - \frac{\pi^2}{16}$$

$$= \frac{1}{2}x\left(\sin x - \frac{\pi}{2}\cos x + \cos x\right)\Big|_0^{\pi/2}$$

$$-\frac{1}{2}\int_0^{\pi/2}\left(\sin x - \frac{\pi}{2}\cos x + \cos x\right)dx - \frac{\pi^2}{16}$$

$$= \frac{\pi}{4} - \frac{1}{2}\left(\sin x - \frac{\pi}{2}\sin x - \cos x\right)\Big|_0^{\pi/2} - \frac{\pi^2}{16}$$

$$= \frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{4} - \frac{1}{2} - \frac{\pi^2}{16} = \frac{8\pi - 16 - \pi^2}{16}.$$

It follows that

$$r_{xy} = -\frac{C_{xy}}{\sigma_x\cdot\sigma_y} = \frac{8\pi - 16 - \pi^2}{\pi^2 + 8\pi - 32} \approx -\frac{0.73688}{3.00232} \approx -0.2454.$$

878. Given the table determining the law of distribution of the system of two random variables (X, Y):

$X \backslash Y$	20	40	60
10	3λ	λ	0
20	2λ	4λ	2λ
30	λ	2λ	5λ

Find: (1) the coefficient λ; (2) the means m_x and m_y; (3) the variance σ_x^2 and σ_y^2; (4) the correlation coefficient r_{xy}.

879. The system of random variables (X, Y) obeys the distribution law with the density

$$f(x, y) = \begin{cases} axy & \text{in the domain } D; \\ 0 & \text{outside that domain.} \end{cases}$$

The domain D is a triangle bounded by the straight lines $x + y - 1 = 0$, $x = 0$, $y = 0$. Find: (1) the coefficient a; (2) the means m_x and m_y; (3) the variances σ_x^2 and σ_y^2; (4) the correlation coefficient r_{xy}.

880. The system of random variables obeys the law of distribution with the density

$$f(x, y) = \begin{cases} a^2 - x^2 - y^2, & \text{if } \ x^2 + y^2 \leqslant a^2 (a > 0); \\ 0, & \text{if } \ x^2 + y^2 > a^2. \end{cases}$$

Find: (1) the coefficient a; (2) the means m_x and m_y; (3) the variances σ_x^2 and σ_y^2; (4) the correlation coefficient r_{xy}.

5.15. Regression Curves. Correlation

Given a system of random variables (X, Y). Suppose n points $(x_1; y_1)$, $(x_2; y_2)$, ..., $(x_n; y_n)$ (some of which may coincide) are obtained as a result of n trials. It is required to calculate the correlation coefficient of that system of random variables.

Taking the law of large numbers into account, we can replace, for a sufficiently large n, the mean values $M(X)$, $M(Y)$ in the formulas specifying σ_x^2, σ_y^2 and C_{xy}, by the arithmetic means of the values of the corresponding random variables. In this case, the following approximations hold true:

$$M(X) \approx \bar{x} = \left(\sum_{i=1}^{n} x_i\right)\Big/n; \quad M(Y) \approx \bar{y} = \left(\sum_{i=1}^{n} y_i\right)\Big/n,$$

$$\sigma_x^2 \approx \left(\sum_{i=1}^{n} x_i^2\right)\Big/n - \bar{x}^2; \quad \sigma_y^2 \approx \left(\sum_{i=1}^{n} y_i^2\right)\Big/n - \bar{y}^2, \quad C_{xy} \approx \left(\sum_{i=1}^{n} x_i y_i\right)\Big/n = \overline{xy}.$$

This allows the correlation coefficient to be found by the formula

$$r_{xy} = \frac{C_{xy}}{\sigma_x \sigma_y}.$$

If $|r_{xy}|\sqrt{n-1} \geqslant 3$, then the relation between the random variables X and Y is sufficiently probable. If the relationship between X and Y has been established, then the linear approximation $\bar{y}_x$ of x is given by the formula of *linear regression*:

$$\bar{y}_x - \bar{y} = r_{xy} \cdot \frac{\sigma_y}{\sigma_x}(x - \bar{x}), \quad \text{or} \quad \bar{y}_x = ax + b.$$

In a similar way, the linear approximation $\bar{x}_y$ of y is given by the formula of linear

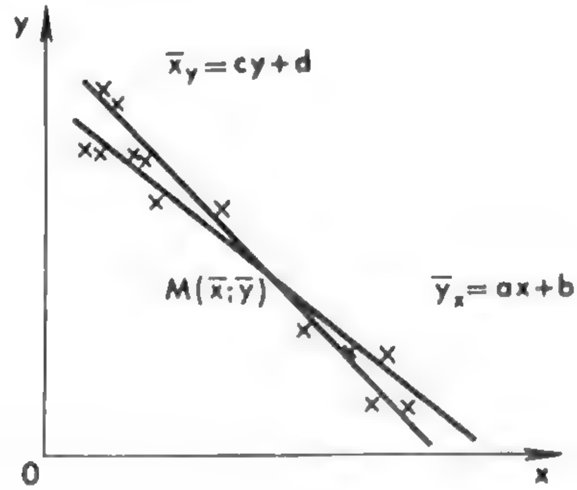


Fig. 46

regression

$$\bar{x}_y - \bar{x} = r_{xy} \cdot \frac{\sigma_x}{\sigma_y}(y - \bar{y}), \quad \text{or} \quad \bar{x}_y = cy + d.$$

It should be borne in mind that $\bar{y}_x = ax + b$ and $\bar{x}_y = cy + d$ are distinct straight lines (Fig. 46). The first straight line is the result of solution of the problem on the minimization of the sum of the squares of deviations from the vertical, and the second is the result of solution of the problem on the minimization of the sum of the squares of deviations from the horizontal.

To construct linear regression equations, it is necessary:

(1) to calculate $\bar{x}$, $\bar{y}$, σ_x, σ_y, C_{xy}, r_{xy} by the original table of values of (X, Y);

(2) to verify the hypothesis concerning the existence of relationship between X and Y;

(3) to derive the equations for the two regression lines and construct the graphs of those equations.

881. Given the table:

i	1	2	3	4	5	6	7	8	9
X	0.25	0.37	0.44	0.55	0.60	0.62	0.68	0.70	0.73
Y	2.57	2.31	2.12	1.92	1.75	1.71	1.60	1.51	1.50

i	10	11	12	13	14	15	16	17
X	0.75	0.82	0.84	0.87	0.88	0.90	0.95	1.00
Y	1.41	1.33	1.31	1.25	1.20	1.19	1.15	1.00

Determine the correlation coefficient r_{xy} and the equations of the regression lines.

Solution. We compile a calculation table:

i	X	Y	X^2	Y^2	XY
1	0.25	2.57	0.0625	6.6049	0.6425
2	0.37	2.31	0.1369	5.3361	0.8547
3	0.44	2.12	0.1936	4.4944	0.9328
4	0.55	1.92	0.3025	3.6864	1.0560
5	0.60	1.75	0.3600	3.0625	1.0500
6	0.62	1.71	0.3844	2.9241	1.0602
7	0.68	1.60	0.4624	2.5600	1.0880
8	0.70	1.51	0.4900	2.2801	1.0570
9	0.73	1.50	0.5329	2.2500	1.0950
10	0.75	1.41	0.5625	1.9881	1.0575
11	0.82	1.33	0.6724	1.7689	1.0906
12	0.84	1.31	0.7056	1.7161	1.1004
13	0.87	1.25	0.7569	1.5625	1.0875
14	0.88	1.20	0.7744	1.4400	1.0560
15	0.90	1.19	0.8100	1.4161	1.0710
16	0.95	1.15	0.9025	1.3225	1.0925
17	1.00	1.00	1.0000	1.0000	1.0000
$\sum$	11.95	26.83	9.1095	45.4127	17.3917

From the table we get: $\sum_{i=1}^{17} x_i = 11.95$, $\sum_{i=1}^{17} y_i = 26.83$, $\sum_{i=1}^{17} x_i^2 = 9.1095$, $\sum_{i=1}^{17} y_i^2 = 45.4127$, $\sum_{i=1}^{17} x_iy_i = 17.3917$. Now we find

$$\bar{x} = 11.95/17 = 0.7029, \quad \bar{y} = 26.83/17 = 1.5782;$$

$$\sigma_x^2 = 9.1095/17 - (0.7029)^2 = 0.0418, \quad \sigma_x = 0.2042,$$

$$\sigma_y^2 = 45.4127/17 - (1.5782)^2 = 0.1806, \quad \sigma_y = 0.4250;$$

$$C_{xy} = 17.3917/17 - 0.7029 \cdot 1.5782 = -0.0863;$$

$$r_{xy} = (-0.0863)/(0.2042 \cdot 0.4250) = -0.09943.$$

We calculate the value of the product $|r_{xy}| \cdot \sqrt{n-1}$; since $|r_{xy}| \cdot \sqrt{n-1} = 0.9943 \cdot 4 = 3.9772 > 3$, the relationship is sufficiently substantiated.

The equations of the regression lines are

$$\bar{y}_x - \bar{y} = r_{xy} \cdot \frac{\sigma_y}{\sigma_x} \cdot (x - \bar{x}),$$

i.e.

$$\bar{y}_x - 1.5782 = -\frac{0.9943 \cdot 0.4250}{0.2042}(x - 0.7029); \quad \bar{y}_x = -2.0695x + 3.0329;$$

and

$$\bar{x}_y - \bar{x} = r_{xy} \cdot \frac{\sigma_x}{\sigma_y}(y - \bar{y}),$$

i.e.

$$\bar{x}_y - 0.7029 = -\frac{0.9943 \cdot 0.2042}{0.4250} \cdot (y - 1.5782); \quad \bar{x}_y = -0.4776y + 1.4566.$$

Having constructed the points determined by the table, and the regression lines, we see that both regression lines pass through the point $M(0.7029; 1.5782)$. The first line intercepts the line segment 3.0329 on the axis of ordinates, the second line, the line segment 1.4566 on the axis of abscissas. The points $(x_i; y_i)$ lie near the regression lines.

882. Seventy-nine trials yielded a correlation table for the variables $X = \sigma_S/\sigma_B$ and Y where σ_S is the yield point of steel, σ_B, the ultimate strength of steel, Y, the percentage of carbon in steel:

X \ Y	0.5	0.6	0.7	0.8
0.5	0	2	0	8
0.6	0	4	2	9
0.7	2	12	3	1
0.8	21	14	0	0
0.9	1	0	0	0

The integers contained in the table are the multiplicities of the values of the corresponding random points. Thus, for instance, the point for which $x = 0.8$, $y = 0.6$ has the multiplicity 14, that is, as a result of 14 trials the value $x = 0.8$ was associated with the value $y = 0.6$.

It is required to determine the correlation coefficient and the equations of the regression lines.

Solution. We find

$$\bar{x} = \frac{0.5(2+8)+0.6(4+2+9)+0.7(2+12+3+1)+0.8(21+14)+0.9\cdot 1}{79} = \frac{55.5}{79} = 0.703;$$

$$\bar{y} = \frac{0.5(2+21+1)+0.6(2+4+12+14)+0.7(2+3)+0.8(8+9+1)}{79} = \frac{49.1}{79} = 0.622.$$

We determine the arithmetic means of the quantities x^2, y^2 and xy:

$$\frac{\sum x^2}{79} = \frac{1}{79}[(0.5)^2 \cdot (2 + 8) + (0.6)^2 \cdot (4 + 2 + 9)$$

$$+ (0.7)^2 \cdot (2 + 12 + 3 + 1) + (0.8)^2 \cdot (21 + 14) + (0.9)^2 \cdot 1] = \frac{39.93}{79} = 0.505;$$

$$\frac{\sum y^2}{79} = \frac{1}{79}[(0.5)^2 \cdot (2 + 21 + 1) + (0.6)^2 \cdot (2 + 4 + 12 + 14)$$

$$+ (0.7)^2 \cdot (2 + 3) + (0.8)^2 \cdot (8 + 9 + 1)] = \frac{31.49}{79} = 0.398;$$

$$\frac{\sum xy}{79} = \frac{1}{79}[0.5 \cdot 0.6 \cdot 2 + 0.5 \cdot 0.8 \cdot 8 + 0.6 \cdot 0.6 \cdot 4 + 0.6 \cdot 0.7 \cdot 2$$

$$+ 0.6 \cdot 0.8 \cdot 9 + 0.7 \cdot 0.5 \cdot 2 + 0.7 \cdot 0.6 \cdot 12 + 0.7 \cdot 0.7 \cdot 3 + 0.7 \cdot 0.8 \cdot 1$$

$$+ 0.8 \cdot 0.5 \cdot 21 + 0.8 \cdot 0.6 \cdot 14 + 0.9 \cdot 0.5 \cdot 1] = \frac{33.74}{79} = 0.427.$$

Next we determine the variances and the covariance;

$$\sigma_x^2 = 0.505 - (0.703)^2 = 0.505 - 0.493 = 0.012; \quad \sigma_x = 0.11;$$
$$\sigma_y^2 = 0.398 - (0.622)^2 = 0.398 - 0.387 = 0.011; \quad \sigma_y = 0.105;$$
$$C_{xy} = 0.427 - 0.703 \cdot 0.622 = 0.427 - 0.437 = -0.01.$$

Now we find the correlation coefficient:

$$r_{xy} = \frac{C_{xy}}{\sigma_x \sigma_y} = -\frac{0.01}{0.11 \cdot 0.105} = -0.867.$$

Finally, we calculate the value of the product $|r_{xy}| \cdot \sqrt{n-1}$; we have

$$|r_{xy}| \cdot \sqrt{n-1} = 0.867\sqrt{78} = 0.867 \cdot 8.84 = 7.66.$$

Since $|r_{xy}| \cdot \sqrt{n-1} > 3$, the relationship is sufficiently probable.

The equations of the regression lines are

$$\bar{y}_x - \bar{y} = r_{xy} \cdot \frac{\sigma_y}{\sigma_x} \cdot (x - \bar{x}),$$

i.e.

$$\bar{y}_x - 0.622 = -0.867 \cdot \frac{0.105}{0.11} \cdot (x - 0.703), \quad \bar{y}_x = -0.828x + 1.204;$$

and

$$\bar{x}_y - \bar{x} = r_{xy} \cdot \frac{\sigma_x}{\sigma_y} \cdot (y - \bar{y}),$$

i.e.

$$\bar{x}_y - 0.703 = -0.867 \cdot \frac{0.11}{0.105}(y - 0.622), \bar{x}_y = -0.908y + 1.268.$$

883. Given the correlation table for the variables X and Y, where X is the service life of a railway car wheel, in years, and Y is the averaged wear value along the thickness of the rim, in millimetres:

X \ Y	0	2	7	12	17	22	27	32	37	42
0	3	6								
1	25	108	44	8	2					
2	3	50	60	21	5	5				
3	1	11	33	32	13	2	3	1		
4		5	5	13	13	7	2			
5			1	2	12	6	3	2	1	
6		1		1			2	1		1
7			1	1				1		

Determine the correlation coefficient and the equations of the regression lines.

884. Given the correlation table for the variables X and Y, where X is the horizontal rail deflection, in centimetres, and Y is the amount of defects in the rail (in centimetres per 25-metre rail):

X \ Y	0	5	10	15	20
7.0	2				
7.5	1	1		1	1
8.0		1			1
8.5	2				
9.0	2		1	1	3
9.5				2	
10.0	3	2	4	3	3
10.5	4	5	1	3	1

X \ Y	0	5	10	15	20
11.0	3		3	2	6
11.5	3	5	1		9
12.0	5	3	6	4	4
12.5	1	1	3	10	6
13.0	1		1	4	5
13.5	1	1		1	6
14.0	2		1		3
14.5			2		1
15.0					
15.5		1	1		
16.0					3

Determine the correlation coefficient and the equations of the regression lines.

5.16. Determining the Characteristics of Random Variables on the Basis of Experimental Data

5.16.1. General and sampling populations. A *sampling population* (or *sample*) is a collection of randomly selected homogeneous items.

A *general population* is a collection of all homogeneous items subject to sampling.

The *volume* of the population (sampling or general) is the number of items of that population.

A sample is said to be *representative* if it shows with a sufficient clarity the quantitative relations of the general population.

5.16.2. Frequency and relative frequency. Suppose we have a sample of volume n. Let us tabulate the results of the sampling as follows:

i	1	2	3	...	n
ξ_i	ξ_1	ξ_2	ξ_3	...	ξ_n

Here $\xi_1, \xi_2, \ldots, \xi_n$ are the values of the random variable X in the 1st, 2nd, 3rd,

. . ., nth trial respectively, among which there may be equal values. Grouping equal values of the random variable, we obtain the table

X	x_1	x_2	x_3	. . .	x_l
n_x	n_1	n_2	n_3	. . .	n_l

where n_i is the number of occurrences of the value $x_i (i = 1, 2, \ldots, l)$. The quantities $n_1, n_2, \ldots, n_l$ are called the *frequencies* of the respective values $x_1, x_2, \ldots, x_l$ of the random variable X. Evidently, $\sum_{i=1}^{l} n_i = n$, that is, the sum of the frequencies of all the values of the random variable is equal to the volume of the sample.

The ratio of the frequency n_i and the volume of the sample n is called the *relative frequency* of the value x_i and is designated as w_i $(i = 1, 2, \ldots, l)$. It is evident that

$$\sum_{i=1}^{l} w_i = \sum_{i=1}^{l} \frac{n_i}{n} = \frac{1}{n} \sum_{i=1}^{l} n_i = \frac{1}{n} \cdot n = 1,$$

that is, the sum of the relative frequencies of all the values of the random variable X is unity.

The table establishing the correspondence between the values of a random variable and their relative frequencies is known as the *statistical distribution* of the random variable X:

X	x_1	x_2	x_3	. . .	x_l
W	w_1	w_2	w_3	. . .	w_l

It should be noted that the term statistical distribution is often applied to the table determining the correspondence between the values of a random variable and their frequencies.

If X is a continuous random variable, then it is expedient to represent its statistical distribution in the form

I	(ξ_0, ξ_1)	(ξ_1, ξ_2)	(ξ_2, ξ_3)	. . .	(ξ_{l-1}, ξ_l)
W	w_1	w_2	w_3	. . .	w_l

Here w_i is the relative frequency of falling of the random variable in the interval (ξ_{i-1}, ξ_i), $i = 1, 2, \ldots, l$.

If a random variable assumes λ values equal to x_i, then, in the case when λ is even, half the values may be placed in the interval (ξ_{i-1}, ξ_i), and the second half in the in-

terval (ξ_i, ξ_{i+1}), $1 \leqslant i \leqslant l-1$. If λ is odd, then $(\lambda + 1)/2$ values may be placed in one of the indicated intervals and $(\lambda - 1)/2$ values in the other. When the volume n of the sample is large, it is not very important which of the intervals will include the greater number of the values.

To make it more pictorial, the statistical distribution of a discrete random variable may be illustrated by a *polygon* of distribution. For that purpose, consecutive points $(x_1; w_1)$, $(x_2; w_2)$, . . ., $(x_l; w_l)$ are constructed on the coordinate plane and are connected by line segments. It should be noted that the points which are not the vertices of the polygon present no interest from the viewpoint of mathematical statistics.

To illustrate the distribution of a continuous random variable, use is made of diagrams called *histograms*.

1. A histogram establishing the relationship between the frequencies and the class intervals in which the values of a random variable fall.

Suppose the continuous random variable X is defined by the table

I	(ξ_0, ξ_1)	(ξ_1, ξ_2)	(ξ_2, ξ_3)	. . .	(ξ_{l-1}, ξ_l)
n_x	n_1	n_2	n_3	. . .	n_l

Assuming the differences $\xi_i - \xi_{i-1}$ to be constant, we put $\xi_i - \xi_{i-1} = h$ for $i = 1, 2, \ldots, l$ (h being the step of the table). We mark the points $\xi_0, \xi_1, \ldots, \xi_l$ on the x-axis and consider the function specified by the equalities $y = n_i/h$ if $x \in (\xi_{i-1}, \xi_i)$, $i = 1, 2, \ldots, l$. Let us calculate the areas S_i of the rectangles, whose lower bases are the segments $[\xi_{i-1}, \xi_i]$ of the x-axis and whose sides are the corresponding segments of the graph of the function $y = n_i/h$ (Fig. 47); we have

$$S_i = (n_i/h) \cdot h = n_i \quad (i = 1, 2, \ldots, l).$$

2. A histogram characterizing the statistical distribution of a random variable.

It establishes the relationship between class-intervals and the relative frequencies of the values of the random variable that have fallen in these class intervals. In this case we consider a function of the form $y = w_i/h$ $(i = 1, 2, \ldots, l)$. By analogy with the foregoing, the area of the respective ith rectangle is numerically equal to w_i. Thus, the area of the figure bounded by the straight lines $x = x_0$, $x = x_l$, $y = 0$, $y = w_i/h$ $(i = 1, 2, \ldots, l)$ is equal to 1 (Fig. 48).

885. As a result of an experiment the random variable X have assumed the following values:

$$\begin{array}{lllll}
\xi_1 = 2, & \xi_2 = 5, & \xi_3 = 7, & \xi_4 = 1, & \xi_5 = 10, \\
\xi_6 = 5, & \xi_7 = 9, & \xi_8 = 6, & \xi_9 = 8, & \xi_{10} = 6, \\
\xi_{11} = 2, & \xi_{12} = 3, & \xi_{13} = 7, & \xi_{14} = 6, & \xi_{15} = 8, \\
\xi_{16} = 3, & \xi_{17} = 8, & \xi_{18} = 10, & \xi_{19} = 6, & \xi_{20} = 7, \\
\xi_{21} = 3, & \xi_{22} = 9, & \xi_{23} = 4, & \xi_{24} = 5, & \xi_{25} = 6.
\end{array}$$

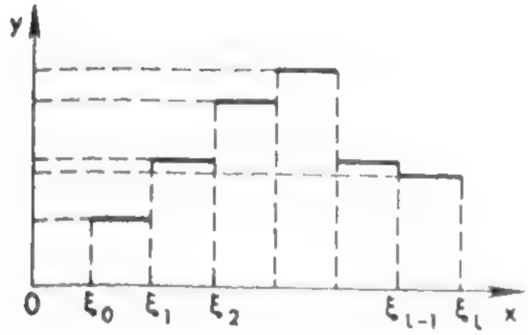


Fig. 47

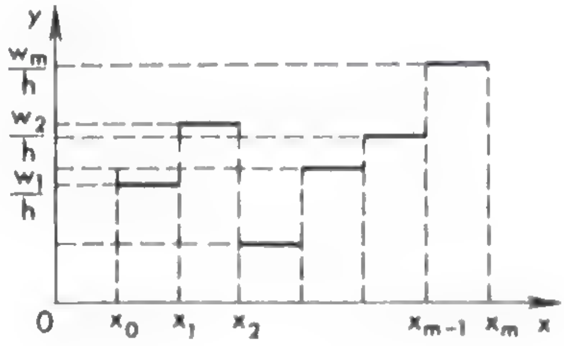


Fig. 48

It is required: (1) to compile a table showing the relationship between the values of the random variable and their frequencies; (2) to construct the statistical distribution; (3) to represent the distribution polygon.

Solution. (1) The volume of the sample is $n = 25$. We compile a table:

X	1	2	3	4	5	6	7	8	9	10
n_x	1	2	3	1	3	5	3	3	2	2

(2) The statistical distribution has the form

X	1	2	3	4	5	6	7	8	9	10
W	1/25	2/25	3/25	1/25	3/25	5/25	3/25	3/25	2/25	2/25

Check it up: $\frac{1}{25} + \frac{2}{25} + \frac{3}{25} + \frac{1}{25} + \frac{3}{25} + \frac{5}{25} + \frac{3}{25} + \frac{3}{25} + \frac{2}{25} + \frac{2}{25} = 1.$

The table can be rewritten in the form

X	1	2	3	4	5	6	7	8	9	10
W	0.04	0.08	0.12	0.04	0.12	0.2	0.12	0.12	0.08	0.08

(3) Let us take the points (1; 0.04), (2; 0.08), (3; 0.12), etc. on the xOw plane. Connecting the consecutive points by line segments, we get a distribution polygon for the random variable X (Fig. 49).

886. As a result of an experiment the random variable X has assumed the following values:

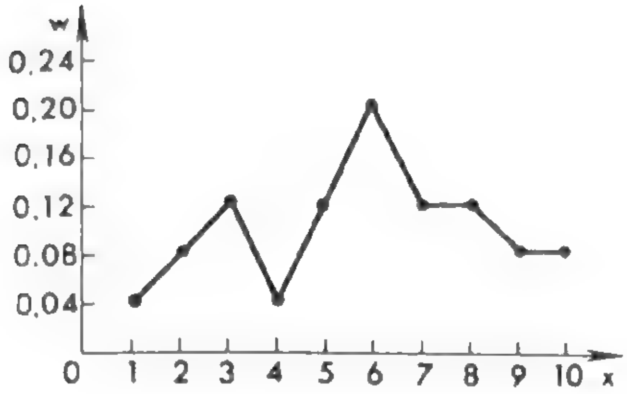


Fig. 49

$$\begin{array}{lllll}
\xi_1 = 16, & \xi_2 = 17, & \xi_3 = 91, & \xi_4 = 13, & \xi_5 = 21, \\
\xi_6 = 11, & \xi_7 = 7, & \xi_8 = 7, & \xi_9 = 19, & \xi_{10} = 5, \\
\xi_{11} = 17, & \xi_{12} = 5, & \xi_{13} = 20, & \xi_{14} = 18, & \xi_{15} = 11, \\
\xi_{16} = 4, & \xi_{17} = 6, & \xi_{18} = 22, & \xi_{19} = 21, & \xi_{20} = 15, \\
\xi_{21} = 15, & \xi_{22} = 23, & \xi_{23} = 19, & \xi_{24} = 25, & \xi_{25} = 1,
\end{array}$$

It is required: to compile a table of statistical distribution, having divided the interval (0, 25) into five class intervals of equal lengths; to construct the histogram of relative frequencies.

Solution. First we compile a table:

I	(0, 5)	(5, 10)	(10, 15)	(15, 20)	(20, 25)
n_x	3	5	4	8	5

The statistical distribution has the form

I	(0.5)	(5, 10)	(10, 15)	(15, 20)	(20, 25)
W	0.12	0.2	0.16	0.32	0.2

The histogram of relative frequencies is shown in Fig. 50.

5.16.3. Frequency function. Suppose $F^*(x)$ is the relative frequency of occurrence of the values of the random variable X satisfying the inequality $X < x$. The function $F^*(x)$ is called a ***frequency function***. Thus,

$$F^*(x) = \begin{cases} 0 & , \text{ if } x < x_1, \\ \sum\limits_{j=1}^{k} w_j, & \text{if } x_k \leqslant x < x_{k+1} \quad (k = 1, 2, \ldots, s-1), \\ 1 & \text{if } x \geqslant x_s. \end{cases}$$

Since, in accordance with Bernoulli's theorem the probabilities of relative frequen-

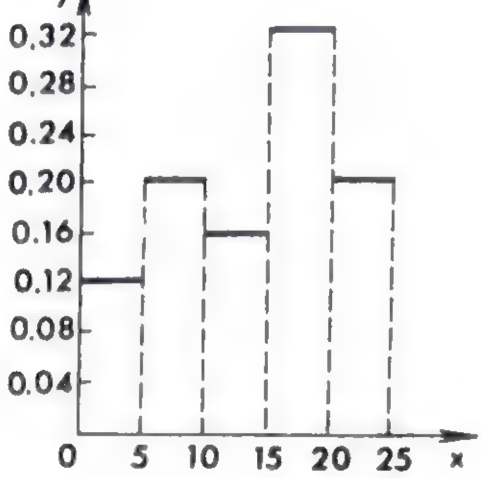


Fig. 50

cies tend to the respective probabilities of the event as n increases indefinitely, it follows that for a large volume of the sample the frequency function $F^*(x)$ is close to the integral distribution function $F(x) = P(X < x)$.

The points $x_1, x_2, \ldots, x_l$ are the points of discontinuity of the first kind for the function $F^*(x)$.

887. Given the statistical distribution

X	11	12	13	14
W_x	0.4	0.1	0.3	0.2

Find the frequency function and construct its graph

Solution. We have

$$F^*(x) = \begin{cases} 0, & \text{if} \quad x \leqslant 11; \\ 0.4 & \text{if} \quad 11 < x \leqslant 12; \\ 0.5, & \text{if} \quad 12 < x \leqslant 13; \\ 0.8, & \text{if} \quad 13 < x \leqslant 14; \\ 1, & \text{if} \quad x > 14. \end{cases}$$

The graph of the function $F^*(x)$ is shown in Fig. 51.

5.16.4. Determining the mean value of a random variable. The *mean value* of the random variable X defined by the statistical distribution, is the expression

$$\bar{x} = w_1x_1 + w_2x_2 + \ldots + w_lx_l = \sum_{i=1}^{l} w_ix_i,$$

or

$$\bar{x} = \frac{n_1x_1 + n_2x_2 + \ldots + n_lx_l}{n} = \frac{1}{n} \sum_{i=1}^{l} n_ix_i. \tag{1}$$

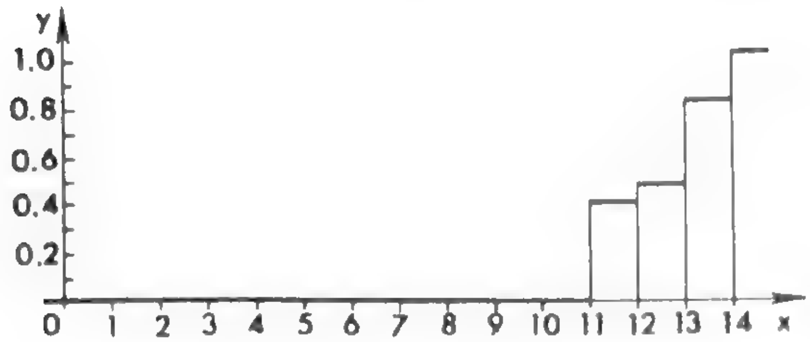


Fig. 51

Equation (1) determines the mean value of X for the sample.

The mean of X for general population is determined in a similar way:

$$\bar{x} = \frac{n_1x_1 + n_2x_2 + \dots + n_Nx_N}{N}, \tag{2}$$

where N is the volume of the general population. Since $n_i/N = p_i$ is the probability of X assuming the value x_i $(1 \leqslant i \leqslant N)$, Eq. (2) can be written in the form

$$\bar{x} = x_1p_1 + x_2p_2 + \dots + x_Np_N = M(X).$$

In accordance with Bernoulli's law of large numbers, it can be assumed that for a sampling population $\bar{x} \approx M(X)$. In what follows, assuming n to be sufficiently large, we shall write $\bar{x} = M(X)$.

If all the values of the random variable X are close to the constant number a, the calculation of $\bar{x}$ is simplified:

$$\bar{x} = \sum_{i=1}^{l} w_ix_i = \sum_{i=1}^{l} w_i(x_i - a + a)$$
$$= \sum_{i=1}^{l} w_i(x_i - a) + a \sum_{i=1}^{l} w_i = \overline{x-a} + a \sum_{i=1}^{l} w_i,$$

i.e.

$$\bar{x} = a + \overline{x-a}, \tag{3}$$

where $\overline{x-a}$ is the mean of the random variable $X - a$. Thus, for a sufficiently large n, the following equality holds true:

$$M(X) = a + M(X - a). \tag{4}$$

888. Find the means of the random variable given by the distribution.

X	13.8	13.9	14	14.1	14.2
n_x	4	3	7	6	5

Solution. All the values of X are close to $a = 14$. We calculate the relative frequencies and compile a table:

$X - 14$	-0.2	-0.1	0	0.1	0.2
W	0.16	0.12	0.28	0.24	0.2

Now we find

$$\overline{x - 14} = \sum_{i=1}^{5} w_i(x_i - 14) = -0.16 \cdot 0.2 - 0.12 \cdot 0.1$$
$$+ 0.28 \cdot 0 + 0.24 \cdot 0.1 + 0.2 \cdot 0.2 = -0.032 - 0.012 + 0.024 + 0.04 = 0.02.$$

Consequently, $\bar{x} = 14 + 0.02 = 14.02$.

5.16.5. Variance and standard deviation. The *statistical variance* of a random variable defined by statistical distribution is the expression

$$D^*(X) = w_1(x_1 - \bar{x})^2 + w_2(x_2 - \bar{x})^2 + \dots + w_l(x_l - \bar{x})^2. \qquad (1)$$

It follows from Eq. (1) that the statistical variance is the mean value of the random variable $(X - \bar{x})^2$. With an increase in n, the mean $\bar{x}$ tends to $M(X)$ in its probability and the relative frequencies $w_1, w_2, \dots w_l$ tend to the corresponding probabilities. Thus, when the volume of the sample is large, the following approximation holds true:

$$D^*(X) \approx D(X).$$

The quantity $\sigma^*(X) = \sqrt{D^*(X)}$ is called a *standard deviation* (or *root mean square deviation*). It has the same dimensionality as the random variable X.

In what follows, assuming the volume of the sample n to be sufficiently large, we shall write $D(X)$ and $\sigma(X)$ instead of $D^*(X)$ and $\sigma^*(X)$ respectively.

If the values of the random variable X are close to the constant quantity a, the calculation of statistical variance is simplified:

$$D^*(X) = \sum_{i=1}^{l} w_i(x_i - \bar{x})^2 = \sum_{i=1}^{l} w_i[(x_i - a) - (\bar{x} - a)]^2$$

$$= \sum_{i=1}^{l} w_i(x_i - a)^2 - 2(\bar{x} - a) \sum_{i=1}^{l} w_i(x_i - a) + (\bar{x} - a)^2 \sum_{i=1}^{l} w_i$$

$$= \sum_{i=1}^{l} w_i(w_i - a)^2 - 2(\bar{x} - a)(\overline{x - a}) + (\bar{x} - a)^2.$$

It follows from Eq. (3) in 5.16.4 that $\bar{x} - a = \overline{x - a}$; therefore

$$D^*(X) = \overline{(x - a)^2} - (\overline{x - a})^2, \tag{2}$$

where $\overline{(x - a)^2}$ is the mean of the random variable $(X - a)^2$, and $(\overline{x - a})^2$ is the standard deviation of the random variable $X - a$. Since the left-hand side of Eq. (2) does not depend on a, the constant quantity a on the right-hand side is discarded as a result of simplification. If, in particular, $a = 0$, we get

$$D(X) = \overline{x^2} - (\bar{x})^2.$$

An analogous formula is often used in the probability theory.

If the random variables X and Y are connected by the linear relation $Y = kX + b$, then their means are connected by the same linear relationship:

$$\bar{y} = k\bar{x} + b \quad \text{or} \quad M(Y) = kM(X) + b. \tag{3}$$

We express the variance of the variable Y in terms of the variance of the variable X, then we get

$$D(Y) = D(kX + b) = D(kX) + D(b) = k^2D(X),$$

since $D(b) = 0$. Consequently,

$$D(Y) = k^2[\overline{x^2} - (\bar{x})^2] \tag{4}$$

889. Calculate $D(X)$ and $\sigma(X)$ for the statistical distribution given in problem 888.
Solution. We compile a table:

$(X - 14)^2$	0.04	0.01	0	0.01	0.04
W	0.16	0.12	0.28	0.24	0.2

Next we have $\bar{x} - 14 = 0.02$, $\overline{(x - 14)^2} = 0.0064 + 0.0012 + 0.0024 + 0.008 = 0.018$.

Consequently, $D(X) = 0.018 - 0.0004 = 0.0176$; $\sigma(X) = \sqrt{0.0176} \approx 0.133$.

890. Determine $\bar{y}$ and $D(Y)$ for the statistical distribution

Y	3	7	11	15	19	23
W	0.02	0.18	0.35	0.3	0.1	0.05

Solution. The values of Y form an arithmetic progression with the first term 3 and the difference 4. Therefore, $Y = 3 + 4(X - 1)$, i.e. $Y = 4X - 1$, $k = 4$, $b = -1$. If X successively assumes the values 1, 2, 3, 4, 5, 6, then Y assumes the corresponding values 3, 7, 11, 15, 19, 23. Thus we can write the statistical distribu-

tion of the variables X and X^2:

X	1	2	3	4	5	6
W	0.02	0.18	0.35	0.3	0.1	0.05
X^2	1	4	9	16	25	36
W	0.02	0.18	0.35	0.3	0.1	0.05

From this we find

$$\bar{x} = 0.02 + 0.36 + 1.05 + 1.2 + 0.5 + 0.3 = 3.43;$$

$$\overline{x^2} = 0.02 + 0.72 + 3.15 + 4.8 + 2.5 + 1.8 = 12.99.$$

Using Eq. (3), we get

$$\bar{y} = 4 \cdot 3.43 - 1 = 12.72,$$

and by Eq. (4) we find

$$D(Y) = 4^2(12.99 - 11.76) = 16 \cdot 1.23 = 19.68.$$

891. Find the mean, the variance and the standard deviation of the random variable defined by the distribution

X	9.8	9.9	10	10.1	10.2
n_x	1	5	8	4	2

892. Determine $\bar{y}$ and $D(Y)$ for the statistical distribution

Y	2	5	8	11	14	17	20	23
W	0.10	0.20	0.15	0.25	0.05	0.12	0.08	0.05

5.16.6. Determining the moments of a random variable from the sampling data. Skewness and excess. The *initial moment* of sth order of the random variable X is the quantity $\alpha_s(X) = M(X^s)$, and the *central moment* is the quantity $\mu_s(X) = M[(X - m_X)^s]$, where m_x is the mathematical expectation of the random variable X.

If we consider the sample to be representative and of sufficient volume, then, to determine $\alpha_s(X)$ and $\mu_x(X)$, we may use the following approximation formulas:

$$\alpha_s(X) \approx \sum_{i=1}^{l} w_i x_i^s, \qquad \mu_s(X) \approx \sum_{i=1}^{l} w_i (x_i - \bar{x})^s.$$

The first-order central moment of any random variable is identically zero. Indeed, $\mu_1 = M(X - m_x) = M(X) - m_x = 0$.

In the case of a symmetric distribution of the random variable X about the mathematical expectation, the other central moments of an odd order are also zero.

It should be also borne in mind that $\alpha_1(X) = M(X)$ and $\mu_2(X) = D(X)$.

If the values of a random variable are close to some number a, then it is expedient to use the following formulas for calculating the central moments of the first four orders:

$$\mu_1(X) = 0,$$

$$\mu_2(X) = \overline{(x - a)^2} - \overline{(x - a)}^2,$$

$$\mu_3(X) = \overline{(x - a)^3} - 3\overline{(x - a)} \cdot \overline{(x - a)^2} + 2\overline{(x - a)}^3,$$

$$\mu_4(X) = \overline{(x - a)^4} - 4\overline{(x - a)}\,\overline{(x - a)^3} + 6\overline{(x - a)}^2\,\overline{(x - a)^2} - 3\overline{(x - a)}^4.$$

By means of the notation $\nu_s = \overline{(x - a)^s}$ these formulas can be reduced to the form

$$\mu_1 = 0, \quad \mu_2 = \nu_2 - \nu_1^2, \quad \mu_3 = \nu_3 - 3\nu_2\nu_1 + 2\nu_1^3,$$
$$\mu_4 = \nu_4 - 4\nu_3\nu_1 + 6\nu_2\nu_1^2 - 3\nu_1^4. \tag{1}$$

If, in particular, $a = 0$, we get the relationships between the central moments μ_s and the initial moments α_s of the first four orders:

$$\mu_1 = 0, \mu_2 = \alpha_2 - \alpha_1^2, \mu_3 = \alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3,$$
$$\mu_4 = \alpha_4 - 4\alpha_1\alpha_3 + 6\alpha_1^2\alpha_2 - 3\alpha_1^4. \tag{2}$$

The initial and central moments of the sth order have the same dimensionality as the sth-degree random variable.

If the random variables X and Y are connected by the linear relation $Y = kX + b$, then the sth-order central moment of the random variable Y is determined as follows:

$$\mu_s(Y) = \mu_s(kX + b) = k^s\mu_s(X + b) = k^s\mu_s(X). \tag{3}$$

It is easy to prove that $\mu_s(X + C) = \mu_s(X)$, where C is an arbitrary constant.

The standard deviation is determined as follows:

$$\sigma(Y) = \sqrt{\mu_2(Y)} = \sqrt{k^2\mu_2(X)} = |k|\sqrt{\mu_2(X)} = |k|\sigma(X). \tag{4}$$

Suppose X is a continuous random variable. To find its numerical values, we shall compile a table:

X	x_1	x_2	...	x_l
W_x	w_1	w_2	...	w_l

where x_i is some number belonging to the interval (ξ_{i-1}, ξ_i), $i = 1, 2, \ldots, l$. It is customary to set $x_i = (\xi_{i-1} + \xi_i)/2$. Expressing the skewness and the excess of the random variable $Y = kX + b$ in terms of the skewness and the excess of the random variable X, whose formulas are given in 5.11, we get

$$S_k(Y) = \frac{\mu_3(Y)}{\sigma^3(Y)} = \frac{k^3\mu_3(X)}{|k|^3\sigma^3(X)} = \operatorname{sign} k \cdot S_k(X); \tag{5}$$

$$E_x(Y) = \frac{\mu_4(Y)}{\sigma^4(Y)} - 3 = \frac{k^4\mu_4(X)}{|k|^4\sigma^4(X)} - 3 = E_x(X). \tag{6}$$

It is evident that if $k > 0$, then $S_k(kX + b) = S_k(X)$; now if $k < 0$, then $S_k(kX + b) = -S_k(X)$.

893. Calculate the central moments of the first four orders of a random variable with the following statistical distribution:

X	11	12	13	14
W	0.35	0.25	0.15	0.25

Solution. We assume $a = 10$. To calculate, $\nu_1, \nu_2, \nu_3, \nu_4$, we compile the following table:

$X - a$	W	$W(X-a)$	$W(X-a)^2$	$W(X-a)^3$	$W(X-a)^4$
1	0.35	0.35	0.35	0.35	0.35
2	0.25	0.50	1.00	2.00	4.00
3	0.15	0.45	1.35	4.05	12.15
4	0.25	1.00	4.00	16.00	64.00
		2.30	6.70	22.40	80.50

Thus we have $\nu_1 = 2.3$, $\nu_2 = 6.7$, $\nu_3 = 22.4$, $\nu_4 = 80.5$. Formulas (1) yield

$$\mu_1(X) = 0;\ \mu_2(X) = 6.7 - 2.3^2 = 1.41;$$
$$\mu_3(X) = 22.4 - 3 \cdot 6.7 \cdot 2.3 + 2 \cdot 2.3^3 = 0.504$$
$$\mu_4(X) = 80.5 - 4 \cdot 22.4 \cdot 2.3 + 6 \cdot 6.7 \cdot 2.3^2 - 3 \cdot 2.3^4 = 3.1257.$$

894. Calculate the central moments of the first four orders of a random variable with the following statistical distribution:

Y	4	9	14	19
W	0.4	0.2	0.3	0.1

Solution. The numbers 4, 9, 14, 19 form an arithmetic progression, therefore $Y = 4 + 5(X - 1)$, i.e. $Y = 5X - 1$, $k = 5$, $b = -1$. Let us compile a table:

X	W	WX	WX^2	WX^3	WX^4
1	0.4	0.4	0.4	0.4	0.4
2	0.2	0.4	0.8	1.6	3.2
3	0.3	0.9	2.7	8.1	24.3
4	0.1	0.4	1.6	6.4	25.6
		2.1	5.5	16.5	53.5

Consequently, $\alpha_1 = 2.1$, $\alpha_2 = 5.5$, $\alpha_3 = 16.5$, $\alpha_4 = 53.5$. Formulas (2) yield

$$\mu_1(x) = 0;\ \mu_2(x) = 5.5 - 4.41 = 1.09;$$
$$\mu_3(x) = 16.5 - 6.3 \cdot 5.5 + 2 \cdot 2.1^3 = 0.372;$$
$$\mu_4(x) = 53.5 - 8.4 \cdot 16.5 + 6 \cdot 4.41 \cdot 5.5 - 3 \cdot 4.41^2 = 2.0857.$$

Using now Eq. (3), we get $\mu_s(Y) = 5^s \mu_s(X)$, i.e.

$$\mu_1(Y) = 0;\quad \mu_2(Y) = 25 \cdot 1.09 = 27.25;$$
$$\mu_3(Y) = 125 \cdot 0.372 = 46.5;\quad \mu_4(Y) = 625 \cdot 2.0857 = 1303.5625.$$

895. Using the sampling data, determine the initial and central moments of the first four orders, skewness and excess, if the random variable X is defined by the table

I	(0, 2)	(2, 4)	(4, 6)	(6, 8)	(8, 10)
n_x	3	4	10	5	3

Solution. The volume of the sample $n = 25$. Let us compile a table:

X	W_x	W_xX	W_xX^2	W_xX^3	W_xX^4
1	0.12	0.12	0.12	0.12	0.12
3	0.16	0.48	1.44	4.32	12.96
5	0.40	2.00	10.00	50.00	250.00
7	0.20	1.40	9.80	68.60	480.20
9	0.12	1.08	9.72	87.48	787.32
		5.08	31.08	210.52	1530.60

Consequently, $\alpha_1 = 5.08$; $\alpha_2 = 31.08$; $\alpha_3 = 210.52$; $\alpha_4 = 1530.60$, i.e. $M(X) = 5.08$; $\mu_1 = 0$.

Using Eqs. (2), we have

$\mu_2 = 31.08 - 25.8064 = 5.1736$, i.e. $D(X) = 5.1736$;

$\mu_3 = 210.52 - 3 \cdot 5.08 \cdot 31.08 + 2 \cdot 5.08^3 = -0.9462$;

$\mu_4 = 1530.60 - 4 \cdot 5.08 \cdot 210.52 + 6 \cdot 5.08^2 \cdot 31.08 - 3 \cdot 5.08^4 = 67.3004$.

from which we find

$$\sigma(X) = \sqrt{5.1736} \approx 2.275;$$

$$S_k(X) = \frac{\mu_3(X)}{\sigma^3(X)} = -\frac{0.9462}{2.275^3} \approx -0.0804;$$

$$E_x(X) = \frac{\mu_4(X)}{\sigma^4(X)} - 3 = \frac{67.3004}{2.275^4} - 3 \approx -0.488.$$

896. Using the sampling data, determine the initial and control moments of the first four orders, skewness and excess for the random variable defined by the table

I	(1, 3)	(3, 5)	(5, 7)	(7, 9)	(9, 11)
n_x	2	4	10	6	3

5.17. Finding the Laws of Distribution of Random Variables on the Basis of Experimental Data

5.17.1. Uniform distribution. Suppose we are given a statistical distribution

I	(ξ_0, ξ_1)	(ξ_1, ξ_2)	...	(ξ_{l-1}, ξ_l)
W	w_1	w_2	...	w_l

If the values $w_1, w_2, \ldots, w_l$ are close to each other, then to process the experimental data, it is convenient to use the law of uniform distribution. As is known (see 5.8), in this case the probability density function is defined as follows:

$$f(x) = \begin{cases} 0 & \text{if} \quad x < a; \\ 1/(b - a), & \text{if } a \leqslant x \leqslant b; \\ 0, & \text{if} \quad x > b. \end{cases}$$

The mean value, the variance and the standard deviation for the uniform distribution are found by the formulas

$$M(X) = (a + b)/2, \; D(X) = (b - a)^2/12, \quad \sigma(X) = (b - a)/(2\sqrt{3}).$$

Solving the system of equations

$$\begin{cases} (a + b)/2 = M(X), \\ (b - a)/(2\sqrt{3}) = \sigma(X), \end{cases}$$

we can find a and b and then the desired distribution density.

897. Perform the fitting of the experimental data by applying the law of uniform

distribution:

I	(0, 10)	(10, 20)	(20, 30)	(30, 40)	(40, 50)	(50, 60)
n_x	11	14	15	10	14	16

Solution. Here $n = 80$. Let us compile a table:

X	5	15	25	35	45	55
W	11/80	14/80	15/80	10/80	14/80	16/80

Setting $X = 5T$, we get the following table:

T	W	WT	WT^2
1	11/80	11/80	11/80
3	7/40	21/40	63/40
5	3/16	15/16	75/16
7	1/8	7/8	49/8
9	7/40	63/40	567/40
11	1/5	11/5	121/5
		25/4	509/10

Now we have

$$M(X) = 5M(T) = 5 \cdot (25/4) = 31.25;$$
$$M(X^2) = 5^2M(T^2) = 25 \cdot (509/10) = 1272.5;$$
$$D(X) = 1272.5 - 976.5625 = 295.9375;$$

$$\begin{cases} (a + b)/2 = 31.25, \\ (b - a)/(2\sqrt{3}) = \sqrt{295.9375}. \end{cases}$$

Solving the last system, we find $a = 1.46$, $b = 61.04$, whence $1/(b - a) = 1/(61.04 - 1.46) = 0.017$. Consequently,

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 1.46; \\ 0.017, & \text{if } 1.46 \leqslant x \leqslant 61.04; \\ 0, & \text{if} \quad x > 61.04. \end{cases}$$

To construct the histogram, we compile the following table (where $h = 10$):

I	(0, 10)	(10, 20)	(20, 30)	(30, 40)	(40, 50)	(50, 60)
W/h	0.0138	0.0175	0.0188	0.0125	0.0175	0.02

Figure 52 depicts the histogram of the relative frequencies of the given statistical distribution and the graph of the probability density function.

Since the uniform distribution is symmetric about the mean, it follows that $\mu_3(X) = 0$, $S_k(X) = 0$. It is also known that with such a distribution the excess is equal to -1.2 irrespective of the values of a and b.

898. Perform the fitting of the experimental data by means of the uniform distribution:

I	(−1, 1)	(1, 3)	(3, 5)	(5, 7)	(7, 9)
n_x	6	7	4	5	8

5.17.2. Poisson's distribution. Poisson's distribution establishes the correlation between the values of the random variable X and the probabilities of these values

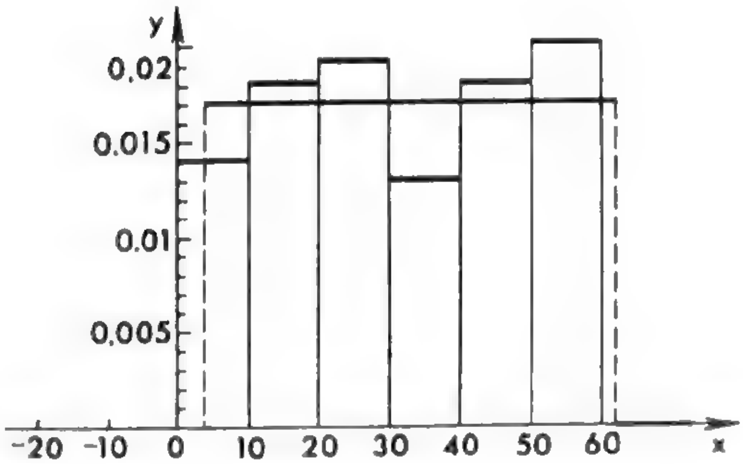


Fig. 52

with the aid of the equation

$$P = \frac{e^{-\lambda}}{x!} \lambda^x, \tag{1}$$

where x assumes the values 0, 1, 2, 3, ...

Thus, the distribution series for the random variable X has the form

X	0	1	2	3	...
P	$e^{-\lambda}$	$e^{-\lambda}\lambda$	$\frac{e^{-\lambda}}{2!}\lambda^2$	$\frac{e^{-\lambda}}{3!}\lambda^3$	...

In practical applications, the random variable X can assume a limited number of values 0, 1, 2, . . ., l, since the quantity $\frac{e^{-\lambda}\lambda^l}{l!}$ is small for a sufficiently large λ.

Recall that for Poisson's distribution $M(X) = D(X) = \lambda$.

Suppose we are given a statistical distribution

X	0	1	2	...	l
n_x	n_0	n_1	n_2	...	n_l

This distribution can also be written as

X	0	1	2	...	l
W	w_0	w_1	w_2	...	w_l

If for the given distribution the quantities $M(X)$ and $D(X)$ are not close to each other, then it is not Poisson's distribution. Now if $M(X) \approx \lambda$ and $D(X) \approx \lambda$, then to decide on the character of the distribution, the value of λ should be substituted into expression (1) and the values of this expression should be calculated for $x = 0$, 1, 2, . . ., l. When the values of P turn out to be close to the respective values of W, it can be assumed that the random variable has a Poisson's distribution.

899. Given the statistical distribution

X	0	1	2	3	4	5	6	7
n_x	7	21	26	21	13	7	3	2

Show that it is close to Poisson's distribution and establish the relationship between the values of the random variables and their probabilities.

Solution. We find $n = \sum_{i=0}^{7} n_i = 7 + 21 + 26 + 21 + 13 + 7 + 3 + 2 = 100$. We compile a table:

X	0	1	2	3	4	5	6	7
W	0.07	0.21	0.26	0.21	0.13	0.07	0.03	0.02

and determine the mean of the random variable:

$$M(X) = 0.21 + 0.52 + 0.63 + 0.52 + 0.35 + 0.18 + 0.14 = 2.55.$$

Now we compile the following table:

X^2	0	1	4	9	16	25	36	49
W	0.07	0.21	0.26	0.21	0.13	0.07	0.03	0.02

Consequently,

$M(X^2) = 0.21 + 1.04 + 1.89 + 2.08 + 1.75 + 1.08 + 0.98 = 9.03$,

whence $\quad D(X) = M(X^2) - [M(X)]^2 = 9.03 - 6.503 = 2.527.$

Setting $\lambda = 2.52$, we can write the relation between the values of the random variable and their probabilities in the form

$$P = \frac{e^{-2.52}}{x!} \cdot 2.52^x.$$

Using this formula to calculate the values of P for $x = 0, 1, 2, \ldots, 7$, we get the table

X	0	1	2	3	4	5	6	7
P	0.08	0.20	0.25	0.21	0.13	0.07	0.03	0.01

900. Solve a problem, similar to the preceding one, for the statistical distribution

X	0	1	2	3	4	5	6	7	8	9	10	11
n_x	1	3	8	14	17	17	15	10	7	5	2	1

5.17.3. Normal distribution. Assume that the statistical distribution

I	(ξ_0, ξ_1)	(ξ_1, ξ_2)	...	(ξ_{l-1}, ξ_l)
W	w_1	w_2	...	w_l

is plotted in histogram fashion as shown in Fig. 53. Let us compile a table

X	x_1	x_2	...	x_l
W	w_1	w_2	...	w_l

setting $x_i = (\xi_{i-1} + \xi_i)/2$, $i = 1, 2, \ldots l$. The smooth curve in the figure connects the points $(x_1; w_1/h)$, $(x_2, w_2/h)$, . . ., $(x_l, w_l/h)$, where h is the step of the table.

If the smooth curve obtained is close to the Gaussian curve, then the statistical data can be processed with the aid of the normal distribution law. Having determined the mean $m = M(X)$ and the standard deviation $\sigma = \sigma(X)$, we consider the function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}. \tag{1}$$

Now we find the values of that function at the points $x_1, x_2, \ldots, x_l$. It is easy to see that the products $hf(x_1)$, $hf(x_2)$, . . ., $hf(x_l)$ are equal to the probabilities of the random variable, having the distribution density (1), falling in the intervals (ξ_0, ξ_1), (ξ_1, ξ_2), . . ., (ξ_{l-1}, ξ_l) respectively. If the given statistical distribution is close to normal,

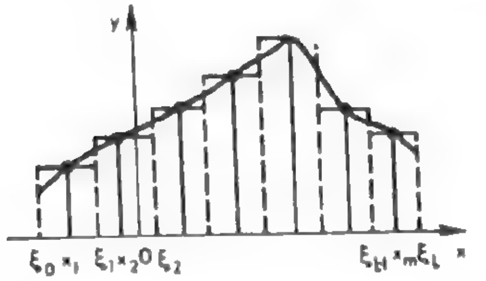


Fig. 53

then the following approximate equalities will hold true: $hf(x_i) \approx w_i$, $i = 1, 2, \ldots, l$. In what follows, we shall present, more exact goodness-of-fit tests for the empirical and theoretical distribution laws.

901. Given the statistical distribution

I	(0, 3)	(3, 6)	(6, 9)	(9, 12)	(12, 15)	(15, 18)	(18, 21)	(21, 24)	(24, 27)	(27, 30)
n_x	1	3	4	6	11	10	7	5	2	1

Show that it is close to the normal distribution and construct the histogram of its relative frequencies.

Solution. Here $n = 50$. We compile a table

X	1.5	4.5	7.5	10.5	13.5	16.5	19.5	22.5	25.5	28.5
W	0.02	0.06	0.08	0.12	0.22	0.2	0.14	0.1	0.04	0.02

Performing a change of variable by the formula $X = 3T - 1.5$, we write the statistical distributions for T and T^2:

T	1	2	3	4	5	6	7	8	9	10
W	0.02	0.06	0.08	0.12	0.22	0.2	0.14	0.1	0.04	0.02
T^2	1	4	9	16	25	36	49	64	81	100
W	0.02	0.06	0.08	0.12	0.22	0.2	0.14	0.1	0.04	0.02

Next we have

$M(T) = 0.02 + 0.12 + 0.24 + 0.48 + 1.1 + 1.2 + 0.98 + 0.8 + 0.36 + 0.2 =$
$= 5.5;$

$M(T^2) = 0.02 + 0.24 + 0.72 + 1.92 + 5.5 + 7.2 + 6.86 + 6.4 + 3.24 + 2 =$
$= 34.1;$

$$M(X) = 3M(T) - 1.5 = 3\cdot 5.5 - 1.5 = 15;$$
$$\sigma^2(X) = 9(34.1 - 30.25) = 34.65;$$
$$\sigma(X) = \sqrt{9\cdot 3.85} = 3\cdot 1.962 \approx 5.9.$$

Consequently,

$$f(x) = \frac{1}{5.9\sqrt{2\pi}}\cdot e^{-(x-15)^2/69.3}.$$

Setting $(x - 15)/5.9 = u$, we get

$$f(x) = \frac{1}{5.9\sqrt{2\pi}}\cdot e^{-u^2/2} \approx 0.17z_u, \quad \text{where} \quad z_u = \frac{1}{\sqrt{2\pi}}\cdot e^{-u^2/2}.$$

The values of the function z_u are given in Table III (see Appendix).

Using these values, we compile a table ($h = 3$):

X	U	z_u	$f(x)$	$hf(x)$
1.5	−2.29	0.029	0.005	0.02
4.5	−1.78	0.082	0.014	0.04
7.5	−1.27	0.178	0.030	0.09
10.5	−0.76	0.299	0.051	0.15
13.5	−0.25	0.387	0.066	0.20
16.5	0.25	0.387	0.066	0.20
19.5	0.76	0.299	0.051	0.15
22.5	1.27	0.178	0.030	0.09
25.5	1.78	0.082	0.014	0.04
28.5	2.29	0.029	0.005	0.02

Note that the results obtained can be compared to the probabilities of the random variable falling in the given interval, the probabilities being calculated by the formula

$$P(a < X < b) = 0.5\left[\Phi\left(\frac{b-m}{\sigma\sqrt{2}}\right) - \Phi\left(\frac{a-m}{\sigma\sqrt{2}}\right)\right],$$

where $\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$ is Laplace's function whose values are given in Table II

(see Appendix), and $m = M(X) = 15$. Using this table, we find

$$P(0 < X < 3) = 0.5\,[-\Phi(1.44) + \Phi(1.80)] = 0.5(-0.9583+0.9891) = 0.0154 \approx 0.02;$$
$$P(3 < X < 6) = 0.5\,[-\Phi(1.08) + \Phi(1.44)] = 0.5(-0.8733+0.9583) = 0.0425 \approx 0.04;$$
$$P(6 < X < 9) = 0.5\,[-\Phi(0.72) + \Phi(1.08)] = 0.5(-0.6914+0.8733) = 0.0905 \approx 0.09;$$
$$P(9 < X < 12) = 0.5\,[-\Phi(0.36)+\Phi(0.72)] = 0.5(-0.3893+0.6914) = 0.151 \approx 0.15;$$
$$P(12 < X < 15) = 0.5\,[-\Phi(0) + \Phi(0.36)] = 0.5\cdot 0.3893 = 0.1946 \approx 0.19;$$
$$P(15 < X < 18) = 0.5\,[\Phi(0.36) - \Phi(0)] = 0.5\cdot 0.3893 = 0.1946 \approx 0.19;$$
$$P(18 < X < 21) = 0.5\,[\Phi(0.72) - \Phi(0.36)] = 0.5(0.6914 - 0.3893) = 0.151 \approx 0.15;$$
$$P(21 < X < 24) = 0.5\,[\Phi(1.08) - \Phi(0.72)] = 0.5(0.8733 - 0.6914) = 0.091 \approx 0.09;$$
$$P(24 < X < 27) = 0.5\,[\Phi(1.44) - \Phi(1.08)] = 0.5(0.9583 + 0.8733) = 0.0425 \approx 0.04;$$
$$P(27 < X < 30) = 0.5\,[\Phi(1.80) - \Phi(1.44)] = 0.5(0.9891 - 0.9583) = 0.0154 \approx 0.02;$$

As a result we get the table

I	(0, 3)	(3, 6)	(6, 9)	(9, 12)	(12, 15)	(15, 18)	(18, 21)	(21, 24)	(24, 27)	(27, 30)
P	0.02	0.04	0.09	0.15	0.19	0.19	0.15	0.09	0.04	0.02

Comparing the values of w and $hf(x)$ (or w and P), we make sure that the given statistical distribution can be assumed to be normal (Fig. 54).

902. Solve the problem, similar to the preceding one, for the statistical distribution

I	(1, 2)	(2, 3)	(3, 4)	(4, 5)	(5, 6)	(6, 7)	(7, 8)	(8, 9)	(9, 10)	(10, 11)	(11, 12)	(12, 13)	(13, 14)	(14, 15)
n_x	4	4	8	16	18	20	30	28	22	18	14	10	4	4

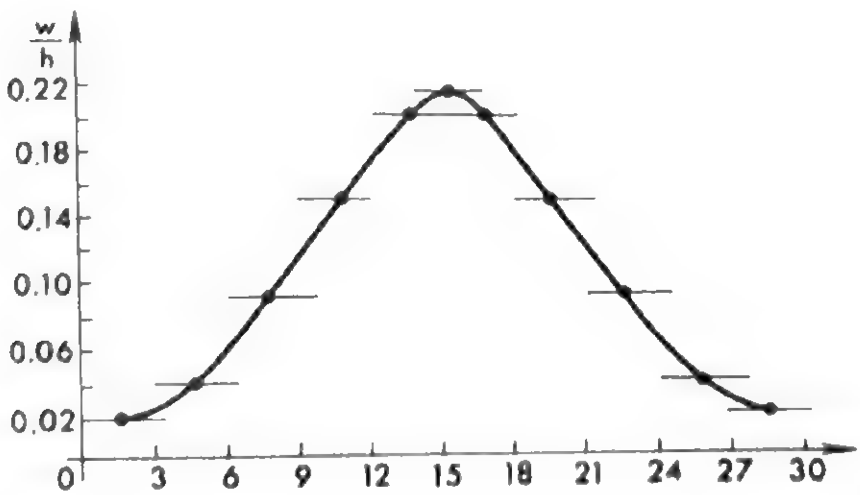


Fig. 54

5.17.4 Sharlier's distribution. The normal distribution is symmetric, that is, the curve $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}$ is symmetric about the straight line $x = m$. However, in practical applications we often come across non-symmetric (skew) distributions. When the absolute value of the skewness is not large, the distribution can be fitted with the aid of the so-called *Sharlier's law*. The density of Sharlier's distribution is specified by the equation

$$f_{\mathrm{Sh}}(x) = f(x) + \frac{1}{\sigma}\left[\frac{S_k(X)}{6} z_u(u^3 - 3u) + \frac{E_x(X)}{24} z_u(u^4 - 6u^2 + 3)\right], \qquad (1)$$

where $f(x)$ is a normal probability density function, $u = (x - m)/\sigma$, $z_u = (1/\sqrt{2\pi})e^{-u^2/2}$, $S_k(X)$ is the skewness, $E_x(X)$ is the excess. Thus, the second summand on the right-hand side of Eq. (1) is the correction to the normal distribution. It can be easily seen that if $S_k(X) = 0$ and $E_x(x) = 0$, then Sharlier's distribution coincides with the normal one. Sharlier's distribution can be written in the form

$$P = \frac{h}{\sigma} \cdot z_u\left[1 + \frac{S_k(X)}{6}(u^3 - 3u) + \frac{E_X(X)}{24}(u^4 - 6u^2 + 3)\right]. \qquad (2)$$

903. Apply Sharlier's distribution to the data of the following statistical table:

I	(0, 3)	(3, 6)	(6, 9)	(9, 12)	(12, 15)	(15, 18)	(18, 21)	(21, 24)	(24, 27)	(27, 30)
n_x	1	5	8	15	28	21	10	6	3	3

Solution. Here $n = 100$. Let us compile a table:

X	1.5	4.5	7.5	10.5	13.5	16.5	19.5	22.5	25.5	28.5
W_x	0.01	0.05	0.08	0.15	0.28	0.21	0.1	0.06	0.03	0.03

We pass to a new variable T related to X as $X = 3T - 1.5$. The statistical distribution of the random variable T has the form

T	1	2	3	4	5	6	7	8	9	10
W	0.01	0.05	0.08	0.15	0.28	0.21	0.1	0.06	0.03	0.03

Here is the calculation table:

T	W	WT	WT^2	WT^3	WT^4
1	0.01	0.01	0.01	0.01	0.01
2	0.05	0.10	0.20	0.40	0.80
3	0.08	0.24	0.72	2.16	6.48
4	0.15	0.60	2.40	9.60	38.40
5	0.28	1.40	7.00	35.00	175.00
6	0.21	1.26	7.56	45.36	272.16
7	0.1	0.70	4.90	34.30	240.10
8	0.06	0.48	3.84	30.72	245.76
9	0.03	0.27	2.43	21.97	197.73
10	0.03	0.30	3.00	30.00	300.00
		5.36	32.06	209.52	1476.44

Next we have

$M(T) = 5.36$; $M(X) = 3 \cdot 5.36 - 1.5 = 14.58$; $M(T^2) = 32.06$;

$D(T) = 32.06 - 28.73 = 3.33$; $\sigma(T) = \sqrt{3.33} = 1.83$; $\sigma(X) = 3\sigma(T) = 5.49$;

$\mu_3(T) = \alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3 = 209.52 - 3 \cdot 5.36 \cdot 32.06 + 2 \cdot 5.36^3 = 1.98$;

$$S_k(T) = \mu_3(T)/\sigma^3(T) = 1.98/1.83^3 = 0.32;$$

$$\mu_4(T) = \alpha_4 - 4\alpha_1\alpha_3 + 6\alpha_1^2\alpha_2 - 3\alpha_1^4$$

$$= 1476.44 - 4 \cdot 5.36 \cdot 209.52 + 6 \cdot 5.36^2 \cdot 32.06 - 3 \cdot 5.36^4 = 34.59;$$

$\sigma^4(T) = 3.33^2 = 11.09$; $E_x(T) = \mu_4(T)/\sigma^4(T) - 3 = 34.59/11.09 - 3 = 0.12.$

Since $h = 3$, $M(X) = 14.58 = m$, $\sigma(X) = 5.49$, $u = (x - 14.58)/5.49$, $S_k(X) = 0.32$, $E_x(X) = 0.12$, the relative frequency of Sharlier's distribution is expressed by the equality

$$w = \frac{3}{5.49} z_u \left[1 + \frac{0.32}{6}(u^3 - 3u) + \frac{0.12}{24}(u^4 - 6u^2 + 3)\right],$$

or $w = 0.55 z_u \cdot S$,

where $S = 1 + 0.05(u^3 - 3u) + 0.005(u^4 - 6u^2 + 3)$.

Now we compile a table to determine the frequencies fitted in accordance with Sharlier's law:

X	U	z_u	U^2	U^3	U^4	$3U$	$6U^2$	$0.05\times(U^3-3U)$	$0.005\times(U^4-6U^2+3)$	S	P
1.5	−2.38	0.02	5.66	−13.48	32.08	−7.41	33.96	−0.32	0.005	0.69	0.01
4.5	−1.84	0.07	3.39	−6.23	11.46	−5.52	20.34	−0.04	−0.03	0.9	0.04
7.5	−1.29	0.17	1.66	−2.15	2.77	−3.87	9.96	0.09	−0.02	1.05	0.09
10.5	−0.74	0.30	0.55	−0.41	0.30	−2.22	3.30	0.09	0.00	1.09	0.18
13.5	−0.19	0.39	0.04	−0.01	0.00	−0.57	0.24	0.03	0.015	1.06	0.23
16.5	0.35	0.38	0.12	0.04	0.01	1.05	0.72	−0.05	0.01	0.97	0.20
19.5	0.90	0.27	0.81	0.73	0.66	2.7	4.86	−0.10	−0.005	0.89	0.13
22.5	1.44	0.14	2.07	2.99	4.30	4.32	12.42	−0.07	−0.025	0.88	0.07
25.5	1.99	0.06	3.96	7.88	15.68	5.97	23.76	0.10	−0.025	1.05	0.03
28.5	2.54	0.02	6.45	16.39	41.62	7.62	38.70	0.44	0.03	1.5	0.02

Comparing the frequencies obtained after fitting by Sharlier's law with the corresponding frequencies given by the statistical table, we infer that they are sufficiently close. However, we can decide whether there is any discrepancy between the statistical and theoretical distributions only after discussing the goodness-of-fit tests (of Pearson, Romanovsky, Kolmogorov).

5.17.5. Goodness-of-fit tests of Pearson and Romanovsky. Let us discuss the goodness of fit of the statistical and theoretical distributions. Suppose the statistical distribution is fitted with the aid of some known distribution (uniform distribution, normal distribution, Sharlier's distribution etc.)

The following goodness-of-fit tests for statistical and theoretical distributions was suggested by Pearson. First we introduce the quantity

$$\chi^2 = n \cdot \sum_{i=1}^{l} \frac{(w_i - p_i)^2}{p_i},$$

where w_i are relative frequencies given in the statistical table, and p_i are the probabilities obtained in accordance with some theoretical distribution. Then we consider the difference $r = l - t$, where l is the number of class intervals of the statistical table and t is the number of conditions to which the frequencies $w_1, w_2, \dots, w_l$ are subject; the number r is called the number of the *degrees of freedom*.

For instance, for the normal distribution $t = 3$ because the following conditions are used:

$$(1)\quad \sum_{i=1}^{l} w_i = 1; \quad (2)\quad \sum_{i=1}^{l} w_i x_i = m_x;$$

$$(3)\quad \sum_{i=1}^{l} (x_i - m_x)^2 w_i = D_x,$$

where m_x and D_x are the mathematical expectation and the variance in the theoretical distribution.

For Sharlier's distribution $t = 5$ because here there are five linear equations relating the values $p_1, p_2, \dots, p_l$:

$$(1)\ \sum_{i=1}^{l} p_i = 1;\quad (2)\ \sum_{i=1}^{l} p_i x_i = m_x;\quad (3)\ \sum_{i=1}^{l} (x_i - m_x)^2 p_i = D_x;$$

$$(4)\ \sum_{i=1}^{l} (x_i - m_x)^3 p_i = \mu_3(X);\quad (5)\ \sum_{i=1}^{l} (x_i - m_x)^4 p_i = \mu_4(X).$$

Employing Table IV (see Appendix), we can use the values of χ^2 and r to determine the quantity P characterizing the probability of goodness of fit of the theoretical and statistical distributions. If $P < 0.1$, we can infer that there is a discrepancy between the theory and the experiment. Now if $P > 0.1$, this means that the hypothesis concerning the accepted theoretical distribution does not contradict the experimental data.

Romanovsky suggested the following goodness-of-fit test: if the quantity $|\chi^2 - r|/\sqrt{2r}$ is larger than or equal to 3, then the discrepancy between the theoretical and experimental frequencies is certainly not accidental, if it is less than 3, then we can assume the discrepancy to be accidental.

904. Verify whether there is goodness of fit of the statistical distribution considered in problem 897 and the theoretical distribution having uniform density.

Solution. The distribution density has been determined from the data of the statistical table in problem 897:

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 1.46; \\ 0.017, & \text{if} \quad 1.46 \leqslant x \leqslant 61.04; \\ 0, & \text{if} \quad x > 61.04. \end{cases}$$

Let us find the probabilities of the uniformly distributed random variable, falling in the intervals $(-10, 0)$, $(0, 10)$, $(10, 20)$, ..., $(60, 70)$, $(70, 80)$:

I	$(-10, 0)$	$(0, 10)$	$(10, 20)$	$(20, 30)$	$(30, 40)$	$(40, 50)$	$(50, 60)$	$(60, 70)$	$(70, 80)$
P	0	0.14	0.17	0.17	0.17	0.17	0.17	0.01	0

It should be noted that $P(0 < X < 10) = P(1.46 < X < 10) = (10 - 1.46) \times 0.017 = 0.14$; $P(60 < X < 70) = P(60 < X < 61.04) = 0.01$. Here

is the table for calculating the values of χ^2:

W	P	$W-P$	$(W-P)^2$	$\frac{(W-P)^2}{P}$
0.14	0.14	0	0	0
0.17	0.17	0	0	0
0.19	0.17	0.02	0.0004	0.0023
0.13	0.17	−0.04	0.0016	0.0094
0.17	0.17	0	0	0
0.2	0.17	0.03	0.0009	0.0052
0	0.01	−0.01	0.0001	0.01
				0.0269

Consequently, $\chi^2 = 80 \cdot 0.0269 = 2.152$; $l = 7$, $t = 3$, $r = 4$. For $r = 4$ we find from Table IV: if $\chi^2 = 2$, then $p = 0.7358$; if $\chi^2 = 3$, then $p = 0.5578$; if $\chi^2 = 2.152$; then $p = 0.7358 - 0.152 \cdot 0.178 = 0.7358 - 0.0271 = 0.7087$.

Thus we can assume that there is no discrepancy between the given statistical distribution and the uniform distribution.

905. Given the statistical distribution:

I	(0, 5)	(5, 10)	(10, 15)	(15, 20)	(20, 25)	(25, 30)	(30, 35)	(35, 40)	(40, 45)	(45, 50)
n_x	2	12	8	4	14	6	10	2	1	11

Verify whether there is goodness of fit of this distribution and the theoretical distribution having uniform density.

Solution. Here $n = 70$. We compile a table:

X	2.5	7.5	12.5	17.5	22.5	27.5	32.5	37.5	42.5	47.5
W	0.029	0.171	0.114	0.057	0.2	0.086	0.143	0.029	0.014	0.157

Next we find

$$M(X) = \sum_{i=1}^{l} w_i x_i = 2.5(0.029 + 3 \cdot 0.171 + 5 \cdot 0.114 + 7 \cdot 0.057 + 9 \cdot 0.2 + 11 \cdot 0.086$$

$$+ 13 \cdot 0.14 + 15 \cdot 0.029 + 17 \cdot 0.014 + 19 \cdot 0.157) = 23.4285;$$

$$M(X^2)=2.5^2(0.029+9\cdot 0.171+25\cdot 0.114+49\cdot 0.057 + 81 \cdot 0.2 + 121 \cdot 0.086$$
$$+ 169 \cdot 0.143 + 225 \cdot 0.029 + 289 \cdot 0.014 + 361 \cdot 0.157) = 782.67;$$
$$D(X) = 782.67 - 596.75 = 185.92; \sigma(X) = \sqrt{185.92} = 13.63.$$

Then we set up and solve the system of equations to determine a and b:

$$\begin{cases} (a + b)/2 = 24.43, \\ (b-a)/(2\sqrt{3}) = 13.63 \end{cases} \Leftrightarrow \begin{cases} a + b = 48.86 \\ b - a = 47.16 \end{cases} \Leftrightarrow (b = 48.01; a = 0.85),$$

$$1/(b - a) = 1/47.16 = 0.0212.$$

Thus we have

$$f(x) = \begin{cases} 0, & \text{if} \quad x < 0.85; \\ 0.0212, & \text{if} \quad 0.85 \leqslant x \leqslant 48.01; \\ 0, & \text{if} \quad x > 48.01. \end{cases}$$

Now we find the probabilities of the normally distributed random variable falling in the intervals (0, 5), (5, 10), (10, 15), ... ,(45, 50):

I	(−5, 0)	(0, 5)	(5, 10)	(10, 15)	(15, 20)	(20, 25)	(25, 30)	(30, 35)	(35, 40)	(40, 45)	(45, 50)	(50, 55)
P	0	0.088	0.106	0.106	0.106	0.106	0.106	0.106	0.106	0.106	0.064	0

We note the fact that

$$P(0 < X < 5) = P(0.85 < X < 5) = 4.15 \cdot 0.0212 = 0.088;$$
$$P(45 < X < 50) = P(45 < X < 48.01) = 3.01 \cdot 0.0212 = 0.064.$$

The table for calculating χ^2 has the form

W	P	$W - P$	$(W - P)^2$	$\dfrac{(W - P)^2}{P}$
0.029	0.088	−0.059	0.003	0.034
0.171	0.106	0.065	0.004	0.038
0.114	0.106	0.008	0.006	0.057
0.057	0.106	−0.049	0.002	0.019
0.2	0.106	0.094	0.009	0.085
0.086	0.106	−0.020	0.000	0.000
0.143	0.106	0.037	0.001	0.009
0.029	0.106	−0.077	0.006	0.057
0.014	0.006	−0.092	0.008	0.075
0.157	0.064	−0.093	0.009	0.141
				0.515

Thus, $\chi^2 = 70 \cdot 0.515 = 36.05$; $l = 10$, $t = 3$, $r = 7$. For the value $r = 7$, we find for $\chi^2 = 30$ from Table IV $P = 0.0001$. Since for the constant value of r the probability P decreases with an increase in χ^2 it follows that for $\chi^2 = 36.05$ the probability $P < 0.0001$. This means that in the given case there is a discrepancy between the theory and the experiment.

The Romanovsky test leads to the same conclusion. Indeed, we find

$$\frac{|\chi^2 - r|}{\sqrt{2r}} = \frac{|36.05 - 7|}{\sqrt{14}} = \frac{29.05}{3.742} \approx 7.763 > 3.$$

Thus we must consider the hypothesis stating that the given statistical distribution is uniform to be incorrect.

906. Apply the Pearson and Romanovsky tests to establish the correctness of the hypothesis concerning the normal distribution of the random variable from problem 901.

Solution. The calculation table has the form

W	P	$W - P$	$(W - P)^2$	$\frac{(W-P)^2}{P}$
0.02	0.02	0	0.0000	0.00
0.06	0.04	0.02	0.0004	0.01
0.08	0.09	−0.01	0.0001	0.001
0.12	0.15	−0.03	0.0009	0.006
0.22	0.20	0.02	0.0004	0.02
0.2	0.20	0.00	0.0000	0.00
0.14	0.15	−0.01	0.0001	0.0007
0.1	0.09	0.01	0.0001	0.001
0.04	0.04	0	0.0000	0.00
0.02	0.02	0	0.0000	0.00

Next we have $n \sum_{i=1}^{10} \frac{(w_i - p_i)^2}{p_i} = 50 \cdot 0.0387 = 1.935$; $l = 10$, $t = 3$, $r = 10 - 3 = 7$. From Table IV we find for $r = 7$ that if $\chi^2 = 1$, then $P = 0.9948$; if $\chi^2 = 2$, then $P = 0.9598$. Consequently, at $\chi^2 = 1.935$ we obtain an intermediate value of P. This value can be found by using the interpolation method. At $\chi^2 = 1$ and $\chi^2 = 2$ the values of P differ by $0.9948 - 0.9598 = 0.035$. With an increase in χ^2 the probability P decreases, therefore at $\chi^2 = 1.935$ we have

$$P = 0.9598 + 0.065 \cdot 0.035 = 0.9621, \text{ or else } P = 0.9948 - 0.935 \cdot 0.035 = 0.9621.$$

The probability obtained exceeds 0.1. In accordance with Pearson's test, this fact enables us to assume that there is no significant discrepancy between the normal distribution and the given statistical distribution.

In accordance with Romanovsky's test we have

$$\frac{|\chi^2 - r|}{\sqrt{2r}} = \frac{|1.935 - 7|}{\sqrt{14}} = \frac{5.065}{3.742} \approx 1.354 < 3.$$

Thus we can consider the discrepancy between the given statistical distribution and the theoretical distribution used for its fitting to be accidental.

907. Fit the data of the following statistical table with the aid of the normal distribution:

I	(4, 1; 4, 2)	(4, 2; 4, 3)	(4, 3; 4, 4)	(4, 4; 4, 5)	(4, 5; 4, 6)	(4, 6; 4, 7)	(4, 7; 4, 8)	(4, 8; 4, 9)	(4, 9; 5)
n_x	1	2	3	4	5	8	8	9	10
I	(5, 0; 5, 1)	(5, 1; 5, 2)	(5, 2; 5, 3)	(5, 3; 5, 4)	(5, 4; 5, 5)	(5, 5; 5, 6)	(5, 6; 5, 7)	(5, 7; 5, 8)	(5,8; 5, 9)
n_x	10	9	9	7	5	4	3	2	1

Verify the goodness of fit of the statistical and theoretical distributions by applying Pearson's and Romanovsky's tests.

Solution. Here $n = 100$. In what follows we shall assume that the values of the random variable coincide with the arithmetic means of the end points of the intervals:

X	4.15	4.25	4.35	4.45	4.55	4.65	4.75	4.85	4.95	5.05	5.15	5.25	5.35	5.45	5.55	5.65	5.75	5.85
W	0.01	0.02	0.03	0.04	0.05	0.08	0.08	0.09	0.1	0.1	0.09	0.09	0.07	0.05	0.04	0.03	0.02	0.01

Since the values of the random variable are close to 5, we compile a table:

$X - 5$	W	$(X - 5)W$	$(X - 5)^2W$
−0.85	0.01	−0.0085	0.0072
−0.75	0.02	−0.0150	0.0113
−0.65	0.03	−0.0195	0.0127
−0.55	0.04	−0.0220	0.0121
−0.45	0.05	−0.0225	0.0101
−0.35	0.08	−0.0280	0.0098
−0.25	0.08	−0.0200	0.0050
−0.15	0.09	−0.0135	0.0020
−0.05	0.1	−0.0500	0.0003
0.05	0.1	0.0500	0.0003
0.15	0.09	0.0135	0.0020
0.25	0.09	0.0225	0.0056

$X-5$	W	$(X-5)W$	$(X-5)^2W$
0.35	0.07	0.0245	0.0086
0.45	0.05	0.0225	0.0101
0.55	0.04	0.0220	0.0121
0.65	0.03	0.0195	0.0127
0.75	0.02	0.0150	0.0113
0.85	0.01	0.0085	0.0072
		-0.001	0.1404

Consequently,

$$M(X-5) = -0.001;\ M[(X-5)^2] = 1.1404;$$
$$M(X) = 5 + M(X-5) = 4.999;$$
$$D(X) = M[(X-5)^2] - [M(X-5)]^2 = 0.1404;\ \sigma(X) = \sqrt{0.1404} = 0.3747 \approx 0.375.$$

The distribution density of the random variable X is specified by the equality

$$f(x) = \frac{1}{0.375\sqrt{2\pi}} \cdot e^{-(x-5)^2/(2\cdot 0.375^2)}, \qquad (*)$$

or

$$f(x) = 2.67 \cdot z_u, \text{ where } u = (x-5)/0.375 \approx 2.67\,(x-5).$$

Let us determine the probability of the normally distributed random variable falling in the intervals (4.1; 4.2), (4. 2; 4.3), ... , (5.8; 5.9) and verify the goodness of fit of the statistical and theoretical distributions using the Pearson and Romanovsky tests. We compile the following tables:

X	U	z_u	$f(x)$	$hf(x)$	W	P	$W-P$	$(W-P)^2$	$\frac{(W-P)^2}{P}$
4.15	-2.27	0.03	0.08	0.01	0.01	0.01	0.00	0.0000	0.000
4.25	-2.00	0.05	0.13	0.02	0.02	0.02	0.00	0.0000	0.000
4.35	-1.74	0.09	0.24	0.02	0.03	0.02	0.01	0.0001	0.005
4.45	-1.47	0.13	0.35	0.04	0.04	0.04	0.00	0.0000	0.000
4.55	-1.20	0.19	0.51	0.05	0.05	0.05	0.00	0.0000	0.000
4.65	-0.93	0.25	0.67	0.07	0.08	0.07	0.01	0.0001	0.001
4.75	-0.66	0.32	0.85	0.09	0.08	0.09	-0.01	0.0001	0.001
4.85	-0.40	0.37	0.99	0.10	0.09	0.10	-0.01	0.0001	0.001
4.95	-0.13	0.39	1.04	0.10	0.10	0.10	0.00	0.0000	0.000
5.05	0.13	0.39	1.04	0.10	0.10	0.10	0.00	0.0000	0.000
5.15	0.40	0.37	0.99	0.10	0.09	0.10	-0.01	0.0001	0.001

X	U	z_u	$f(x)$	$hf(x)$	W	P	$W-P$	$(W-P)^2$	$\frac{(W-P)^2}{P}$
5.25	0.66	0.32	0.85	0.09	0.09	0.09	0.00	0.0000	0.000
5.35	0.93	0.25	0.67	0.07	0.07	0.07	0.00	0.0000	0.000
5.45	1.20	0.19	0.51	0.05	0.05	0.05	0.00	0.0000	0.000
5.55	1.47	0.13	0.35	0.04	0.04	0.04	0.00	0.0000	0.000
5.65	1.74	0.09	0.24	0.02	0.03	0.02	0.01	0.0001	0.005
5.7.	2.00	0.05	0.13	0.02	0.02	0.02	0.00	0.0000	0.000
5.85	2.27	0.03	0.08	0.01	0.01	0.01	0.00	0.0000	0.000

Consequently, $\chi^2 = 100 \cdot 0.014 = 1.4$, $l = 18$, $t = 3$, $r = 18 - 3 = 15$. From Table IV we find for $r = 15$ that if $\chi^2 = 1$, then $P = 1.000$, if $\chi^2 = 2$, then $P = 1.000$. Therefore, for $\chi^2 = 1.4$ the desired probability* $P = 1.000$. Thus, in accordance with Pearson's test, the hypothesis stating that the statistical distribution is a normal distribution with the mean value equal to 5 and the variance equal to 0.14 is correct.

Let us now use Romanovsky's test:

$$\frac{|\chi^2 - r|}{\sqrt{2r}} = \frac{|1.4 - 15|}{\sqrt{30}} = \frac{13.6}{5.477} \approx 2.483 < 3.$$

This is another confirmation that there is no discrepancy between the given statistical distribution and the normal distribution having the density specified by equality (*).

908. Verify the hypothesis stating that there is no discrepancy between the statistical distribution considered in problem 903 and Sharlier's distribution.

Solution. The calculation table has the form

W	P	$W-P$	$(W-P)^2$	$\frac{(W-P)^2}{P}$
0.01	0.01	0	0.0000	0
0.05	0.04	0.01	0.0001	0.003
0.08	0.09	−0.01	0.0001	0.001
0.15	0.18	−0.03	0.0009	0.005
0.28	0.23	0.05	0.0025	0.011

* Since the table presents the values with an accuracy to 0.001, the desired value of P is a little less than unity.

W	P	$W - P$	$(W - P)^2$	$\frac{(W - P)^2}{P}$
0.21	0.20	0.01	0.0001	0.001
0.10	0.13	−0.03	0.0009	0.007
0.06	0.07	−0.01	0.0001	0.001
0.03	0.03	0.00	0.0000	0.000
0.03	0.02	0.01	0.0001	0.005
				0.034

Consequently, $\chi^2 = 100 \cdot 0.034 = 3.4$, $l = 10$, $t = 5$, i.e. $r = 10 - 5 = 5$. From Table IV we find for $r = 5$ that if $\chi^2 = 3$, then $P = 0.7000$; if $\chi^2 = 4$, then $P = 0.5494$. Therefore, for $\chi^2 = 3.4$ we have

$$P = 0.700 - 0.4 \cdot 0.1506 = 0.63976 > 0.1.$$

Applying Romanovsky's test, we find

$$\frac{|\chi^2 - r|}{\sqrt{2r}} = \frac{|3.4 - 5|}{\sqrt{10}} = \frac{1.6}{3.162} = 0.506 < 3.$$

Thus, in accordance with Pearson's and Romanovsky's tests, we can consider the hypothesis stating that there is goodness of fit of the statistical distribution in question and Sharlier's distribution to be correct.

5.17.6. Kolmogorov goodness-of-fit test. Suppose we are given the statistical distribution

X	x_1	x_2	x_3	...	x_l
W_x	w_1	w_2	w_3	...	w_l

where $x_1, x_2, \ldots, x_l$ are the mean values of the corresponding intervals of the random variable. As a measure of discrepancy between the statistical and theoretical distributions Kolmogorov uses in his test the maximum of the values of the modulus of the difference between the statistical distribution function $F^*(x)$ and the corresponding theoretical (integral) distribution function $F(x)$.

As is known, the integral distribution function is defined by the relations

$$F(x) = \begin{cases} 0, & \text{if } x < x_1, \\ \sum\limits_{j=1}^{k} p_j, & \text{if } x_k \leqslant x \leqslant x_{k+1}\ (k = 1, 2, \dots, l-1); \\ 1, & \text{if } x \geqslant x_l, \end{cases}$$

where $p_j = hf(x_j)$ $(j = 1, 2, \dots, l)$, and $f(x)$ is the distribution density of the random variable X.

First we find the quantity

$$\lambda = D\sqrt{n}, \tag{1}$$

where $D = \max |F^*(x) - F(x)|$, and n is the volume of the sample. Then, from the equality

$$P(\lambda) = 1 - \sum_{j=-\infty}^{\infty} (-1)^j e^{-2j^2\lambda^2} \tag{2}$$

we determine the probability of the fact that due to purely accidental causes the maximum discrepancy between $F^*(x)$ and $F(x)$ will turn out to be not smaller than that actually observed.

If the probability $P(\lambda)$ is small (less than 0.05), then the hypothesis must be rejected as incorrect, for comparatively large values of $P(\lambda)$ the hypothesis can be considered compatible with the experimental data.

To find the values of $P(\lambda)$, it is convenient to use Table V (see Appendix).

909. Evaluate goodness of fit of the statistical distribution considered in problem 899 and Poisson's distribution.

Solution. We compile a table:

X	W	P	$F^*(x)$	$F(x)$	$F^*(x) - F(x)$
0	0.07	0.08	0.07	0.08	0.01
1	0.21	0.20	0.28	0.28	0
2	0.26	0.25	0.54	0.53	0.01
3	0.21	0.21	0.75	0.74	0.01
4	0.13	0.13	0.88	0.87	0.01
5	0.07	0.07	0.95	0.94	0.01
6	0.03	0.03	0.98	0.97	0.01
7	0.02	0.01	1	0.98	0.02

It is evident that $D = \max |F^*(x) - F(x)| = 0.02$. Since $n = 100$, we can use formula (1) and find $\lambda = 0.02 \cdot \sqrt{100} = 0.2$. We obtain from Table V

$P(0.2) = 1.000$. Thus, the statistical distribution in question does not contradict the theoretical Poisson distribution.

910. Using Kolmogorov's test, check whether there is goodness of fit of the following statistical distribution and the normal distribution:

I	(0, 2)	(2, 4)	(4, 6)	(6, 8)	(8, 10)	(10, 12)	(12, 14)	(14, 16)	(16, 18)	(18, 20)
n_x	10	29	51	58	122	90	81	39	30	10

Solution. Let us write the given distribution as

X	1	3	5	7	9	11	13	15	17	19
W	0.02	0.06	0.10	0.12	0.20	0.18	0.16	0.08	0.06	0.02

Passing to the new variable T by the formula $X = 2T - 1$, we compile a calculation table:

T	W	WT	WT^2
1	0.02	0.02	0.02
2	0.06	0.12	0.24
3	0.10	0.30	0.90
4	0.12	0.48	1.92
5	0.20	1.00	5.00
6	0.18	1.08	6.48
7	0.16	1.12	7.84
8	0.08	0.64	5.12
9	0.06	0.54	4.86
10	0.02	0.20	2.00
		5.50	34.38

Next we have

$$M(T) = 5.50, \quad M(T^2) = 34.38, \quad D(T) = 34.38 - 30.25 = 4.13;$$

$$\sigma(T) = \sqrt{4.13} = 2.032;$$

$$M(X) = 2M(T) - 1 = 2 \cdot 5.5 - 1 = 10; \quad \sigma(X) = 2\sigma(T) = 2 \cdot 2.032 = 4.064.$$

Then the distribution density will be written as

$$f(x) = \frac{1}{4.064\sqrt{2\pi}} \cdot e^{-(x-10)^2/(2 \cdot 4.064^2)} \qquad (*)$$

or $f(x) = 0.246 \cdot z_u$, where $u = (x - 10)/4.064$.

We compile two tables

X	U	z_u	$f(x)$	$hf(x)$
1	−2.214	0.035	0.009	0.02
3	−1.722	0.091	0.022	0.04
5	−1.230	0.187	0.046	0.09
7	−0.738	0.303	0.075	0.15
9	−0.246	0.387	0.095	0.19
11	0.246	0.387	0.095	0.19
13	0.738	0.303	0.075	0.15
15	1.230	0.187	0.046	0.09
17	1.722	0.091	0.022	0.04
19	2.214	0.035	0.009	0.02

X	W	$hf(x)$	$F^*(x)$	$F(x)$	$F^*(x) - F(x)$
1	0.02	0.02	0.02	0.02	0.00
3	0.06	0.04	0.08	0.06	0.02
5	0.10	0.09	0.18	0.15	0.03
7	0.12	0.15	0.30	0.30	0.00
9	0.20	0.19	0.50	0.49	0.01
11	0.18	0.19	0.68	0.68	0.00
13	0.16	0.15	0.84	0.83	0.01
15	0.08	0.09	0.92	0.92	0.00
17	0.06	0.04	0.98	0.96	0.02
19	0.02	0.02	1	0.98	0.02

It can be seen from the second table that almost all the values of the relative frequencies are close to the corresponding values of the probabilities found with the aid of the density function specified by equation (*). It follows immediately that the given statistical distribution is normal. However, to make a final decision, we shall apply Kolmogorov's test.

As can be seen from the second table, $D = \max |F^*(x) - F(x)| = 0.03$. Since $n = 500$, we have $\lambda = 0.03 \cdot \sqrt{500} \approx 0.67$. From Table V we find $P(0.65) = 0.7920$, $P(0.70) = 0.7112$. Since with an increase in λ the probability $P(\lambda)$ decreases, we have $0.7112 < P(0.67) < 0.7920$.

Thus, we can assert that the upper limit of an absolute error of the approximation $F^*(x) \approx F(x)$ will be not less than 0.03 for any value of x.

Chapter 6

The Concept of a Partial Differential Equation

6.1. Partial Differential Equations

6.1.1. Examples of the simplest partial differential equations. Let us consider some examples of partial differential equations.

911. Find the function $z = z(x, y)$ satisfying the differential equation $\dfrac{\partial z}{\partial x} = 1$.

Solution. Integration yields $z = x + \varphi(y)$, where $\varphi(y)$ is an arbitrary function. This is the general solution of the given differential equation.

912. Solve the equation $\dfrac{\partial^2 z}{\partial y^2} = 6y$, where $z = z(x, y)$.

Solution. Integrating twice with respect to y, we obtain $\dfrac{\partial z}{\partial y} = 3y^2 + \varphi(x), z = y^3 +$ $+ y \cdot \varphi(x) + \psi(x)$, where $\varphi(x)$ and $\psi(x)$ are arbitrary functions.

913. Solve the equation $\dfrac{\partial^2 z}{\partial x \partial y} = 0$.

Solution. Integrating the equation with respect to x, we obtain $\dfrac{\partial z}{\partial y} = f(y)$. Integrating the result with respect to y, we find $z = \varphi(x) + \psi(y)$, where $\psi(y) =$ $= \int f(y) dy$.

914. Find the general solution of the equation $\dfrac{\partial^2 z}{\partial x \partial y} = 1$.

915. Find the general solution of the equation $\dfrac{\partial^4 z}{\partial x^2 \partial y^2} = 0$.

6.1.2. First-order differential equations linear with respect to their partial derivatives. Let us consider the differential equation

$$X \frac{\partial z}{\partial x} + Y \frac{\partial z}{\partial y} = Z, \qquad (1)$$

where X, Y and Z are the functions of x, y and z.

First we solve a system of ordinary differential equations

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}.$$

Assume that the solution of the system is specified by the equations

$$\omega_1(x, y, z) = C_1, \quad \omega_2(x, y, z) = C_2.$$

Then the general solution of differential equation (1) will assume the form

$$\Phi[\omega_1(x, y, z), \omega_2(x, y, z)] = 0,$$

where $\Phi(\omega_1, \omega_2)$ is an arbitrary continuously differentiable function.

916. Find the general solution of the equation $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$.

Solution. Let us consider the system of equations $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{z}$. Solving the equation $\dfrac{dx}{x} = \dfrac{dy}{y}$, we obtain $\dfrac{y}{x} = C_1$; the solution of the equation $\dfrac{dx}{x} = \dfrac{dz}{z}$ is $\dfrac{z}{x} = C_2$. Now we can find the general solution of the given equation:

$$\Phi(y/x, z/x) = 0 \quad \text{or} \quad z/x = \psi(y/x),$$

that is, $z = x\psi(y/x)$, where ψ is an arbitrary function.

917. Find the general solution of the equation $(x^2 + y^2)\dfrac{\partial z}{\partial x} + 2xy\dfrac{\partial z}{\partial y} = 0$.

Solution. We write the system of equations $\dfrac{dx}{x^2 + y^2} = \dfrac{dy}{2xy} = \dfrac{dz}{0}$. Making use of the property of the proportion, we represent the equation $\dfrac{dx}{x^2 + y^2} = \dfrac{dy}{2xy}$ in the form

$$\frac{dx + dy}{x^2 + y^2 + 2xy} = \frac{dx - dy}{x^2 + y^2 - 2xy}, \quad \text{or} \quad \frac{d(x + y)}{(x + y)^2} = \frac{d(x - y)}{(x - y)^2}.$$

Integration yields

$$-\frac{1}{x + y} = -\frac{1}{x - y} + C, \quad \frac{1}{x - y} - \frac{1}{x + y} = C, \quad \frac{2y}{x^2 - y^2} = C.$$

The last equality can be rewritten as $\dfrac{y}{x^2 - y^2} = C_1$.

The second equation of the system is $dz = 0$. Hence $z = C_2$.
The general solution has the form

$$\Phi\left(\frac{y}{x^2 - y^2}, z\right) = 0, \quad \text{or} \quad z = \psi\left(\frac{y}{x^2 - y^2}\right).$$

918. Find the surface satisfying the equation $yz\dfrac{\partial z}{\partial x} + xz\dfrac{\partial z}{\partial y} = -2xy$ and passing through the circle $x^2 + y^2 = 16$, $z = 3$.

Solution. Let us solve the system of equations $\frac{dx}{yz} = \frac{dy}{xz} = -\frac{dz}{2xy}$. Clearing the equation of fractions, we get

$$xdx = ydy, \quad 2xdx = -zdz.$$

Integrating both equations, we get

$$x^2 - y^2 = C_1, \quad x^2 + \frac{z^2}{2} = C_2.$$

The general solution of the given equation has the form

$$x^2 + \frac{z^2}{2} = \psi(x^2 - y^2). \qquad (*)$$

From the family of surfaces specified by this equation we must isolate the surface passing through the circle $x^2 + y^2 = 16$, $z = 3$. To find the function ψ, we set $x^2 = 16 - y^2$, $z = 3$ in equation $(*)$ and get $16 - y^2 + 9/2 = \psi(16 - 2y^2)$. Assuming $16 - 2y^2 = t$, we obtain $y^2 = 8 - t/2$. Consequently, $\psi(t) = (t + 25)/2$, i.e. $\psi(x^2 - y^2) = (x^2 - y^2 + 25)/2$. Substituting the expression obtained into relation $(*)$, we get

$$x^2 + \frac{z^2}{2} = \frac{x^2 - y^2 + 25}{2}, \quad \text{or} \quad x^2 + y^2 + z^2 = 25.$$

Thus, the desired surface is a sphere.

919. Find the general solution of the equation $\frac{\partial z}{\partial x}\sin x + \frac{\partial z}{\partial y}\sin y = \sin z$.

920. Find the general solution of the equation $yz\frac{\partial z}{\partial x} + xz\frac{\partial z}{\partial y} = xy$.

921. Find the surface satisfying the equation $\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = 4$ and passing through the parabola $y^2 = z$, $x = 0$.

6.2. Types of Second-Order Partial Differential Equations. Their Reduction to Canonical Equations

Let us consider the following second-order equation

$$a\frac{\partial^2 z}{\partial x^2} + 2b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} + F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0, \qquad (1)$$

where a, b, c are the functions of x and y.

The indicated equation is said to be of *hyperbolic type* in the domain D if in this domain $b^2 - ac > 0$. Now if $b^2 - ac = 0$, then the equation is of *parabolic type*, and if $b^2 - ac < 0$, it is of *elliptic type*.

The equation

$$\frac{\partial^2 z}{\partial x \partial y} = F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

is a *canonical equation of hyperbolic type*; the equation

$$\frac{\partial^2 z}{\partial y^2} = F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

is a *canonical equation of parabolic type*; and the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

is a *canonical equation of elliptic type.*

The differential equation

$$a(dy)^2 - 2b\,dx\,dy + c(dx)^2 = 0$$

is the equation for the *characteristics* of equation (1).

For an equation of hyperbolic type, the equation of characteristics possesses two solutions: $\varphi(x, y) = C_1$, $\psi(x, y) = C_2$, that is, there are two families of real characteristics. A change of variables $\xi = \varphi(x, y)$, $\eta = \psi(x, y)$ reduces differential equation (1) to canonical form.

For an equation of parabolic type both families of characteristics coincide, that is, the equation of characteristics gives only one integral $\varphi(x, y) = C$. In that case, a change of variables $\xi = \varphi(x, y)$, $\eta = \psi(x, y)$ must be performed, where $\psi(x, y)$ is some function for which $\frac{\partial \xi}{\partial x} \cdot \frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y} \cdot \frac{\partial \eta}{\partial x} \neq 0$. Such a change of variables leads to a canonical equation.

For an equation of elliptic type, the solutions of the equation of characteristics have the form $\varphi(x, y) \pm i\psi(x, y) = C_{1,2}$, where $\varphi(x, y)$ and $\psi(x, y)$ are real functions. The substitution $\xi = \varphi(x, y)$, $\eta = \psi(x, y)$ reduces equation (1) to canonical form.

922. Reduce to canonical form the equation $x^2 \cdot \frac{\partial^2 u}{\partial x^2} - y^2 \cdot \frac{\partial^2 u}{\partial y^2} = 0.$

Solution. Here $a = x^2$, $b = 0$, $c = -y^2$, $b^2 - ac = x^2y^2 > 0$; consequently, the equation is of hyperbolic type.

We derive an equation of characteristics:

$$x^2(dy)^2 - y^2(dx)^2 = 0, \quad \text{or} \quad (x\,dy + y\,dx)(x\,dy - y\,dx) = 0$$

and obtain two differential equations

$$xdy + ydx = 0 \quad \text{and} \quad xdy - ydx = 0;$$

separating the variables and integrating, we get

$$\frac{dy}{y} + \frac{dx}{x} = 0, \quad \text{i.e.} \quad \ln y + \ln x = \ln C_1,$$
$$\frac{dy}{y} - \frac{dx}{x} = 0, \quad \text{i.e.} \quad \ln y - \ln x = \ln C_2.$$

Taking antilogarithms, we find that $xy = C_1$ and $y/x = C_2$ are the equations of two families of characteristics. We introduce new variables $\xi = xy$, $\eta = y/x$. Let us express the partial derivatives with respect to the old variables in terms of the partial derivatives with respect to the new variables:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi}\cdot y - \frac{\partial u}{\partial \eta}\cdot\frac{y}{x^2};$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi}\cdot x + \frac{\partial u}{\partial \eta}\cdot\frac{1}{x};$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\cdot y\right) - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\cdot\frac{y}{x^2}\right)\\
&= \left(\frac{\partial^2 u}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right)y\\
&\quad - \left(\frac{\partial^2 u}{\partial \eta\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial x}\right)\cdot\frac{y}{x^2} + \frac{\partial u}{\partial \eta}\cdot\frac{2y}{x^3}\\
&= \left(\frac{\partial^2 u}{\partial \xi^2}\cdot y - \frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{y}{x^2}\right)y\\
&\quad - \left(\frac{\partial^2 u}{\partial \xi\partial \eta}\cdot y - \frac{\partial^2 u}{\partial \eta^2}\cdot\frac{y}{x^2}\right)\frac{y}{x^2} + \frac{\partial u}{\partial \eta}\cdot\frac{2y}{x^3}\\
&= \frac{\partial^2 u}{\partial \xi^2}\cdot y^2 - 2\frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{y^2}{x^2} + \frac{\partial^2 u}{\partial \eta^2}\cdot\frac{y^2}{x^4} + 2\frac{\partial u}{\partial \eta}\cdot\frac{y}{x^3};
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= x\left(\frac{\partial^2 u}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)\\
&\quad + \frac{1}{x}\left(\frac{\partial^2 u}{\partial \eta\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial y}\right)\\
&= x\left(\frac{\partial^2 u}{\partial \xi^2}\cdot x + \frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{1}{x}\right) + \frac{1}{x}\left(\frac{\partial^2 u}{\partial \xi\partial \eta}\cdot x + \frac{\partial^2 u}{\partial \eta^2}\cdot\frac{1}{x}\right)\\
&= x^2\cdot\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi\partial \eta} + \frac{1}{x^2}\cdot\frac{\partial^2 u}{\partial \eta^2}.
\end{aligned}$$

Substituting the expressions obtained for the second partial derivatives into the given differential equation, we get

$$x^2\left(\frac{\partial^2 u}{\partial\xi^2}\cdot y^2-2\frac{\partial^2 u}{\partial\xi\,\partial\eta}\cdot\frac{y^2}{x^2}+\frac{\partial^2 u}{\partial\eta^2}\cdot\frac{y^2}{x^4}+2\frac{\partial u}{\partial\eta}\cdot\frac{y}{x^3}\right)$$
$$-y^2\left(\frac{\partial^2 u}{\partial\xi^2}x^2+2\frac{\partial^2 u}{\partial\xi\,\partial\eta}+\frac{1}{x^2}\cdot\frac{\partial^2 u}{\partial\eta^2}\right)=0;$$

$$-4\frac{\partial^2 u}{\partial\xi\,\partial\eta}\cdot y^2+2\frac{\partial u}{\partial\eta}\cdot\frac{y}{x}=0;\quad \frac{\partial^2 u}{\partial\xi\,\partial\eta}-\frac{1}{2}\cdot\frac{\partial u}{\partial\eta}\cdot\frac{1}{xy}=0;$$

$$\frac{\partial^2 u}{\partial\xi\,\partial\eta}-\frac{1}{2\xi}\cdot\frac{\partial u}{\partial\eta}=0,$$

that is, we have reduced the equation to canonical form.

923. Reduce to canonical form the equation

$$\frac{\partial^2 z}{\partial x^2}\cdot\sin^3 x\ \S\text{`}2y\sin x\cdot\frac{\partial^2 z}{\partial x\partial y}+y^2\cdot\frac{\partial^2 z}{\partial y^2}=0.$$

Solution. Here $a=\sin^2 x$, $b=-y\sin x$, $c=y^2$. Since $b^2-ac=y^2\sin^2 x-$ $-y^2\sin^2 x=0$, the given equation is of parabolic type.

The equation of the characteristics has the form

$$\sin^2 x(dy)^2+2y\sin x\,dx\,dy+y^2(dx)^2=0,\quad\text{or}\quad(\sin x\,dy+y\,dx)^2=0.$$

Separating the variables in the equation $\sin x\,dy+y\,dx=0$ and integrating, we obtain

$$\frac{dy}{y}+\frac{dx}{\sin x}=0;\quad \ln y+\ln\tan\frac{x}{2}=\ln C;\quad y\cdot\tan\frac{x}{2}=C.$$

Performing the change of variables $\xi=y\tan\dfrac{x}{2}$, $\eta=y$ (an arbitrary function), we get

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial\xi}\cdot\frac{\partial\xi}{\partial x}+\frac{\partial z}{\partial\eta}\cdot\frac{\partial\eta}{\partial x}=\frac{1}{2}\cdot\frac{\partial z}{\partial\xi}\cdot y\cdot\sec^2\frac{x}{2};$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial\xi}\cdot\frac{\partial\xi}{\partial y}+\frac{\partial z}{\partial\eta}\cdot\frac{\partial\eta}{\partial y}=\frac{\partial z}{\partial\xi}\cdot\tan\frac{x}{2}+\frac{\partial z}{\partial\eta};$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{1}{2}\left(\frac{\partial^2 z}{\partial\xi^2}\cdot\frac{\partial\xi}{\partial x}+\frac{\partial^2 z}{\partial\xi\partial\eta}\cdot\frac{\partial\eta}{\partial x}\right)\cdot y\sec^2\frac{x}{2}$$
$$+\frac{1}{2}\cdot\frac{\partial z}{\partial\xi}\cdot y\sec^2\frac{x}{2}\cdot\tan\frac{x}{2}$$
$$=\frac{1}{4}\cdot\frac{\partial^2 z}{\partial\xi^2}\cdot y^2\cdot\sec^4\frac{x}{2}+\frac{1}{2}\cdot y\cdot\frac{\partial z}{\partial\xi}\cdot\sec^2\frac{x}{2}\cdot\tan\frac{x}{2};$$

$$\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial^2 z}{\partial\xi^2}\cdot\frac{\partial\xi}{\partial y}+\frac{\partial^2 z}{\partial\xi\partial\eta}\cdot\frac{\partial\eta}{\partial y}\right)\cdot\tan\frac{x}{2}$$

$$+ \frac{\partial^2 z}{\partial\eta\partial\xi} \cdot \frac{\partial\xi}{\partial y} + \frac{\partial^2 z}{\partial\eta^2} \cdot \frac{\partial\eta}{\partial y}$$
$$= \frac{\partial^2 z}{\partial\xi^2} \cdot \tan^2\frac{x}{2} + 2 \cdot \frac{\partial^2 z}{\partial\xi\partial\eta} \cdot \tan\frac{x}{2} + \frac{\partial^2 z}{\partial\eta^2};$$
$$\frac{\partial^2 z}{\partial x\partial y} = \frac{1}{2}\left(\frac{\partial^2 z}{\partial\xi^2} \cdot \frac{\partial\xi}{\partial y} + \frac{\partial^2 z}{\partial\xi\partial\eta} \cdot \frac{\partial\eta}{\partial y}\right) y \cdot \sec^2\frac{x}{2} + \frac{1}{2} \cdot \frac{\partial z}{\partial\xi} \cdot \sec^2\frac{x}{2}$$
$$= \frac{1}{2}\left(\frac{\partial^2 z}{\partial\xi^2} \cdot \tan\frac{x}{2} + \frac{\partial^2 z}{\partial\xi\partial\eta}\right) y \cdot \sec^2\frac{x}{2} + \frac{1}{2} \cdot \frac{\partial z}{\partial\xi} \cdot \sec^2\frac{x}{2}.$$

Substituting the expressions obtained for the second partial derivatives into the given differential equation, we get

$$\frac{1}{4}\frac{\partial^2 z}{\partial\xi^2} \cdot y^2 \cdot \sec^4\frac{x}{2} \cdot \sin^2 x + \frac{1}{2}\frac{\partial z}{\partial\xi} \cdot y \cdot \sec^2\frac{x}{2} \cdot \tan\frac{x}{2} \cdot \sin^2 x$$
$$- \left(\frac{\partial^2 z}{\partial\xi^2} \cdot \tan\frac{x}{2} + \frac{\partial^2 z}{\partial\xi\partial\eta}\right) \cdot y^2 \cdot \sec^2\frac{x}{2} \cdot \sin x - \frac{\partial z}{\partial\xi} \cdot y \cdot \sec^2\frac{x}{2} \cdot \sin x$$
$$+ y^2 \cdot \left(\frac{\partial^2 z}{\partial\xi^2} \cdot \tan^2\frac{x}{2} + 2\frac{\partial^2 z}{\partial\xi\partial\eta} \cdot \tan\frac{x}{2} + \frac{\partial^2 z}{\partial\eta^2}\right) = 0.$$

It can be easily shown that the terms containing $\frac{\partial^2 z}{\partial\xi^2}$ and $\frac{\partial^2 z}{\partial\xi\partial\eta}$ cancel out, and the equation assumes the form

$$\frac{1}{2} \cdot \frac{\partial z}{\partial\xi} \cdot y\sec^2\frac{x}{2} \cdot \tan\frac{x}{2} \cdot \sin^2 x + y^2 \cdot \frac{\partial^2 z}{\partial\eta^2} - \frac{\partial z}{\partial\xi} \cdot y\sec^2\frac{x}{2} \cdot \sin x = 0,$$

or

$$y \cdot \frac{\partial^2 z}{\partial\eta^2} = \frac{\partial z}{\partial\xi} \cdot \sin x.$$

Since $\sin x = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$, $\tan\frac{x}{2} = \frac{\xi}{\eta}$, it follows that $\sin x = \frac{2\xi\eta}{\xi^2 + \eta^2}$. The final result is

$$\frac{\partial^2 z}{\partial\eta^2} = \frac{2\xi}{\xi^2 + \eta^2} \cdot \frac{\partial z}{\partial\xi}.$$

924. Reduce to canonical form the equation

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x\partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0.$$

Solution. Here $a = 1, b = -1, c = 2, b^2 - ac = -1 < 0$, i.e. the equation is of elliptic type.

The equation of the characteristics has the form

$$(dy)^2 + 2dxdy + 2(dx)^2 = 0, \quad \text{or} \quad y'^2 + 2y' + 2 = 0.$$

Hence $y' = -1 \pm i$; we obtain two families of imaginary characteristics: $y + x - ix = C_1$ and $y + x + ix = C_2$. We perform a change of variables $\xi = y + x$, $\eta = x$ and get

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta};$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{\partial z}{\partial \xi};$$

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial^2 z}{\partial \xi^2} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial^2 z}{\partial \xi \partial \eta} \cdot \frac{\partial \eta}{\partial x}\right) + \left(\frac{\partial^2 z}{\partial \eta \partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial^2 z}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial x}\right)$$
$$= \frac{\partial^2 z}{\partial \xi^2} + 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2};$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial \xi^2} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial^2 z}{\partial \xi \partial \eta} \cdot \frac{\partial \eta}{\partial x} = \frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \xi \partial \eta};$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial \xi^2} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial^2 z}{\partial \xi \partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{\partial^2 z}{\partial \xi^2}.$$

Substituting the expressions obtained into the differential equation, we get

$$\frac{\partial^2 z}{\partial \xi^2} + 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2} - 2\frac{\partial^2 z}{\partial \xi^2} - 2\frac{\partial^2 z}{\partial \xi \partial \eta} + 2\frac{\partial^2 z}{\partial \xi^2} = 0, \text{ i.e. } \frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2} = 0.$$

Reduce to canonical form the following equations:

925. $x^2 \cdot \dfrac{\partial^2 z}{\partial x^2} + 2xy \cdot \dfrac{\partial^2 z}{\partial x \partial z} + y^2 \cdot \dfrac{\partial^2 z}{\partial y^2} = 0.$

926. $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} - 3\dfrac{\partial^2 z}{\partial y^2} - 2\dfrac{\partial z}{\partial x} + 6\dfrac{\partial z}{\partial y} = 0.$

927. $\dfrac{1}{x^2} \cdot \dfrac{\partial^2 z}{\partial x^2} + \dfrac{1}{y^2} \cdot \dfrac{\partial^2 z}{\partial y^2} = 0.$

6.3. Equation of Vibrating String

6.3.1. Solution of the equation of vibrating string by the method of characteristics (D'Alembert's method). A *string* is a thin thread which can be easily bent. Suppose the string is under the action of strong initial tension T_0. If it is disturbed from the equilibrium position and is subjected to the action of some force, the string begins to vibrate (Fig. 55).

We shall restrict the discussion to small, transverse and plane vibrations of the string, that is, the vibrations for which the deviations of the points of the string from the state of rest are small, at any time moment all the points of the string are in

the same plane and each point of the string vibrates remaining on the same perpendicular to the straight line corresponding to the state of rest of the string.

Assuming this straight line to be the x-axis, we denote by $u = u(x, t)$ the deviation of the points of the string from the position of equilibrium at the time moment t. For each fixed value of t the graph of the function $u = u(x, t)$ on the plane xOu represents the shape of the string at the moment of time t.

The function $u = u(x, t)$ satisfies the differential equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \cdot \frac{\partial^2 u}{\partial x^2} + f,$$

where $a^2 = T_0/\rho, f = F/\rho$, ρ is the mass of the unit length (linear density of the string), F is the force acting on the string at right angles to the abscissa axis and calculated for a unit length.

If there is no external force, i.e. $f = 0$, then we have an *equation of free vibrations of the string*

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

To complete the definition of the motion of the string, we must assign the shape and the velocity of the string at the initial moment, that is, the position of its points and their velocity, as the functions of the abscisses x of those points. Assume $u\Big|_{t=0} = \varphi(x)$, $\frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x)$. These conditions are called the *initial conditions* of the problem.

Reducing the equation $\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0$ to canonical form, we get the equation $\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$, where $\xi = x - at$, $\eta = x + at$. The general solution of the last equation can be written as $u = \Theta_1(\xi) + \Theta_2(\eta)$, where $\xi = x - at$, $\eta = x + at$, Θ_1, Θ_2 are arbitrary functions.

Thus, the general solution of the differential equation of free vibrations is of the form

$$y = \Theta_1(x - at) + \Theta_2(x + at).$$

Choosing the functions Θ_1 and Θ_2 such that the function $u = u(x, t)$ satisfies the

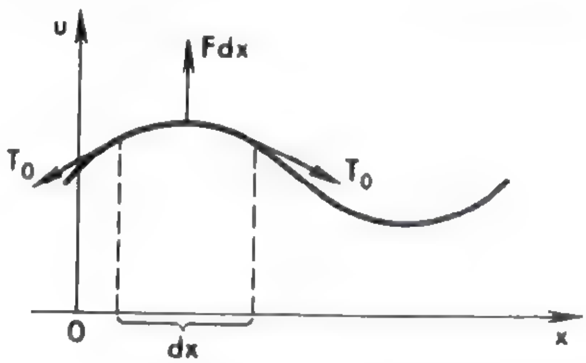


Fig. 55

initial conditions, we arrive at the solution of the original differential equation in the form

$$u = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(z)dz.$$

928. Find the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \text{ if } u\Big|_{t=0} = x^2, \ \frac{\partial u}{\partial t}\Big|_{t=0} = 0.$$

Solution. Since $a = 1$, and $\psi(x) = 0$, it follows that $u = \dfrac{\varphi(x-at)+\varphi(x+at)}{2}$, where $\varphi(x) = x^2$. Thus we have

$$u = \frac{(x-t)^2+(x+t)^2}{2}, \quad \text{or} \quad u = x^2 + t^2.$$

929. Find the solution of the equation $\dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2}$, if $u\Big|_{t=0} = 0, \ \dfrac{\partial u}{\partial t}\Big|_{t=0} = x.$

Solution. Here $a = 2$, $\varphi(x) = 0$, $\psi(x) = x$. Hence

$$u = \frac{1}{4}\int_{x-2t}^{x+2t} z\,dz = \frac{1}{8}z^2\Big|_{x-2t}^{x+2t} = \frac{1}{8}[(x+2t)^2-(x-2t)^2], \text{ i.e. } u = xt.$$

930. Find the shape of the string specified by the equation $\dfrac{\partial^2 u}{\partial t^2} = a^2 \cdot \dfrac{\partial^2 u}{\partial x^2}$ at the time moment $t = \dfrac{\pi}{2a}$, if $u\Big|_{t=0} = \sin x, \ \dfrac{\partial u}{\partial t}\Big|_{t=0} = 1.$

Solution. We have

$$u = \frac{\sin(x+at)+\sin(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} dz,$$

that is,

$$u = \sin x \cdot \cos at + \frac{1}{2a}\cdot z\Big|_{x-at}^{x+at}, \quad \text{or} \quad u = \sin x \cdot \cos at + t.$$

If $t = \pi/2a$, then $u = \pi/2a$, that is, the string is parallel to the abscissa axis.

931. Find the solution of the equation $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$, if $u\Big|_{t=0} = x, \ \dfrac{\partial u}{\partial t}\Big|_{t=0} = -x.$

932. Find the solution of the equation $\dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2}$, if $u\Big|_{t=0} = 0, \ \dfrac{\partial u}{\partial t}\Big|_{t=0} = \cos x.$

933. Find the shape of the string specified by the equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ at the time moment $t = \pi$, if $u\Big|_{t=0} = \sin x$, $\frac{\partial u}{\partial t}\Big|_{t=0} = \cos x$.

6.3.2. Fourier method of solving the equation of vibrating string. The solution of the differential equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ satisfying the initial conditions

$$u\Big|_{t=0} = \varphi(x), \ \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x)$$

and the boundary conditions

$$u\big|_{x=0} = 0, \ u\big|_{x=l} = 0$$

can be represented as the sum of the infinite series

$$u(x, t) = \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi a t}{l} + b_k \sin \frac{k\pi a t}{l}\right) \sin \frac{k\pi x}{l}, \quad (1)$$

where

$$a_k = \frac{2}{l}\int_0^l \varphi(x) \sin \frac{k\pi x}{l}\,dx, \ b_k = \frac{2}{k\pi a}\int_0^l \psi(x) \sin \frac{k\pi x}{l}\,dx.$$

The indicated boundary conditions are introduced when the discussion concerns the vibrations of a string of length l, fixed at two points: $x = 0$ and $x = l$.

934. A string fixed at the end points $x = 0$ and $x = l$ has at the initial moment the shape of the parabola $u = (4h/l^2) \cdot x(l - x)$. Determine the displacement of the points of the string from the abscissa axis if their initial velocities are zero (Fig. 56).

Solution. Here $\varphi(x) = (4h/l^2) \cdot x(l - x)$, $\psi(x) = 0$. We find the coefficients of the series determining the solution of the equation of the string vibration:

$$a_k = \frac{2}{l} \cdot \int_0^l \varphi(x) \cdot \sin \frac{k\pi x}{l}\,dx = \frac{8h}{l^3} \cdot \int_0^l (lx - x^2) \cdot \sin \frac{k\pi x}{l}\,dx; \ b_k = 0.$$

To find the coefficients a_k, we twice perform integration by parts:

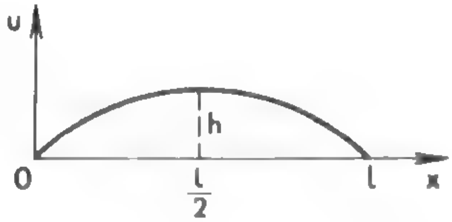


Fig. 56

$$u_1 = lx - x^2, \quad dv_1 = \sin\frac{k\pi x}{l}\,dx, \quad du_1 = (l - 2x)dx,$$

$$v_1 = -\frac{l}{k\pi}\cdot\cos\frac{k\pi x}{l};$$

$$a_k = -\frac{8h}{l^3}(lx - x^2)\cdot\frac{l}{k\pi}\cdot\cos\frac{k\pi x}{l}\Bigg|_0^l + \frac{8h}{k\pi l^2}\cdot\int_0^l (l - 2x)\cdot\cos\frac{k\pi x}{l}\,dx,$$

that is

$$a_k = \frac{8h}{k\pi l^2}\cdot\int_0^l (l - 2x)\cdot\cos\frac{k\pi x}{l}\,dx;$$

$$u_2 = l - 2x, dv_2 = \cos\frac{k\pi x}{l}\,dx, du_2 = -2dx, v_2 = \frac{l}{k\pi}\cdot\sin\frac{k\pi x}{l};$$

$$a_k = \frac{8h}{k^2\pi^2 l}(l - 2x)\cdot\sin\frac{k\pi x}{l}\Bigg|_0^l + \frac{16h}{k^2\pi^2 l}\cdot\int_0^l \sin\frac{k\pi x}{l}\,dx = -\frac{16h}{k^3\pi^3}\cdot\cos\frac{k\pi x}{l}\Bigg|_0^l$$

$$= -\frac{16h}{k^3\pi^3}(\cos k\pi - 1) = \frac{16h}{k^3\pi^3}\cdot[1 - (-1)^k].$$

Substituting the expressions for a_k and b_k into equation (1), we obtain

$$u(x, t) = \sum_{k=1}^{\infty}\frac{16h}{k^3\pi^3}\cdot[1 - (-1)^k]\cdot\cos\frac{k\pi at}{l}\cdot\sin\frac{k\pi x}{l}.$$

But if $k = 2n$, then $1 - (-1)^k = 0$, and if $k = 2n + 1$, then $1 - (-1)^k = 2$; therefore, we finally have

$$u(x, t) = \frac{32h}{\pi^3}\cdot\sum_{n=1}^{\infty}\frac{1}{(2n + 1)^3}\cdot\cos\frac{(2n + 1)\pi at}{l}\cdot\sin\frac{(2n + 1)\pi x}{l}.$$

935. Given the string fixed at the end points $x = 0$, $x = l$. Suppose at the initial moment the string is shaped as a polygonal line OAB shown in Fig. 57. Find the shape of the string for any moment of time t, provided there are no initial velocities.

Solution. The slope of the straight line OA is equal to $h/(l/2)$, i.e. $2h/l$. Consequently, the equation of that line is $u = (2h/l)x$. The line AB intercepts the line segments l and $2h$ on the axes of coordinates and, therefore, the equation of that straight line has the form $x/l + u/2h = 1$, or $u = (2h/l)(l - x)$. Thus we have

$$\varphi(x) = \begin{cases}(2h/l)x, & \text{if } 0 \leqslant x \leqslant l/2;\\ (2h/l)(l - x), & \text{if } l/2 \leqslant x \leqslant l;\end{cases} \quad \psi(x) = 0.$$

We find

$$a_k = \frac{2}{l}\int_0^l \varphi(x)\cdot\sin\frac{k\pi x}{l}\,dx = \frac{4h}{l^2}\cdot\int_0^{l/2} x\cdot\sin\frac{k\pi x}{l}\,dx + \frac{4h}{l^2}\cdot\int_{l/2}^{l}(l-x)\sin\frac{k\pi x}{l}\,dx,$$

$$b_k = 0.$$

Integrating by parts, we get

$$a_k = -\frac{4h}{k\pi l}\cdot x\cdot\cos\frac{k\pi x}{l}\Big|_0^{l/2} + \frac{4h}{k\pi l}\cdot\int_0^{l/2}\cos\frac{k\pi x}{l}\,dx$$

$$-\frac{4h}{k\pi l}(l-x)\cos\frac{k\pi x}{l}\Big|_{l/2}^{l} - \frac{4h}{k\pi l}\cdot\int_{l/2}^{l}\cos\frac{k\pi x}{l}\,dx$$

$$= -\frac{2h}{k\pi}\cdot\cos\frac{k\pi}{2} + \frac{4h}{k^2\pi^2}\cdot\sin\frac{k\pi x}{l}\Big|_0^{l/2} + \frac{2h}{k\pi}\cdot\cos\frac{k\pi}{2}$$

$$-\frac{4h}{k^2\pi^2}\cdot\sin\frac{k\pi x}{l}\Big|_{l/2}^{l} = \frac{4h}{k^2\pi^2}\cdot\sin\frac{k\pi}{2} + \frac{4h}{k^2\pi^2}\cdot\sin\frac{k\pi}{2} = \frac{8h}{k^2\pi^2}\cdot\sin\frac{k\pi}{2}.$$

Consequently,

$$u(x,t) = \frac{8h}{\pi^2}\cdot\sum_{k=1}^{\infty}\frac{1}{k^2}\cdot\sin\frac{k\pi}{2}\cdot\sin\frac{k\pi x}{l}\cdot\cos\frac{k\pi at}{l}.$$

We write out several members of the series:

$$u(x,t) = \frac{8h}{\pi^2}\left(\sin\frac{\pi x}{l}\cdot\cos\frac{\pi at}{l} - \frac{1}{3^2}\cdot\sin\frac{3\pi x}{l}\cdot\cos\frac{3\pi at}{l}\right.$$

$$\left.+\frac{1}{5^2}\cdot\sin\frac{5\pi x}{l}\cdot\cos\frac{5\pi at}{l} - \frac{1}{7^2}\cdot\sin\frac{7\pi x}{l}\cdot\cos\frac{7\pi at}{l} + \ldots\right).$$

936. Suppose the initial deviations of the string fixed at the end points $x = 0$ and $x = l$ are zero, and the initial velocity is expressed by the formula

$$\frac{\partial u}{\partial t} = \begin{cases} v_0\,(\text{const}), & \text{if } |x - l/2| < h/2; \\ 0, & \text{if } |x - l/2| > h/2. \end{cases}$$

Determine the shape of the string for any moment of time t.

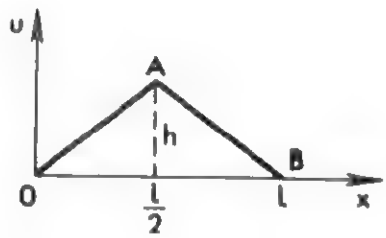


Fig. 57

Solution. Here $\varphi(x) = 0$ and $\psi(x) = v_0$ in the interval $((l - h)/2, (l + h)/2)$, and $\psi(x) = 0$ outside this interval.

Consequently, $a_k = 0$;

$$b_k = \frac{2}{k\pi a} \cdot \int\limits_{(l-h)/2}^{(l+h)/2} v_0 \cdot \sin\frac{k\pi x}{l}\,dx$$

$$= -\frac{2v_0}{k\pi a} \cdot \frac{l}{k\pi} \cos\frac{k\pi x}{l}\Bigg|_{(l-h)/2}^{(l+h)/2}$$

$$= \frac{2v_0 l}{k^2\pi^2 a} \cdot \left[\cos\frac{k\pi(l-h)}{2l} - \cos\frac{k\pi(l+h)}{2}\right] = \frac{4v_0 l}{k^2\pi^2 a} \cdot \sin\frac{k\pi}{2} \cdot \sin\frac{k\pi h}{2l}.$$

Hence

$$u(x, t) = \frac{4v_0 l}{\pi^2 a} \cdot \sum_{k=1}^{\infty} \frac{1}{k^2} \cdot \sin\frac{k\pi}{2} \cdot \sin\frac{k\pi h}{2} \cdot \cos\frac{k\pi at}{l} \cdot \sin\frac{k\pi x}{l},$$

or

$$u(x, t) = \frac{4v_0 l}{\pi^2 a} \cdot \left(\sin\frac{\pi h}{2l} \cdot \sin\frac{\pi at}{l} \cdot \sin\frac{\pi x}{l} - \frac{1}{3^2} \cdot \sin\frac{3\pi h}{2l} \cdot \sin\frac{3\pi at}{l} \cdot \sin\frac{3\pi x}{l} + \frac{1}{5^2} \cdot \sin\frac{5\pi h}{2l} \cdot \sin\frac{5\pi at}{l} \cdot \sin\frac{5\pi x}{l} - \ldots\right).$$

937. The string is fixed at the end points $x = 0$ and $x = 3$. At the initial instant the string is shaped as a polygonal line OAB, where $O(0; 0)$, $A(2; -0.1)$, $B(3; 0)$ (Fig. 58). Find the shape of the string for any time moment t if the initial velocities of the points of the string are zero.

938. The string, fixed at the end points $x = 0$ and $x = l$, has the form $u = h(x^4 - 2x^3 + x)$ at the initial instant. Find the shape of the string for any time moment t if the initial velocities are absent.

939. The string is fixed at the end points $x = 0$ and $x = l$. The initial deviations of the points of the string are zero and the initial velocity is expressed by the formula

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \begin{cases} \cos\dfrac{\pi(x - l/2)}{h}, & \text{if } \left|x - \dfrac{l}{2}\right| < \dfrac{h}{2}; \\ 0, & \text{if } \left|x - \dfrac{l}{2}\right| > \dfrac{l}{2}. \end{cases}$$

Find the shape of the string for any time moment t.

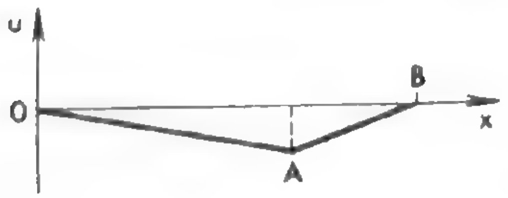


Fig. 58

6.4. Heat Equation

6.4.1. Heat equation in the non-stationary case. Let us denote by $u = u(M, t)$ the temperature at the point M of a homogeneous body, bounded by the surface S, at the moment of time t. It is known that the quantity of heat dQ absorbed by the body during the time dt is expressed by the equation

$$dQ = k \cdot \frac{\partial u}{\partial n}\, dS dt, \tag{1}$$

where dS is an element of the surface, k is the so-called thermal conductivity, $\frac{\partial u}{\partial n}$ is the derivative of the function u in the direction of the outer normal to the surface S. Since the heat flows in the direction of temperature decrease, we have $dQ > 0$ if $\frac{\partial u}{\partial n} > 0$ and $dQ < 0$ if $\frac{\partial u}{\partial n} < 0$. It follows from equality (1) that

$$Q = dt \cdot \iint_S k \cdot \frac{\partial u}{\partial n}\, dS.$$

Let us use another method to calculate Q. We isolate an element dV of the volume V bounded by the surface S. The quantity of heat dQ received by the element dV during the time dt is proportional to the temperature rise in that element and to the mass of the element itself, i.e.

$$dQ = \gamma \cdot \frac{\partial u}{\partial t}\, dt \cdot \rho dV, \tag{2}$$

where ρ is the density of the substance and γ is a proportionality factor known as the heat capacity of the substance. It follows from equation (2) that

$$Q = dt \cdot \iiint_V \gamma \cdot \rho \frac{\partial u}{\partial t}\, dV.$$

Thus we obtain

$$\iiint_V \frac{\partial u}{\partial t}\, dV = a^2 \cdot \iint_S \frac{\partial u}{\partial n}\, dS,$$

where $a^2 = \frac{k}{\rho\gamma}$. Taking into account that $\frac{\partial u}{\partial n} = |\operatorname{grad} u|$ and $\operatorname{grad} u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}$, we transform the resulting equality as follows:

$$\iiint_V \frac{\partial u}{\partial t}\, dV = a^2 \iint_S \left[\frac{\partial u}{\partial x}\cos(n, x) + \frac{\partial u}{\partial y}\cos(n, y) + \frac{\partial u}{\partial z}\cos(n, z)\right] dS.$$

Using Ostrogradsky-Gauss' formula to transform the right-hand side of the equation, we get

$$\iiint\limits_V \frac{\partial u}{\partial t}\, dV = a^2 \iiint\limits_V \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) dV.$$

From this we obtain the differential equation

$$\frac{\partial u}{\partial t} = a\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$$

known as the *heat equation for the non-stationary case.*

If the body is a bar directed along the Ox axis, the heat equation has the form

$$\frac{\partial u}{\partial t} = a^2 \cdot \frac{\partial^2 u}{\partial x^2}.$$

Let us consider Cauchy's problem for the following three cases:

1. The case of an infinite bar. It is required to find the solution $u(x, t)$ of the equation

$$\frac{\partial u}{\partial t} = a^2 \cdot \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad -\infty < x < +\infty,$$

satisfying the initial condition $u(x, 0) = f(x)$, $-\infty < x < +\infty$. Applying Fourier's method, we obtain the solution of the equation in the form

$$u(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} f(\xi) \cdot e^{-(\xi - x)^2/(4a^2 t)} d\xi.$$

2. The case of a bar bounded on one side. The solution of the equation $\dfrac{\partial u}{\partial t} = a^2 \cdot \dfrac{\partial^2 u}{\partial x^2}$ satisfying the initial condition $u(x, 0) = f(x)$ and the boundary condition $u(0, t) = \varphi(t)$ is expressed by the formula

$$u(x, t) = \frac{1}{2a\sqrt{\pi t}} \cdot \int_0^{\infty} f(\xi) \cdot \left[e^{-(\xi - x)^2/(4a^2 t)} - e^{-(\xi + x)^2/(4a^2 t)}\right] d\xi$$

$$+ \frac{1}{2a\sqrt{\pi t}} \cdot \int_0^{t} \varphi(\eta) \cdot e^{x^2/(4a^2(t-\eta))}(t - \eta)^{-3/2} d\eta.$$

3. The case of a bar bounded on both sides, $x = 0, x = l$. Cauchy's problem here is to find the solution of the equation $\dfrac{\partial u}{\partial t} = a^2 \cdot \dfrac{\partial^2 u}{\partial x^2}$ satisfying the initial condition $u(x, t)\big|_{t=0} = f(x)$ and two boundary conditions, for instance, $u\big|_{x=0} = u\big|_{x=l} = 0$ or $\dfrac{\partial u}{\partial x}\Big|_{x=0} = \dfrac{\partial u}{\partial x}\Big|_{x=l} = 0$. In this case the particular solution is sought in the form of the series

$$u(x, t) = \sum_{k=1}^{\infty} b_k \cdot e^{-(k\pi a/l)^2 \cdot t} \sin\frac{k\pi x}{l},$$

where

$$b_k = \frac{2}{l} \int_0^l f(x) \sin \frac{k\pi x}{l} dx$$

(for the boundary conditions $u|_{x=0} = u|_{x=l} \equiv 0$), and in the form of the series

$$u(x, t) = \sum_{k=1}^{\infty} a_k \cdot e^{-(k\pi a/l)^2 \cdot t} \cdot \cos \frac{k\pi x}{l} + a_0,$$

where

$$a_k = \frac{2}{l} \int_0^l f(x) \cdot \cos \frac{k\pi x}{l} dx,$$

$$a_0 = \frac{1}{l} \int_0^l f(x) dx$$

$$\left(\text{for the boundary conditions } \frac{\partial u}{\partial x}\bigg|_{x=0} = \frac{\partial u}{\partial x}\bigg|_{x=l} \equiv 0\right).$$

940. Solve the equation $\frac{\partial u}{\partial t} = a^2 \cdot \frac{\partial^2 u}{\partial x^2}$ for the following initial distribution of the temperatures of the bar:

$$u(x, t)\bigg|_{t=0} = f(x) = \begin{cases} u_0, & \text{if} \quad x_1 < x < x_2; \\ 0, & \text{if} \quad x < x_1 \text{ or } x > x_2. \end{cases}$$

Solution. The bar is infinite and, therefore, the solution is written as Poisson's integral:

$$u(x, t) = \frac{1}{2a\sqrt{\pi t}} \cdot \int_{-\infty}^{+\infty} f(\xi) \cdot e^{-(\xi - x)^2/(4a^2 t)} d\xi.$$

Since in the interval (x_1, x_2) the function $f(x)$ is equal to the constant temperature u_0 and outside the interval the temperature is zero, the solution has the form

$$u(x, t) = \frac{u_0}{2\, a \sqrt{\pi\, t}} \cdot \int_{x_1}^{x_2} e^{-(\xi - x)^2/(4a^2 t)} d\xi.$$

The result obtained can be reduced to the probability integral (see p. 232):

$$\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\mu^2} d\mu.$$

Indeed, setting $(x - \xi)/(2a\sqrt{t}) = \mu$, $d\xi = -2a\sqrt{t} \cdot d\mu$, we get

$$u(x, t) = -\frac{u_0}{\sqrt{\pi}} \cdot \int_{(x-x_1)/(2a\sqrt{t})}^{(x-x_2)/(2a\sqrt{t})} e^{-\mu^2} d\mu$$

$$= \frac{u_0}{\sqrt{\pi}} \cdot \int_0^{(x-x_1)/(2a\sqrt{t})} e^{-\mu^2} d\mu - \frac{u_0}{\sqrt{\pi}} \int_0^{(x-x_2)/(2a\sqrt{t})} e^{-\mu^2} d\mu.$$

Thus, the solution is expressed by the formula

$$u(x, t) = \frac{u_0}{2} \cdot \left[\Phi\left(\frac{x - x_1}{2a\sqrt{t}}\right) - \Phi\left(\frac{x - x_2}{2a\sqrt{t}}\right)\right].$$

The graph of the function $\Phi(z)$ is the curve shown in Fig. 59.

941. Find the solution of the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ satisfying the initial condition $u|_{t=0} = f(x) = u_0$ and the boundary condition $u|_{x=0} = 0$.

Solution. Here we have a differential heat equation for a semi-infinite bar. The solution satisfying the indicated conditions has the form

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \cdot \int_0^{\infty} u_0 \cdot [e^{-(\xi - x)^2/(4t)} - e^{-(\xi + x)^2/(4t)}] d\xi,$$

or

$$u(x, t) = \frac{u_0}{2\sqrt{\pi t}} \cdot \int_0^{\infty} [e^{-(\xi - x)^2/(4t)} - e^{-(\xi + x)^2/(4t)}] d\xi.$$

Setting $(x - \xi)/(2\sqrt{t}) = \mu$, $d\xi = -2\sqrt{t}\, d\mu$, we transform the first integral using the probability integral, that is,

$$\frac{u_0}{2\sqrt{\pi t}} \cdot \int_0^{\infty} e^{-(\xi - x)^2/(4t)} d\xi$$

$$= \frac{u_0}{\sqrt{\pi}} \cdot \int_{-\infty}^{x/(2\sqrt{t})} e^{-\mu^2} d\mu = \frac{u_0}{2}\left[1 + \Phi\left(\frac{x}{2\sqrt{t}}\right)\right].$$

Setting $(x + \xi)/(2\sqrt{t}) = \mu$, $d\xi = 2\sqrt{t}\, d\mu$, we get

$$\frac{u_0}{2\sqrt{\pi t}} \cdot \int_0^{\infty} e^{-(x+\xi)^2/(4t)} d\xi = \frac{u_0}{\sqrt{\pi}} \cdot \int_{x/(2\sqrt{t})}^{+\infty} e^{-\mu^2} d\mu = \frac{u_0}{2}\left[1 - \Phi\left(\frac{x}{2\sqrt{t}}\right)\right].$$

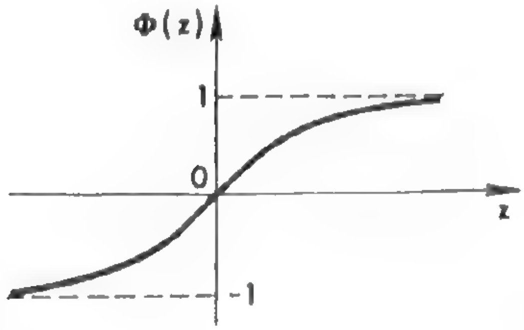


Fig. 59

Thus, the solution assumes the form

$$u(x, t) = u_0 \cdot \Phi\left(\frac{x}{2\sqrt{t}}\right).$$

942. Find the solution of the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ $(0 < x < l)$, $t > 0$, satisfying the initial conditions

$$u\Big|_{t=0} = f(x) = \begin{cases} x, & \text{if } 0 < x \leqslant l/2; \\ l - x, & \text{if } l/2 \leqslant x < l \end{cases}$$

and the boundary conditions $u\big|_{x=0} = u\big|_{x=l} = 0$.

Solution. The solution of Cauchy's problem satisfying the indicated boundary conditions will be sought in the form

$$u(x, t) = \sum_{k=1}^{\infty} b_k e^{-(k\pi/l)^2 \cdot t} \sin \frac{k\pi x}{l},$$

where

$$b_k = \frac{2}{l} \int_0^l f(x) \sin \frac{k\pi x}{l}\, dx = \frac{2}{l} \int_0^{l/2} x \sin \frac{k\pi x}{l}\, dx + \frac{2}{l} \int_{l/2}^{l} (l - x) \sin \frac{k\pi x}{l}\, dx.$$

We perform integration by parts, setting $u = x$, $dv = \sin \dfrac{k\pi x}{l}\, dx$, $du = dx$ and $v = -\dfrac{l}{k\pi} \cdot \cos \dfrac{k\pi x}{l}$, and get

$$b_k = \frac{2}{l}\left(-\frac{lx}{k\pi} \cdot \cos \frac{k\pi x}{l} + \frac{l^2}{k^2\pi^2} \cdot \sin \frac{k\pi x}{l}\right)\Bigg|_0^{l/2}$$

$$+\frac{2}{l}\left(-\frac{l^2}{k\pi} \cdot \cos \frac{k\pi x}{l} + \frac{lx}{k\pi} \cdot \cos \frac{k\pi x}{l} - \frac{l^2}{k^2\pi^2} \cdot \sin \frac{k\pi x}{l}\right)\Bigg|_{l/2}^{l}$$

$$= \frac{4l}{k^2\pi^2} \cdot \sin \frac{k\pi}{2}.$$

Consequently, the desired solution has the form

$$u(x, t) = \frac{4l}{\pi^2} \cdot \sum_{k=1}^{\infty} \frac{1}{k^2} \cdot \sin \frac{k\pi}{2} \cdot e^{-k^2\pi^2 t/l^2} \cdot \sin \frac{k\pi x}{l},$$

or

$$u(x, t) = \frac{4l}{\pi^2} \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2n+1}\, e^{-(2n+1)^2\pi^2 t/l^2} \cdot \sin \frac{(2n + 1)\pi x}{l}.$$

943. Find the solution of the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ satisfying the initial conditions

$$u(x, t)\big|_{t=0} = f(x) = \begin{cases} 1 - x/l, & \text{if} \quad 0 \leqslant x \leqslant l; \\ 1 + x/l, & \text{if} \quad -l \leqslant x \leqslant 0; \\ 0, & \text{if} \quad x \geqslant l \text{ and } x \leqslant -l. \end{cases}$$

Hint. The solution will be expressed by the formula

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \cdot \int_{-l}^{0} \left(1 + \frac{\xi}{l}\right) e^{-(x-\xi)^2/(4t)} d\xi$$

$$+ \frac{1}{2\sqrt{\pi t}} \cdot \int_{0}^{l} \left(1 - \frac{\xi}{l}\right) e^{-(x-\xi)^2/(4t)} d\xi.$$

Simplify the result by the substitution $x - \xi/(2\sqrt{t}) = \mu$.

944. Find the solution of the heat equation if the left end point $x = 0$ of a semi-infinite bar is insulated and the initial temperature distribution is

$$u\big|_{t=0} = f(x) = \begin{cases} 0, & \text{if} \quad x < 0; \\ u_0, & \text{if} \quad 0 < x < l; \\ 0, & \text{if} \quad l < x. \end{cases}$$

945. Given a thin homogeneous bar of length l isolated from space and having the initial temperature $f(x) = cx(l - x)/l^2$. The temperature at the ends of the bar is zero. Determine the temperature of the bar at the time moment $t > 0$.

Hint. The law of distribution of the bar temperature is described by the equation $\frac{\partial u}{\partial t} = a^2 \cdot \frac{\partial^2 u}{\partial x^2}$, by the initial condition $u\big|_{t=0} = f(x) = \frac{cx(l - x)}{l^2}$ and by the boundary conditions $u\big|_{x=0} = u\big|_{x=l} = 0$.

6.4.2. Heat equation for the stationary case. The heat equation for the stationary case reduces to *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \tag{1}$$

since $\frac{\partial u}{\partial t} \equiv 0$. The Laplace equation can be written in the form $\Delta u = 0$, where u is the function of the point alone and does not depend on time.

For problems concerned with plane figures the Laplace equation is written in the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{2}$$

Laplace's equation for space is of the same form if u does not depend on the coordinate z, i.e. $u(M)$ retains constant value when the point M moves along the straight line parallel to the Oz axis. The substitution $x = r\cos\Theta$, $y = r\sin\Theta$ can transform equation (2) to polar coordinates:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\Theta + \frac{\partial u}{\partial y}\sin\Theta,$$

$$\frac{\partial^2 u}{\partial r} = \frac{\partial^2 u}{\partial x^2}\cos^2\Theta + 2\frac{\partial^2 u}{\partial x \partial y}\sin\Theta\cos\Theta + \frac{\partial^2 u}{\partial y^2}\sin^2\Theta,$$

$$\frac{\partial u}{\partial \Theta} = -\frac{\partial u}{\partial x}r\sin\Theta + \frac{\partial u}{\partial y}r\cos\Theta,$$

$$\frac{\partial^2 u}{\partial \Theta^2} = \frac{\partial^2 u}{\partial x^2}r^2\sin^2\Theta - 2\frac{\partial^2 u}{\partial x \partial y}r^2\sin\Theta\cos\Theta$$
$$+ \frac{\partial^2 u}{\partial y^2}r^2\cos^2\Theta - \frac{\partial u}{\partial x}r\cos\Theta - \frac{\partial u}{\partial y}r\sin\Theta.$$

Hence

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \Theta^2} = r^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

or

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \Theta^2} = 0.$$

The concept of a harmonic function is related to the Laplace equation. A function is said to be *harmonic* in the domain D if in that domain it is continuous together with its derivatives up to the second order inclusive and satisfies Laplace's equation. Thus, for equation (1) the function $u = 1/r$, where $r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$, is harmonic in any domain, excluding the point $M_0(x_0; y_0; z_0)$. For any plane domain the expression $u = \ln(1/r)$ (or $u = \ln r$) can serve as such a function, that is, this function satisfies equation (2).

The problem of finding the function u which is harmonic in the domain D, continuous in D, including the surface S bounding the domain, and satisfies the boundary condition $u|_{\text{sur}\,S} = f(M)$, where $f(M) = f(x, y, z)$ is a given function, continuous on S, is known as the *Dirichlet problem.*

946. Find the stationary distribution of temperature in a thin bar with an insulated lateral surface if at the end points of the bar $u|_{x=0} = u_0$, $u|_{x=l} = u_l$.

Solution. The Dirichlet problem for a one-dimensional case consists in finding from the Laplace equation $\dfrac{d^2u}{dx^2} = 0$ the function u satisfying the boundary conditions $u|_{x=0} = u_0$, $u|_{x=l} = u_l$. The general solution of the indicated equation is

$u = Ax + B$, and, taking into account the boundary conditions, we obtain

$$u = \frac{u_l - u_0}{l} x + u_0,$$

that is, a stationary distribution of temperature in a thin bar with the insulated lateral surface.

947. Find the stationary distribution of temperature in the space between two cylinders with the common axis Oz under the condition that constant temperature is held on the surfaces of the cylinders.

Hint. Pass to cylindrical coordinates assuming that u does not depend on Θ or z.

6.5. Dirichlet Problem for a Circle

Suppose we are given a circle of radius R with centre at the pole O of the polar system of coordinates. We shall seek the function $u(r, \Theta)$ harmonic in the circle and satisfying on its boundary the condition $u|_{r=R} = f(\Theta)$, where $f(\Theta)$ is the given function continuous on the boundary. The desired function must satisfy in the circle the Laplace equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \Theta^2} = 0. \tag{1}$$

In solving the problem, we shall only apply Fourier's method. We shall assume that the particular solution is sought in the form

$$u = Q(r) \cdot T(\Theta).$$

Then we get

$$r^2 \cdot Q''(r) \cdot T(\Theta) + r \cdot Q'(r) \cdot T(\Theta) + Q(r) \cdot T''(\Theta) = 0.$$

We separate the variables:

$$\frac{T''(\Theta)}{T(\Theta)} = -\frac{r^2 \cdot Q''(r) + r \cdot Q'(r)}{Q(r)}.$$

Equating every part of the equality obtained to the constant $-k^2$, we obtain two ordinary differential equations:

$$T''(\Theta) + k^2 \cdot T(\Theta) = 0,$$

$$r^2 \cdot Q''(r) + r \cdot Q'(r) - k^2 \cdot Q(r) = 0.$$

For $k = 0$ this equation yields

$$T(\Theta) = A + B\Theta, \tag{2}$$

$$Q(r) = C + D\ln r. \tag{3}$$

Now if $k > 0$, then

$$T(\Theta) = A\cos k\Theta = B\sin k\Theta, \tag{4}$$

and the solution of the second equation will be sought in the form $Q(r) = r^m$, which yields $r^2m(m-1)r^{m-2} + rmr^{m-1} - k^2r^m = 0$, or $r^m(m^2 - k^2) = 0$, i.e. $m = \pm k$.

Consequently,

$$Q(r) = Ck^k + Dr^{-k}. \tag{5}$$

Note that as a function of Θ, $u(r, \Theta)$ is a periodic function with period 2π, since for a one-valued function the quantities $u(r, \Theta)$ and $u(r, \Theta + 2\pi)$ coincide. Therefore, it follows from equation (1) that $B = 0$, and in equation (4) k can assume one of the values 1, 2, 3, ... ($k > 0$). Furthermore, in equations (3) and (5) D must be zero since otherwise the function would possess a discontinuity at the point $r = 0$ and would not be harmonic in the circle. Thus we have obtained an infinite number of particular solutions of equation (1), continuous in the circle, which we can write (slightly changing the notation) in the form

$$u_0(r, \Theta) = A_0/2; \quad u_n(r, \Theta) = (A_n\cos n\Theta + B_n\sin n\Theta)r^n \quad (n = 1, 2, \dots).$$

Let us now derive a function

$$u(r, \Theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n\cos n\Theta + B_n\sin n\Theta)r^n,$$

which also serves as a solution of the Laplace equation due to its linearity and homogeneity. It remains to determine the quantities A_0, A_n, B_n so that this function would satisfy the condition $u|_{r=R} = f(\Theta)$, i.e.

$$f(\Theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n\cos n\Theta + B_n\sin n\Theta)R^n.$$

Here we have an expansion of the function $f(\Theta)$ into the Fourier series in the interval $[-\pi, \pi]$. By virtue of the known formulas, we find

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)d\tau, \quad A_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\tau)\cos n\tau d\tau,$$

$$B_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\tau)\sin n\tau d\tau.$$

Thus we have

$$u(r,\Theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cdot \cos n(\tau - \Theta)\right]d\tau.$$

Let us simplify the result obtained. Setting $r/R = \rho$, $\tau - \Theta = t$, we represent the expression in brackets in the form

$$\frac{1}{2} + \sum_{n=1}^{\infty} \rho^n \cos nt = \sum_{n=0}^{\infty} \rho^n \cos nt - \frac{1}{2}.$$

Now we consider the series

$$\sum_{n=0}^{\infty} (\rho e^{it})^n = \sum_{n=0}^{\infty} \rho^n \cos nt + i \sum_{n=0}^{\infty} \rho^n \sin nt.$$

The series converges at $\rho < 1$ and its sum is equal to

$$\frac{1}{1 - \rho e^{it}} = \frac{1}{1 - \rho\cos t - i\rho\sin t} = \frac{1 - \rho\cos t + i\rho\sin t}{1 - 2\rho\cos t + \rho^2}.$$

Consequently,

$$\sum_{n=0}^{\infty} \rho^n \cos nt - \frac{1}{2} = \frac{1 - \rho\cos t}{1 - 2\rho\cos t + \rho^2} - \frac{1}{2}$$

$$= \frac{1 - \rho^2}{2(1 - 2\rho\cos t + \rho^2)},$$

or, returning to the previous notation, we get

$$u(r, \Theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\tau) \frac{R^2 - r^2}{R^2 - 2Rr\cos(\tau - \Theta) + r^2} d\tau.$$

We have obtained a solution of the Dirichlet problem for a circle. The integral appearing on the right-hand side is known as *Poisson's integral.*

948. Find the stationary distribution of temperature on a homogeneous thin circular plate of radius R whose upper part is at the temperature of $1°$ and the lower part, at $0°$.

Solution. If $-\pi < \tau < 0$, then $f(\tau) = 0$, and if $0 < \tau < \pi$, then $f(\tau) = 1$. The temperature distribution is expressed by the integral

$$u(r, \Theta) = \frac{1}{2\pi} \int_{0}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\tau - \Theta) + r^2} d\tau.$$

Suppose the point $(r; \Theta)$ is located in the upper semi-circle, i.e. $0 < \Theta < \pi$ then $\tau - \Theta$ varies from $-\Theta$ to $\pi - \Theta$, and this interval, of length π, does not contain the points $\pm\pi$. Therefore, we introduce the substitution $\tan\dfrac{\tau - \Theta}{2} = t$, whence

we have $\cos(\tau - \Theta) = \dfrac{1-t^2}{1+t^2}$, $d\tau = \dfrac{2dt}{1+t^2}$. Then we get

$$u(r, \Theta) = \frac{1}{\pi}\int\limits_{-\tan(\Theta/2)}^{\cot(\Theta/2)} \frac{R^2 - r^2}{(R-r)^2+(R+r)^2\cdot t^2}\, dt = \frac{1}{\pi} \arctan\left(\frac{R+r}{R-r}\cdot t\right)\Bigg|_{-\tan(\Theta/2)}^{\cot(\Theta/2)}$$

$$= \frac{1}{\pi}\left[\arctan\left(\frac{R+r}{R-r}\cot\frac{\Theta}{2}\right) + \arctan\left(\frac{R+r}{R-r}\tan\frac{\Theta}{2}\right)\right]$$

$$= \frac{1}{\pi} \arctan \frac{\dfrac{R+r}{R-r}\left(\cot\dfrac{\Theta}{2} + \tan\dfrac{\Theta}{2}\right)}{1 - \left(\dfrac{R+r}{R-r}\right)^2} = -\frac{1}{\pi}\cdot \arctan\frac{R^2 - r^2}{2Rr\sin\Theta},$$

or

$$\tan(u\pi) = -\frac{R^2 - r^2}{2Rr\sin\Theta}.$$

The right-hand side being negative, this means that for $0 < \Theta < \pi$ u satisfies the inequalities $1/2 < u < 1$. For that case we get the solution

$$\tan(\pi - u\pi) = \frac{R^2 - r^2}{2Rr\sin\Theta}, \quad \text{or} \quad u = 1 - \frac{1}{\pi}\arctan\frac{R^2 - r^2}{2Rr\sin\Theta} \quad (0 < \Theta < \pi).$$

Now if the point lies in the lower semi-circle, i.e. $\pi < \Theta < 2\pi$, then the interval $(-\Theta, \pi - \Theta)$ of variation of $\tau - \Theta$ contains the point $-\pi$, but does not contain 0, and we can make the substitution $\cot\dfrac{\tau - \Theta}{2} = t$, whence $\cos(\tau - \Theta) = \dfrac{t^2 - 1}{t^2 + 1}$, $d\tau = -\dfrac{2dt}{1 + t^2}$. Then, for these values of Θ we have

$$u(r, \Theta) = -\frac{1}{\pi}\int\limits_{-\cot(\Theta/2)}^{\tan(\Theta/2)} \frac{R^2-r^2}{(R+r)^2+(R-r)^2t^2}\, dt$$

$$= -\frac{1}{\pi}\left[\arctan\left(\frac{R-r}{R+r}\cdot\tan\frac{\Theta}{2}\right) + \arctan\left(\frac{R-r}{R+r}\cdot\cot\frac{\Theta}{2}\right)\right].$$

Performing similar transformations, we find

$$u = -\frac{1}{\pi}\arctan\frac{R^2 - r^2}{2Rr\sin\Theta} \quad (\pi < \Theta < 2\pi).$$

Since the right-hand side is now positive ($\sin\Theta < 0$), we have $0 < u < 1/2$.

949. Find the solution of the Laplace equation for the interior of the annulus $1 \leqslant r \leqslant 2$, satisfying the boundary conditions $u\big|_{r=1} = 0$, $u\big|_{r=2} = y$.

Hint. Introduce polar coordinates.

Chapter 7

Elements of the Theory of Functions of a Complex Variable

7.1. Functions of a Complex Variable

Assume that the complex variable $z = x + yi$ takes on various values from a certain set Z. If every value of z from Z can be put into correspondence with one or several values of another complex variable $w = u + vi$, then the complex variable w is called a *function of* z in the domain Z and is written as $w = f(z)$.

The function $w = f(z)$ is called *one-valued* if every value of z from the set Z can be put into correspondence with only one value of w. Now if there are values of z each of which can be put into correspondence with several values of w, then the function $w = f(z)$ is called *multiple-valued*.

If $w = u + vi$ is a function of $z = x + yi$, then each of the variables u and v is a function of x and y, that is, $u = u(x, y)$, $v = v(x, y)$. Conversely, if $w = u(x, y) + v(x, y)i$, where $u(x, y)$ and $v(x, y)$ are real functions of x and y, then w can be regarded as a function of the complex variable $z = x + yi$. Indeed, to every complex number $z = x + yi$ there corresponds a definite pair of real numbers $(x; y)$, and this pair of numbers is associated with one or several values of w.

The one-valued function $w = f(z)$ is said to have a definite *limit* C as $z \to c$ (c and C being complex numbers), if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that the inequality $|z - c| < \delta$ yields the inequality $|f(z) - C| < \varepsilon$. In this case we write $\lim\limits_{z \to c} f(z) = C$.

The function $w = f(z)$ is said to be *continuous at the point* z_0 if $\lim\limits_{z \to z_0} f(z) = f(z_0)$. A function continuous at every point of a certain domain D is said to be *continuous in that domain*.

Let us consider a domain D bounded by a closed nonself-intersecting curve Γ This domain is called *simple connected* (Fig. 60).

If the domain D is bounded by two non-intersecting and nonself-intersecting curves Γ_1 and Γ_2, it is called *doubly connected* (Fig. 61).

Assume that Γ_1 is the outer curve and Γ_2 is the inner curve. The domain is also doubly connected in the case when Γ_2 degenerates into a point or into an arc of a continuous curve. Trebly connected and four-connected, and so on, domains can be defined in a similar way. Figure 62 shows a four-connected domain.

The functions e^z, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$ of a complex variable are defined as the sums of the following series convergent throughout the plane of the complex variable:

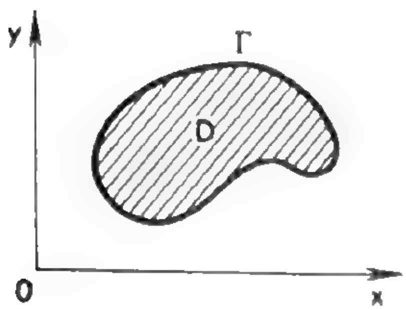


Fig. 60

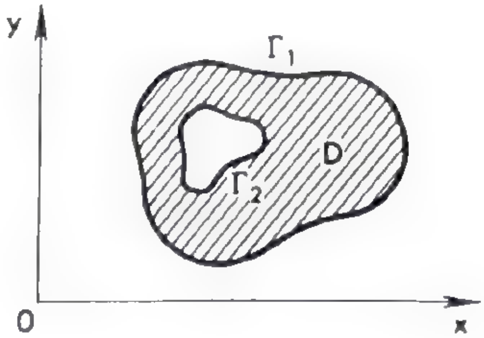


Fig. 61

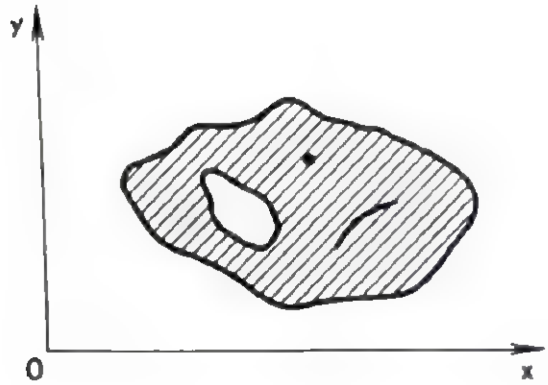


Fig. 62

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots;$$

$$\sin z = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots;$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \ldots;$$

$$\sinh z = \frac{e^z - e^{-z}}{2} = \frac{z}{1!} + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots;$$

$$\cosh z = \frac{e^z + e^{-z}}{2} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots.$$

Euler's formula holds true for functions of a complex variable:

$$e^{zi} = \cos z + i \sin z.$$

It follows from this formula that

$$\sinh zi = i \sin z, \quad \cosh zi = \cos z.$$

The formulas

$$e^{z_1}\cdot e^{z_2}=e^{z_1+z_2},\ e^{z_1}/e^{z_2}=e^{z_1-z_2},$$
$$\sin(z_1\pm z_2)=\sin z_1\cos z_2\pm\cos z_1\sin z_2,$$
$$\cos(z_1\pm z_2)=\cos z_1\cos z_2\pm\sin z_1\sin z_2$$

known from elementary mathematics are also valid for complex values of the arguments z_1 and z_2.

The functions $z^{1/n}$ $(n\in N)$, $\operatorname{Ln} z$, $\arcsin z$, $\arccos z$, $\arctan z$ are said to be the inverse functions of z^n, e^z, $\sin z$, $\cos z$, $\tan z=\sin z/\cos z$ respectively. In this case the functions $z^{1/n}$, $\operatorname{Ln} z$, $\arcsin z$, $\arccos z$, $\arctan z$ are multiple-valued.

It may be shown that

$$\operatorname{Ln} z=\ln\rho+(\varphi+2k\pi)i\quad(k\in Z),$$

where $\rho=|z|$ and $\varphi=\arg z$.

950. Given the function $w=z^2+z$. Find the values of the function for (1) $z=1+$ $+\ i$; (2) $z=2-i$; (3) $z=i$; (4) $z=-1$.

Solution. We have

(1) $w=(1+i)^2+1+i=1+2i-1+1+\ i=1+3i$;

(2) $w=(2-i)^2+2-i=4-4i-1+2-i=5(1-i)$;

(3) $w=i^2+i=-1+i$;

(4) $w=1-1=0$.

951. Given the function $f(z)=x^2+y^2i$, where $z=x+yi$. Find: (1) $f(1+2i)$; (2) $f(2-3i)$; (3) $f(0)$; (4) $f(-i)$.

Solution. We have

(1) $x=1,\ y=2,\ f(1+2i)=1+4i$;

(2) $x=2,\ y=-3,\ f(2-3i)=4+9i$;

(3) $x=0,\ y=0,\ f(0)=0+0\cdot i=0$;

(4) $x=0,\ y=-1,\ f(-i)=i$.

952. Show that the function $w=|z|$ is continuous for any value of z.

Solution. Since the difference between two sides of a triangle does not exceed the third side, we have

$$||z|-|z_0||\leqslant|z-z_0|$$

(Fig. 63). Suppose $0<\delta<\varepsilon$. Then the inequality $|z-z_0|<\delta$ yields the inequality $||z|-|z_0||<\varepsilon$, i.e. $\lim\limits_{z\to z_0}|z|=|z_0|$. Thus, $|z|$ is a continuous function.

953. Show that $w=z^2$ is a continuous function for any value of z.

Solution. We have $z^2-z_0^2=(z-z_0)(z+z_0)$. If $z\to z_0$, then there is a positive number M such that the inequalities $|z|<M$, $|z_0|<M$ hold true. But

$$|z^2-z_0^2|=|z-z_0|\cdot|z+z_0|<|z-z_0|\cdot(|z|+|z_0|)<2M|z-z_0|.$$

Let us take $\delta<\varepsilon/2M$. It follows from the inequality $|z-z_0|<\delta$ that

$$|z^2-z_0^2|<2M\delta<2M\cdot\varepsilon/2M,\quad\text{i.e.}\quad|z^2-z_0^2|<\varepsilon.$$

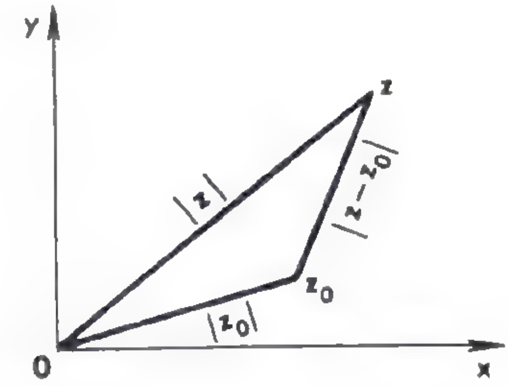


Fig. 63

Thus, $\lim\limits_{z\to z_0} z^2 = z_0^2$, that is, $w = z^2$ is a continuous function.

954. Find $\ln(\sqrt{3} + i)$.

Solution. We have $z = \sqrt{3} + i$, $\rho = |z| = 2$, $\varphi = \arg z = \arctan(1/\sqrt{3}) = \pi/6$, i.e. $\ln(\sqrt{3} + i) = \ln 2 + (\pi/6 + 2k\pi)i$, $k \in Z$.

955. Calculate $\cos(i/2)$ with an accuracy to within 0.0001.

Solution. Since

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots;$$

we find

$$\cos\frac{i}{2} = 1 + \frac{1}{2!2^2} + \frac{1}{4!2^4} + \frac{1}{6!2^6} + \ldots = 1.1276.$$

956. Given the function $w = e^z$. Find its value for (1) $z = \pi i/2$; (2) $z = \pi(1 - i)$; (3) $z = 1 + (\pi/2 + 2\pi k)i$, where $k \in Z$.

957. Given the function $f(z) = 1/(x - yi)$, where $z = x + yi$. Find $f(1+i)$, $f(i)$, $f(3 - 2i)$.

958. Show that $w = 2z^3$ is a continuous function.

959. Find $\ln(1 - i)$.

960. Prove the validity of the equality $\sin i \cdot \cosh 1 = i \cos i \cdot \sinh 1$.

961. Solve the equation $\cos z = 2$.

962. Find $\arcsin i$.

963. Calculate $\sin i$ by calculating the real and the imaginary part with an accuracy to within 0.0001.

964. What is $\sin(\pi/6 + i)$ equal to? Calculate the real and the imaginary part to within 0.001.

965. Given the function $f(z) = e^{e^z}$. Find its values at the points: (1) $z = i$; (2) $z = 1 + \pi i/2$.

7.2. The Derivative of a Function of a Complex Variable

The *derivative* of the one-valued function of the complex variable $w = f(z)$ is the limit of the ratio $\dfrac{\Delta w}{\Delta z} = \dfrac{f(z + \Delta z) - f(z)}{\Delta z}$, if Δz tends to zero in an arbitrary way.

Thus,

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

The function possessing a derivative at the given value of z is called *differentiable* (or *monogenic*) for the given value of z. If the function $w = f(z)$ is one-valued and possesses a finite derivative at every point of the domain D, then this function is called *analytic* in the domain D.

If the function $w = f(z) = u(x, y) + iv(x, y)$ is differentiable at the point $z = x + yi$, then there exist at that point the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$, the derivatives being connected by the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

called *Cauchy-Riemann conditions.*

The Cauchy-Riemann conditions are the necessary conditions for the differentiability of the function $w = f(z)$ at the point $z = x + yi$.

Conversely, if the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ are continuous at the point $z = x + yi$ and the Cauchy-Riemann conditions $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$, are satisfied, then the function $w = f(z)$ is differentiable at the point $z = x + yi$.

The derivative of the function $f(z)$ is expressed in terms of the partial derivatives of the functions $u(x, y)$ and $v(x, y)$ by the formulas

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}.$$

The derivatives of the elementary functions z^n, e^z, $\cos z$, $\sin z$, $\ln z$, $\arcsin z$, $\arccos z$, $\arctan z$, $\sinh z$, $\cosh z$ can be found from the same formulas as in the case of the real argument:

$$(z^n)' = n \cdot z^{n-1}, \qquad (\arcsin z)' = 1/\sqrt{1 - z^2},$$

$$(e^z)' = e^z, \qquad (\arccos z)' = -1\sqrt{1 - z},$$

$$(\cos z)' = -\sin z, \qquad (\arctan z)' = 1/(1 + z^2),$$

$$(\sin z)' = \cos z, \qquad (\sinh z)' = \cosh z,$$

$$(\ln z)' = 1/z \qquad (\cosh z)' = \sinh z.$$

966. Is the function $f(z) = y + xi$ differentiable?

Solution. We find

$$u = y, \quad v = x, \quad \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 0.$$

One of the Cauchy-Riemann conditions is not satisfied. Thus we see that the given function is not differentiable.

967. Is the function $f(z) = (x^2 - y^2) + 2xyi$ differentiable?

Solution. We have

$$u = x^2 - y^2, \quad v = 2xy; \quad \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y,$$

$$\frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x; \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

The Cauchy-Riemann conditions are satisfied. Consequently, the function is differentiable. Since $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}$, we have

$$f'(z) = 2x + 2yi = 2(x + yi) = 2z.$$

Another procedure can be used to find the derivative $f'(z)$:

$$f(z) = (x + yi)^2 = z^2, \quad f'(z) = 2z.$$

968. Is the function $f(z) = e^x \cos y + i \cdot e^x \sin y$ differentiable?

Solution. We find

$$u = e^x \cos y, \quad v = e^x \sin y;$$

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y,$$

$$\frac{\partial v}{\partial x} = e^x \sin y, \quad \frac{\partial v}{\partial y} = e^x \cos y; \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

The Cauchy-Riemann conditions are satisfied. Next we have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y)$$

$$= e^x \cdot e^{yi} = e^{x+yi} = e^z,$$

or

$$f(z) = e^x(\cos y + i \sin y) = e^x \cdot e^{yi} = e^{x+yi} = e^z, \quad f'(z) = e^z.$$

969. Given the real part $u(x, y) = x^2 - y^2 - x$ of the differentiable function $f(z)$, where $z = x + yi$. Find the function $f(z)$.

Solution. We find $\frac{\partial u}{\partial x} = 2x - 1$. Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (by virtue of one of Cauchy-Riemann conditions), we have $\frac{\partial v}{\partial y} = 2x - 1$. Integration yields

$$v(x, y) = 2xy - y + \varphi(x),$$

where $\varphi(x)$ is an arbitrary function.

Let us apply another Cauchy-Riemann condition: $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Since $\frac{\partial v}{\partial x} = 2y + \varphi(x)$, it follows that $\frac{\partial u}{\partial y} = -2y - \varphi'(x)$.

But we find from the hypothesis that $\frac{\partial u}{\partial y} = -2y$. Consequently,

$$-2y - \varphi'(x) = -2y, \quad \varphi'(x) = 0, \quad \varphi(x) = C,$$

whence

$$f(z) = x^2 - y^2 - x + i(2xy - y + C) = x^2 - y^2 + 2xyi - (x + yi) + Ci$$

or

$$f(z) = (x + yi)^2 - (x + yi) + Ci, \quad \text{i.e.} \quad f(z) = z^2 - z + C_1.$$

970. Given the imaginary part $v(x, y) = x + y$ of the differentiable function $f(z)$. Find the function.

Solution. We have $\frac{\partial v}{\partial y} = 1$; consequently, $\frac{\partial u}{\partial x} = 1$ (in accordance with Cauchy-Riemann condition). Hence

$$u = x + \varphi(y), \quad \frac{\partial u}{\partial y} = \varphi'(y), \quad \frac{\partial v}{\partial x} = 1.$$

But $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Consequently, $\varphi'(y) = -1$. Integration yields $\varphi(y) = -y + C$. Hence $u = x - y + C$. Thus we have

$$f(z) = x - y + C + i(x + y) = (1 + i)(x + yi) + C, \quad \text{i.e.} \quad f(z) = (1 + i)z + C.$$

971. Is the function $f(z) = (x^2 + y^2) - 2xyi$ differentiable?

972. Show that the function $f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3)$ is differentiable and find its derivative.

973. Is the function $f(z) = \sin x \cosh y + i \cos x \sinh y$ differentiable? If it is, find its derivative.

974. Determine the real functions $\varphi(y)$ and $\psi(x)$ so that the function $f(z) = \varphi(y) + i\psi(x)$ be differentiable.

975. At what value of λ is the function $f(z) = y + \lambda xi$ differentiable?

976. At what value of a is the function $f(z) = a\bar{z}$ (where $\bar{z} = x - yi$) differentiable?

977. Given the real part $u = 2^x \cos(y \ln 2)$ of the differentiable function $f(z)$. Find the function.

978. Given the imaginary part $v = \sin x \sinh y$ of the differentiable function $f(z)$. Find the function.

7.3. The Notion of Conformal Mapping

Suppose we are given the funcion $w = f(z)$ analytic in the domain D. Let us assign to z a definite value $z = x + yi$ which is associated with a definite value $w = u + vi$. Thus, every point $(x; y)$ on the plane xOy is associated with a definite point $(u; v)$ on the plane uOv.

If the point $(x; y)$ on the plane xOy describes some curve Γ lying in the domain D, then the point $(u; v)$ on the plane uOv will describe the curve Γ'. This curve Γ' is called the *image* of Γ under the mapping onto the plane uOv by means of the analytic function $w = f(z)$.

Let us take a point $z_0 = x_0 + y_0 i$ on the curve Γ. This point is associated with the point $w_0 = u_0 + v_0 i$ on the curve Γ . We draw a tangent line L to the curve Γ at the point $(x_0; y_0)$ and a tangent line L' to the curve Γ' at the point $(u_0; v_0)$. Assume that α is the angle through which the curve L should be rotated for its direction to coincide with the direction of the straight line L' (the angle between the original and mapped directions). It is proved in the theory of analytic functions that $\alpha = \arg f'(z_0)$ if $f(z_0) \neq 0$.

Let us consider another curve γ which also passes through the point $(x_0; y_0)$, and its image, the curve γ', passing through the point $(u_0; v_0)$. Suppose that l is a tangent to the curve γ at the point $(x_0; y_0)$ and l' is a tangent to the curve γ' at the point $(u_0; v_0)$.

For the direction of the line l to coincide with that of the line l', the straight line l must be rotated through the angle α in this case as well since the rotation angle is equal to $f'(z_0)$ (the value of the derivative does not depend on the choice of the curve passing through the point $(x_0; y_0)$; see Fig. 64).

If φ and ψ are the angles which the tangents L and l form with the Ox axis, and φ' and ψ' are the angles which the tangents L' and l' form with the Ou axis, then $\varphi' - \varphi = \alpha$, $\psi' - \psi = \alpha$ and $\varphi' - \varphi = \psi' - \psi$. Consequently, $\psi - \varphi = \psi' - \varphi'$. But $\psi - \varphi$ is the angle between the tangents L and l, and $\psi' - \varphi'$ is the angle between the tangents L' and l'. Thus, two arbitrary curves intersecting at the point $(x_0; y_0)$ are mapped into two corresponding curves intersecting at the point $(u_0; v_0)$ so that the angle β between the tangents to the given and mapped curves is the same.

It is easy to prove that the absolute value of the derivative at the point $(x_0; y_0)$, i.e $|f'(z_0)|$, expresses the limit of the ratio the distances between the mapped points $w_0 + \Delta w_0$ and w_0 and the original points $z_0 + \Delta z_0$ and z_0 (Fig. 65).

Having considered another curve and its image, we infer that $|f'(z_0)|$ expresses the limit of the ratio of the distances between the mapped points $w_0 + \Delta' w_0$ and w_0 and the original points $z_0 + \Delta' z_0$ and z_0.

Thus, $|f'(z_0)|$ is the value of the distortion in scale at the point z_0 upon the mapping with the aid of the function $w = f(z)$.

It follows that if we map an infinitesimal triangle lying in the plane xOy onto the plane uOv by means of the function $w = f(z)$, we obtain an infinitesimal curvilinear triangle similar to the original triangle because of the equality of the respective angles and the proportionality of the analogous sides (in the limit).

The mapping with the aid of the analytic function $w = f(z)$ is known as the *conformal mapping*.

979. By means of the function $w = 1/z$ map the points (1) $(1; 1)$; (2) $(0; -2)$; (3) $(2; 0)$ onto the plane uOv.

Solution. (1) The point $(1; 1)$ is associated with the value $z = 1 + i$; consequently,

$$w = \frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i.$$

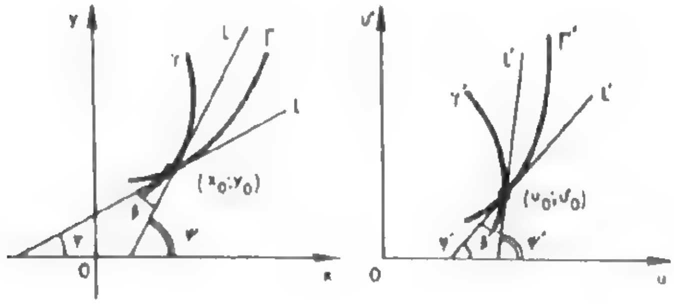


Fig. 64

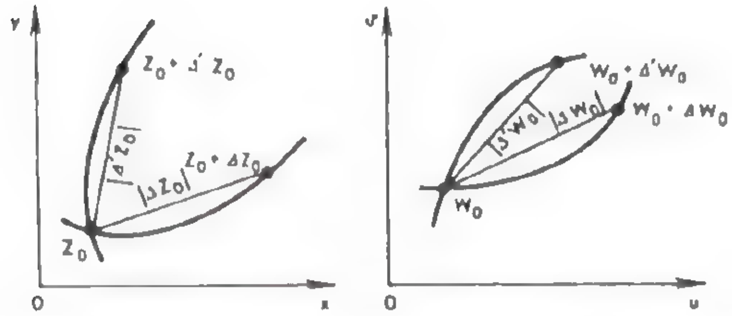


Fig. 65

On the plane uOv we obtain a point $(1/2;\ -1/2)$;
(2) $z = -2i$, $w = 1/(-2i) = (1/2)i$; we get a point $(0;\ 1/2)$;
(3) $z = 2$, $w = 1/2$; we get a point $(1/2;\ 0)$.

980. By the function $w = z^3$ map the curve $y = x$ onto the plane uOv.

Solution. We have

$$w = (x + yi)^3 = x^3 + 3x^2yi - 3xy^2 - y^3i = (x^3 - 3xy^2) + (3x^2y - y^3)i.$$

Thus we have

$$u = x^3 - 3xy^2, \quad v = 3x^2y - y^3.$$

We eliminate x and y from the equations obtained and from the equation $y = x$:

$$u = -2x^3, \quad v = 2x^3, \quad \text{i.e.} \quad v = -u.$$

Thus, the image of the bisector of the 1st and 3rd quadrants of the system xOy is the bisector of the 2nd and 4th quadrants of the system uOv.

981. Assume that $w = z^2$ and z describes a square specified by the inequalities $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. What domain is described by w?

Solution. We have

$$w = (x + yi)^2 = x^2 - y^2 + 2xyi, \quad u = x^2 - y^2, \quad v = 2xy.$$

Let us find the images of the vertices of the square (Fig. 66). If $x = 0, y = 0$, then $u = 0, v = 0$; if $x = 0, y = 1$, then $u = -1, v = 0$; if $x = 1, y = 0$, then $u = 1, v = 0$; if $x = 1, y = 1$, then $u = 0, v = 2$.

Let us find the images of the sides of the square.

OB: $y = 0$, $u = x^2$, $v = 0$, i.e. $v = 0$, $u \geqslant 0$ is the intercept OB_1 on the abscissa axis Ou.

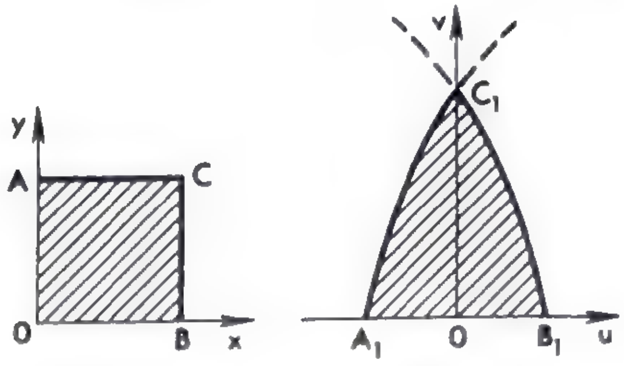


Fig. 66

OA: $x = 0$, $u = -y^2$, $v = 0$, i.e. $v = 0$, $u \leqslant 0$ is the intercept OA_1 on the abscissa axis Ou.

AC: $y = 1$, $u = x^2 - 1$, $v = 2x$; eliminating x, we get $u = v^2/4 - 1$ which is the arc of the parabola connecting the points $A_1(-1; 0)$ and $C_1(0; 2)$.

BC: $x = 1$, $u = 1 - y^2$, $v = 2y$; eliminating y, we get $u = 1 - v^2/4$ which is the arc of the parabola connecting the points $B_1(1; 0)$ and $C_1(0; 2)$.

Thus, the image of the square is a curvilinear triangle bounded by the curves $v = 0$, $u = v^2/4 - 1$, $u = 1 - v^2/4$ and lying in the upper half-plane.

982. Find the image of the circle $x^2 + y^2 = 1$ onto the plane uOv under the mapping $w = 2z + 1$.

Solution. We have

$$w = 2(x + yi) + 1 = (2x + 1) + 2yi,$$

whence $u = 2x + 1$, $v = 2y$. These equations yield $x = (u - 1)/2$, $y = v/2$; substituting these expressions into the equation of the circle, we get

$$(u - 1)^2 + v^2 = 4.$$

Thus, the desired image is a circle whose radius is 2 and whose centre is the point $0(1; 0)$.

983. Find the rotation angle and the extension coefficient at the point $z = -2i$ under the mapping $w = \dfrac{(z + i)^2}{z - i}$.

Solution. Since the rotation angle and the extension coefficient can be found from the derivative at the given point, we differentiate the given function:

$$w' = \frac{2(z + i)(z - i) - (z + i)^2}{(z - i)^2} = \frac{4 + (z - i)^2}{(z - i)^2}.$$

At the point $z_0 = -2i$ we have

$$\alpha = \arg\left[\frac{4 + (z - i)^2}{(z - i)^2}\right] = \arg\frac{5}{9} = 0,$$

$$k = \left|\frac{4 + (z - i)^2}{(z - i)^2}\right| = \frac{5}{9} < 1 \text{ (compression)}.$$

984. At what points of the plane is the rotation angle under the mapping $w = \frac{1+iz}{1-iz}$ equal to zero? At what points is the coefficient of extension equal to 1?

Solution. The hypothesis presupposes that we must first seek the points where the given mapping is conformal since only then can we speak of the rotation angle and the extension coefficient.

We find

$$w' = \frac{i(1-iz) + i(1+iz)}{(1-iz)^2} = \frac{2i}{(1-iz)^2} = -\frac{2i}{(z+i)^2}.$$

Since $w'(z)$ is not equal to zero at any value of z, the given mapping is conformal throughout the plane with the deleted point $z = -i$. The rotation angle α of that mapping at the point z is

$$\alpha = \arg w'(z) = \arg\left[\frac{-2i}{(z+i)^2}\right] = \arg \frac{-4x(y+1) - 2i\,[x^2 - (y+1)^2]}{[x^2+(y+1)^2]^2}.$$

The number $w'(z)$ is real if $\operatorname{Im} w'(z) = 0$ and positive if, in addition, $\operatorname{Re} w'(z) > 0$:

$$\begin{cases}\operatorname{Im} w'(z) = 0\\ \operatorname{Re} w'(z) > 0\end{cases} \leftrightarrow \begin{cases}(y+1)^2 = x^2\\ x(y+1) < 0\end{cases} \leftrightarrow y = -x-1 \quad (x \neq 0).$$

Thus, the rotation angle under the given mapping is equal to zero at the points of the straight line $y = -x - 1$ (with the deleted point $z = -i$).

The coefficient of extension at the point z is equal to $k = |w'(z)|$ and must be equal to unity by the hypothesis; consequently,

$$|w'(z)| = 1 \leftrightarrow \left|\frac{-2i}{(z+i)^2}\right| = 1 \leftrightarrow |(z+i)^2| = 2 \leftrightarrow |z+i| = \sqrt{2}$$

is the equation of a circle whose centre is at the point $z = -i$ and whose radius is equal to $\sqrt{2}$.

985. By the function (a) $w = z^2$ map the straight lines $x = 2, y = 1$ onto the plane uOv.

(b) $w = -z^2$ map the straight line $x + y = 1$ onto the plane uOv.

(c) $w = iz + 1$ find the mapping of the coordinate axes onto the plane uOv.

986. Elucidate the meaning of the mapping onto the plane uOv by the function $w = e^{\varphi i}z$, where φ is a constant quantity.

987. Map the parabola $y = x^2$ onto the plane uOv by the function $w = z^2$.

988. Show that the angle between the straight lines $y = 1$ and $y = x - 1$ does not change under the mapping $w = (1 + i)z + (1 - i)$.

989. Find the rotation angle and the coefficient of extension at the point z_0 under the mapping $w = f(z)$:

(a) $w = z^3, z_0 = 1 - i$;

(b) $w = \frac{1}{z}, \quad z_0 = 2i$;

(c) $w = u = iv$, where $u = e^y \cos x$, $v = -e^y \sin x$; $z_0 = i$.

990. Find the points of the plane at which the extension coefficient is unity under the following mappings:
(a) $w = z^2$;
(b) $w = z^2 - 2z$.

991. Find the points of the plane at which the rotation angle is zero under the following mappings:
(a) $w = z^3$;
(b) $w = iz^2$.

7.4. Integral with Respect to a Complex Variable

As is known, the curve Γ is called *smooth* if it possesses a continuously varying tangent.

A curve is called *piecewise smooth* if it consists of a finite number of smooth arcs.

Given the function of the complex variable $w = f(z)$ continuous in a certain domain D. Suppose Γ is an arbitrary smooth curve lying in that domain. Let us consider the arc of the curve starting at the point z_0 and terminating at the point z. We partition the arc into n parts by the arbitrary points $z_0, z_1, z_2, \ldots, z_{n-1}, z_n = z$ lying consecutively along the line Γ.

We compose the sum

$$S_n = f(z_0)\,\Delta z_0 + f(z_1)\Delta z_1 + \ldots + f(z_{n-1})\Delta z_{n-1},$$

where $\Delta z_k = z_{k+1} - z_k$ $(k = 0, 1, \ldots, n-1)$. Suppose λ is the greatest of the values $|\Delta z_k|$. If $\lambda \to 0$, then $n \to \infty$ and the sum S_n tends to a definite limit. This limit is called the *integral* of the function $f(z)$ over the arc of the curve Γ contained between the points z_0 and z, that is,

$$\int_\Gamma f(z)\,dz = \lim_{\lambda \to 0} [f(z_0)\Delta z_0 + f(z_1)\Delta z_1 + \ldots + f(z_{n-1})\Delta z_{n-1}].$$

If $f(z) = u(x, y) + iv(x, y)$, then the integral $\int_\Gamma f(z)\,dz$ reduces to two line integrals of the real functions by the formula

$$\int_\Gamma f(z)\,dz = \int_\Gamma u(x, y)\,dx - v(x, y)\,dy + i\int_\Gamma v(x, y)\,dx + u(x, y)\,dy.$$

Assume that Γ is a piecewise smooth curve consisting of the smooth parts $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$; then the integral over this curve can be defined by means of the equation

$$\int_\Gamma f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz + \ldots + \int_{\Gamma_m} f(z)\,dz.$$

If $f(z)$ is an analytic function in the simply connected domain D, then the value of the integral $\int_\Gamma f(z)\,dz$, taken along the arbitrary piecewise smooth curve Γ, belonging to the domain D, is independent of the curve Γ and is defined only by the positions of the initial and the terminal point of that curve.

For every analytic function $f(z)$ in some simply connected domain D the integral $\int_\gamma f(z)\,dz$ taken over any closed piecewise smooth contour γ, lying in the interior of the domain D, is equal to zero (**Cauchy's theorem**).

Let us consider the expression $F(z) = \int_{z_0}^{z} f(t)\,dt$.

Here the path of integration is an arbitrary piecewise smooth curve Γ lying in the domain D and connecting the points z_0 and z. The function $f(t)$ is assumed to be analytic in the domain D. It is easy to show that $F'(z) = f(z)$. The function $F(z)$ whose derivative is $f(z)$ is called an *antiderivative* (*primitive*) of the function $f(z)$. If one of the antiderivatives of $F(z)$ is known, then all the other antiderivatives enter into the expression $F(z) + C$, where C is an arbitrary constant. The expression $F(z) + C$ is called an *indefinite integral* of the function $f(z)$. As in the case of real functions, there holds here the equality

$$\int_{z_0}^{z} f(t)\,dt = \Phi(z) - \Phi(z_0)$$

(Newton-Leibniz formula), where $\Phi(z)$ is some antiderivative of the function $f(z)$.

The antiderivative of the analytic function $f(z)$ can be found by means of oridinary integration formulas.

Let us consider $n + 1$ closed piecewise smooth curves $\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_n$, such that every curve $\gamma_1, \gamma_2, \ldots, \gamma_n$ lies in the exterior of all the others and they all lie in the interior of γ_0. The set of points lying simultaneously in the interior of γ_0 and in the exterior of $\gamma_1, \gamma_2, \ldots, \gamma_n$ is an $(n + 1)$-connected domain D.

Assume that $f(z)$ is an analytic function in the domain D (the values belonging to the contours $\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_n$ inclusive). In that case there holds the equality

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \ldots + \int_{\gamma_n} f(z)\,dz.$$

992. Calculate the integral $\int_{AB} f(z)\,dz$, where $f(z) = (y + 1) - xi$, AB is a line segment connecting the points $z_A = 1$ and $z_B = -i$

Solution. We have $u = y + 1$, $v = -x$. Hence

$$\int_{AB} f(z)\,dz = \int_{AB} (y + 1)\,dx + x\,dy - i\int_{AB} x\,dx - (y + 1)\,dy$$

$$= (y + 1)\cdot x\Big|_{x=1,\,y=0}^{x=0,\,y=-1} - i\cdot\frac{x^2}{2}\Big|_1^0 + i\cdot\frac{(y+1)^2}{2}\Big|_0^{-1} = -1 + \frac{1}{2}i - \frac{1}{2}i = -1.$$

There is another way of solving the problem. It is easy to see that $f(z) = 1 - iz$ and

$$\int_{AB} f(z)\,dz = \int_1^{-i} (1 - iz)\,dz = \left.\frac{(1 - iz)^2}{-2i}\right|_1^{-i}$$
$$= \frac{(1 + i^2)^2}{-2i} + \frac{(1 - i)^2}{2i} = \frac{1 - 2i + i^2}{2i} = -1.$$

993. Calculate the integral $\int_{AB} f(z)\,dz$, where $f(z) = x^2 + y^2 i$, AB is a line segment connecting the points $A(1 + i)$ and $B(2 + 3i)$.

Solution. We have $u = x^2$, $v = y^2$; this means that

$$\int_{AB} f(z)\,dx = \int_{AB} x^2dx - y^2dy + i\int_{AB} y^2dx + x^2\,dy.$$

Since the expression $x^2dx - y^2dy$ is a total differential, the first integral on the right-hand side of the equality can be calculated as a definite integral:

$$\int_{AB} x^2dx - y^2dy = \int_1^2 x^2dx - \int_1^3 y^2dy = \left.\frac{x^3}{3}\right|_1^2 - \left.\frac{y^3}{3}\right|_1^3 = \frac{7}{3} - \frac{26}{3} = -\frac{19}{3}.$$

To calculate the second integral, it is necessary to derive the equation of the straight line AB:

$$\frac{y - 1}{3 - 1} = \frac{x - 1}{2 - 1}, \text{ i.e. } y = 2x - 1.$$

From this we get $dy = 2\,dx$ and

$$\int_{AB} y^2dx + x^2\,dy \int_1^2 [(2x - 1)^2 + 2x^2]\,dx = \int_1^2 (6x^2 - 4x + 1)\,dx$$
$$= (2x^3 - 2x^2 + x)\Big|_1^2 = 10 - 1 = 9.$$

Thus, $\int_{AB} f(z)\,dz = -\frac{19}{3} + 9i$.

994. Calculate the integral $\int_i^{1+i} z\,dz$.

Solution. The integrand is an analytic function. Applying the Newton-Leibniz formula, we find

$$\int_i^{1+i} z\,dz = \left.\frac{z^2}{2}\right|_i^{1+i} = \frac{1}{2}\,[(1 + i)^2 - i^2] = \frac{1}{2}\,(1 + 2i - 1 + 1) = \frac{1}{2} + i.$$

995. Calculate $\int\limits_{\gamma} \bar{z}\,dz$, where γ is a closed contour $x = \cos t, y = \sin t$.

Solution. Since $\bar{z} = x - yi$, $dz = dx + i\,dy$, we have

$$\int\limits_{\gamma} \bar{z}\,dz = \int\limits_{\gamma} x\,dx + y\,dy + i\int\limits_{\gamma} x\,dy - y\,dx.$$

The first integral on the right-hand side is zero as an integral of a total differential over a closed contour.

When calculating the second integral, we must take into account that $dx = -\sin t\,dt$, $dy = \cos t\,dt$. Hence $x\,dy - y\,dx = \cos^2 t\,dt + \sin^2 t\,dt$, and we finally get

$$\int\limits_{\gamma} \bar{z}\,dz = i\int\limits_{0}^{2\pi} dt = 2\pi i.$$

996. Calculate $\int\limits_{\gamma} \frac{dz}{z-4}$, where γ is an ellipse $x = 3\cos t$, $y = 2\sin t$.

Solution. The integrand is an analytic function in the domain bounded by that ellipse, therefore, $\int\limits_{\gamma} \frac{dz}{z-4} = 0$.

997. Calculate $\int\limits_{\gamma} \frac{dt}{z-(1+i)}$, where γ is a circle $|z - (i + 1)| = 1$.

Solution. The equation of the circle can be written in the form $(x-1)^2 + (y-1)^2 = 1$, or $x = 1 + \cos t$, $y = 1 + \sin t$, or else $z = 1 + i + e^{it}$. In the domain bounded by the circle γ the integrand is not an analytic function since at the point $z = 1 + i$ serving as the centre of the circle the function turns into infinity.

Since $dz = i \times e^{it}dt$, it follows that

$$\int\limits_{\gamma} \frac{dz}{z-(1+i)} = \int\limits_{0}^{2\pi} \frac{ie^{it}dt}{e^{it}} = i\int\limits_{0}^{2\pi} dt = 2\pi i.$$

998. Calculate $\int\limits_{\gamma} \frac{2z-1-i}{(z-1)(z-i)}\,dz$, where γ is a circle $|z| = 2$.

Solution. The integrand posseses discontinuities only at the points $z = 1$ and $z = i$. The function $f(z)$ is analytic in the the triply connected domain which is a circle with the boundary γ from which two circles $|z - 1| < r$, $|z - i| < r$ are cut, where $r > 0$ is a sufficiently small quantity (Fig. 67). Consequently,

$$\int\limits_{\gamma} f(z)\,dz = \int\limits_{\gamma_1} f(z)\,dx + \int\limits_{\gamma_2} f(z)\,dz,$$

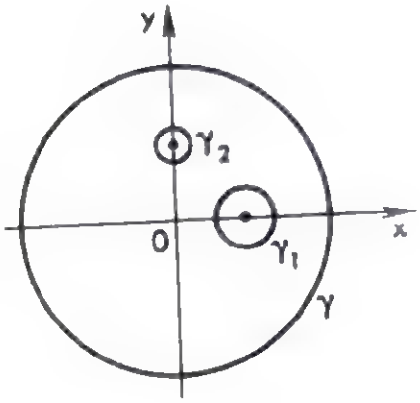


Fig. 67

where γ_1 is a circle $|z - 1| = r$ and γ_2 is a circle $|z - i| = r$.

Since

$$f(z) = \frac{z - 1 + z - i}{(z - 1)(z - i)} = \frac{1}{z - i} + \frac{1}{z - 1},$$

it follows that

$$\int f(z)dz = \int_{\gamma_1} \frac{dz}{z - i} + \int_{\gamma_1} \frac{dz}{z - 1} + \int_{\gamma_2} \frac{dz}{z - i} + \int_{\gamma_2} \frac{dz}{z - 1}.$$

The first and the fourth summands on the right-hand side are zero, since the integrands are analytic functions in the respective domains.

Consequently,

$$\int_{\gamma} f(z)\, dz = \int_{\gamma_1} \frac{dz}{z - 1} + \int_{\gamma_2} \frac{dz}{z - i}.$$

The circle γ_1 is specified by the equation $z = 1 + re^{i\varphi}$, and γ_2 by the equation $z = i + re^{i\varphi}$. Hence

$$\int_{\gamma} f(z)\, dz = \int_0^{2\pi} \frac{ire^{i\varphi}d\varphi}{re^{i\varphi}} + \int_0^{2\pi} \frac{ire^{i\varphi}d\varphi}{re^{i\varphi}} = 4\pi i.$$

999. Calculate the integral (a) $\int\limits_{\Gamma} f(z)\, dz$, if $f(z) = y + xi$, Γ is a polygonal line OAB with vertices at the points $z_o = 0$, $z_A = i$, $z_B = 1 + i$;

(b) $\int\limits_{AB} z^2\, dz$, if AB is a line segment connecting the points $z_A = 1$, $z_B = i$;

(c) $\int\limits_{\gamma} z^{10}\, dz$, where γ is an ellipse $x^2/a^2 + y^2/b^2 = 1$;

(d) $\int\limits_{\gamma} \frac{dz}{z^2}$, where γ is a circle $(x - 4)^2 + (y - 3)^2 = 1$.

1000. Calculate the integral $\int\limits_{\gamma} \frac{dz}{z}$, where γ is a circle $z = e^{it}$.

1001. Calculate the integral $\int\limits_{\gamma} \frac{(a + b)z - az_1 - az_2}{(z - z_1)(z - z_2)} dz$, where γ is a circle $|z| \leqslant R$, and z_1 and z_2 are interior points of that circle, with $z_1 \neq z_2$.

7.5. The Taylor and the Laurent Series

Suppose the function $f(z)$ is analytic in some neighbourhood of the point a. Let us consider the series

$$f(a) + \frac{f'(a)}{1!}(z - a) + \frac{f''(a)}{2!}(z - a)^2 + \frac{f'''(a)}{3!}(z - a)^3 + \dots .$$

This series is called the *Taylor series* of the function $f(z)$ and represents the function $f(z)$ inside its circle of convergence, that is, the following equality holds in the convergence circle:

$$f(z) = f(a) + \frac{f'(a)}{1!}(z - a) + \frac{f''(a)}{2!}(z - a)^2 + \dots .$$

If $a = 0$, the last equality is written as

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \dots .$$

In that case we say that the function $f(z)$ is expanded into the *Maclaurin series*.
Let us now consider two series:

$$\frac{A_{-1}}{z - a} + \frac{A_{-2}}{(z - a)^2} + \frac{A_{-3}}{(z - a)^3} + \dots \tag{1}$$

and

$$A_0 + A_1(z - a) + A_2(z - a)^2 + A_3(z - a)^3 + \dots . \tag{2}$$

The domain of convergence of the first series (provided it exists) is specified by the inequality $|z - a| > r$. If there exists a domain of convergence of the second series, it is specified by the inequality $|z - a| < R$. Then, under the condition that $r < R$, the domain of convergence of the series

$$\dots + \frac{A_{-3}}{(z - a)^3} + \frac{A_{-2}}{(z - a)^2} + \frac{A_{-1}}{z - a} + A_0 + A_1(z - a) + A_2(z - a)^2 + A_3(z - a)^3 + \dots ,$$

obtained by the summation of series (1) and (2), is an annulus $r < |z - a| < R$ bounded by concentric circles with centre at the point a and radii r and R (Fig. 68).

Assume that $f(z)$ is a one-valued analytic function in the annulus $r < |z - a| < R$, in which this function can be represented as the sum of the series

$$f(z) = \dots + \frac{A_{-3}}{(z-a)^3} + \frac{A_{-2}}{(z-a)^2} + \frac{A_{-1}}{z-a}$$

$$+ A_0 + A_1(z-a) + A_2(z-a)^2 + A_3(z-a)^3 + \dots .$$

The series on the right-hand side is called the *Laurent series* of the function $f(z)$. The coefficients in this series can be calculated by the formula

$$A_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{n+1}} dz \quad (n \in Z).$$

Series (1) is called the *principal part* of Laurent's series and series (2) is called the *regular part* of Laurent's series.

If the Laurent series contains the principal part, then a is called an *isolated singular point*. In the case when the principal part of the Laurent series contains a finite number of terms, that is, has the form

$$\frac{A_{-1}}{z-a} + \frac{A_{-2}}{(z-a)^2} + \dots + \frac{A_{-n}}{(z-a)^n} \quad (A_{-n} \neq 0),$$

the isolated singular point a is called the *pole of the nth order* of the function $f(z)$. In this case the coefficient A_{-1} is called the *residue* of the function $f(z)$ about the pole a.

The singular point $z = a$ is called an isolated singular point of a one-valued character if a circle of a sufficiently small radius can be described about it, such that after its centre $z = a$ is removed, a doubly connected domain appears in which the

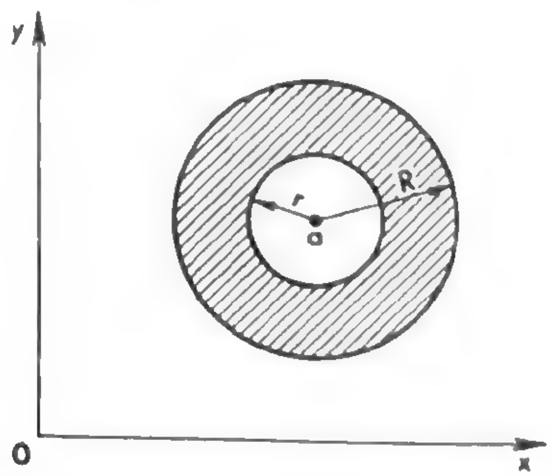


Fig. 68

function is analytic. An isolated singular point of a one-valued character is called:

(a) *removable* if the principal part of the expansion into the Laurent series is absent. For instance, for the function $f(z) = \frac{\sin z}{z}$ the point $z = 0$ is a removable singular point since

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots ;$$

(b) a *pole*, if the principal part contains a finite number of terms. For instance, for the function $f(z) = \frac{\sin z}{z^2}$ the point $z = 0$ is a pole of the first order since

$$f(z) = \frac{1}{z_2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \dots ;$$

(c) *essential*, if the principal part contains an infinite number of terms. For instance, at the point $z = 0$, the function $f(z) = e^{1/z}$ possesses an essential singular point since

$$f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \dots .$$

There is the following connection between the zero and the pole of a function. If $z = a$ is a zero of multiplicity k of the function $f(z)$, then $z = a$ is a pole of the same order of the function $1/f(z)$; conversely, if $z = b$ is a pole of order k of the function $f(z)$, then $z = b$ is a zero of the same multiplicity of the function $1/f(z)$.

It should be noted that if $\lim\limits_{z \to a} (z - a)^k f(z) = c \neq 0$, then $z = a$ is a pole of the kth order of the function $f(z)$.

1002. **Expand** the function $f(z) = z^5$ into the Taylor series in the powers of the binomial $z - i$.

Solution. We find the derivatives of the function $f(z) = z^5$:

$$f'(z) = 5z^4, f''(z) = 20z^3, f'''(z) = 60z^2,$$
$$f^{\mathrm{IV}}(z) = 120z, f^{\mathrm{V}}(z) = 120, f^{\mathrm{VI}}(z) = f^{\mathrm{VII}}(z) = \dots = 0.$$

Next we determine the values of the derivatives at the point $a = i$:

$$f(i) = i, f'(i) = 5, f''(i) = -20i, f'''(i) = -60,$$
$$f^{\mathrm{IV}}(i) = 120i, f^{\mathrm{V}}(i) = 120.$$

Hence we have

$$f(z) = i + 5(z - i) - 10i(z - i)^2 - 10(z - i)^3 + 5i(z - i)^4 + (z - i)^5.$$

The Taylor series of the function $f(z) = z^5$ is a polynomial of the fifth degree.

1003. Expand the function $f(z) = \cosh(1 - z)$ into the Taylor series in the powers of the binomial $z - (1 - \pi i/2)$.

Solution. We find

$$\begin{aligned} f(z) &= \cosh(1 - z), & f(a) &= \cosh(\pi i/2) = \cos(\pi/2) = 0, \\ f'(z) &= -\sinh(1 - z), & f'(a) &= -\sinh(\pi i/2) = -i \sin(\pi/2) = -i, \\ f''(z) &= \cosh(1 - z), & f''(a) &= 0, \\ f'''(z) &= -\sinh(1 - z), & f'''(a) &= -i. \end{aligned}$$

Consequently,

$$f(z) = -i\left[\left(z - 1 + \frac{\pi}{2}i\right) + \frac{1}{3!}\left(z - 1 + \frac{\pi}{2}i\right)^3 + \frac{1}{5!}\left(z - 1 + \frac{\pi}{2}i\right)^5 + \dots\right].$$

1004. Investigate the convergence of the series

$$\dots + \frac{1}{2^3(z - 1)^3} + \frac{1}{2^2(z - 1)^2} + \frac{1}{2(z - 1)} + 1 + \frac{z - 1}{5} + \frac{(z - 1)^2}{5^2} + \frac{(z - 1)^3}{5^3} + \dots .$$

Solution. Let us consider two series

$$\frac{1}{2(z - 1)} + \frac{1}{2^2(z - 1)^2} + \frac{1}{2^3(z - 1)^3} + \dots , \tag{a}$$

$$1 + \frac{z - 1}{5} + \frac{(z - 1)^2}{5^2} + \frac{(z - 1)^3}{5^3} + \dots . \tag{b}$$

If we put $z - 1 = 1/z'$ in series (a), we obtain a power series

$$\frac{z'}{2} + \frac{z'^2}{2^2} + \frac{z'^3}{2^3} + \dots . \tag{c}$$

The radius of convergence of the last series can be found by means of D'Alembert's test:

$$\rho = \lim_{n \to \infty} \frac{1/2^{n-1}}{1/2^n} = 2.$$

Thus we see that power series (c) converges if $|z'| < 2$. Consequently, series (a) converges if $|1/(z-1)| < 2$. From this we get $|z-1| > 1/2$. Hence, series (a) converges outside the circle of radius $r = 1/2$ with centre at the point $z = 1$. Let us determine the radius of convergence of series (b):

$$R = \lim_{n\to\infty} \frac{1/5^{n-1}}{1/5^n} = 5.$$

Thus, the domain of convergence of series (b) is defined by the inequality $|z-1| < 5$.

We infer from the above-said that the domain of convergence of the given series is an annulus $1/2 < |z-1| < 5$.

The solution of the problem can be simplified. Series (a) and (b) are geometric progression with the denominators $\dfrac{1}{2(z-1)}$ and $\dfrac{z-1}{5}$ respectively. They converge if $\left|\dfrac{1}{2(z-1)}\right| < 1$ and $\left|\dfrac{z-1}{5}\right| < 1$. Consequently, $|z-1| > 1/2$ and $|z-1| < 5$. Thus, the domain of convergence is an annulus specified by the double inequality $1/2 < |z-1| < 5$.

1005. Investigate the convergence of the series

$$\ldots + \frac{(3+4i)^3}{z^3} + \frac{(3+4i)^3}{z^2} + \frac{3+4i}{z} + 1 + \frac{z}{i} + \frac{z^2}{i^2} + \frac{z^3}{i^3} + \ldots .$$

Solution. Let us consider two series

$$\frac{3+4i}{z} + \frac{(3+4i)^2}{z^2} + \frac{(3+4i)^3}{z^3} + \ldots; \tag{a}$$

$$1 + \frac{z}{i} + \frac{z^2}{i^2} + \frac{z^3}{i^3} + \ldots . \tag{b}$$

Series (a) and (b) are geometric progressions with the denominators $(3+4i)/z$ and z/i respectively. They converge if $|(3+4i)/z| < 1$ and $|z/i| < 1$. Since $|3+4i| = \sqrt{9+16} = 5$, $|i| = 1$, we have $5/|z| < 1$ and $|z| < 1$, or $|z| > 5$ and $|z| < 1$. But these inequalities are inconsistent and, therefore, the given series is not convergent at any point of the plane.

1006. Expand the function $f(z) = 1/(2z-5)$ into the Laurent series in the powers of z in the neighbourhood of the point $z = 0$.

Solution. Let us represent the given function in the form

$$f(z) = \frac{-1/5}{1 - 2z/5}.$$

The inequality $|2z/5| < 1$ is satisfied in the neighbourhood of the point $z = 0$, therefore, the fraction $\dfrac{-1/5}{1 - 2z/5}$ can be regarded as the sum of an infinitely decreasing geometric progression with the first term $a = -1/5$ and the denominator $q = 2z/5$. From this we get

$$f(z) = -\frac{1}{5} - \frac{2z}{5^2} - \frac{2^2z^2}{5^3} - \frac{2^3z^3}{5^4} - \dots, \text{ or } f(z) = -\sum_{n=1}^{\infty} \frac{2^{n-1}z^{n-1}}{5^n}.$$

This expansion contains only the regular part. We infer from the inequality $|2z/5| < 1$ that the domain of convergence of the series is the circle $|z| < 5/2$.

1007. Expand the function $f(z) = 1/(2z - 5)$ into the Laurent series in the powers of z in the neighbourhood of the point $z = \infty$.

Solution. We have

$$f(z) = \frac{1}{2z - 5} = \frac{1/2z}{1 - 5/2z}.$$

The inequality $|5/2z| < 1$ is satisfied in the neighbourhood of the point $z = \infty$, therefore, $f(z)$ can be represented as the sum of an infinitely decreasing geometric progression with the first term $a = 1/2z$ and the denominator $q = 5/2z$. Consequently,

$$f(z) = \frac{1}{2z} + \frac{5}{2^2z^2} + \frac{5^2}{2^3z^3} + \frac{5^3}{2^4z^4} + \dots, \text{ or } f(z) = \sum_{n=1}^{\infty} \frac{5^{n-1}}{2^nz^n}.$$

The expansion does not include the regular part. The series converges in the domain $|z| > 5/2$ (outside the circle).

1008. Expand the function $f(z) = \dfrac{1}{(z - 1)(z - 3)}$ into the Laurent series in the powers of z in the annulus $1 < |z| < 3$.

Solution. Let us expand the given function into partial fractions:

$$\frac{1}{(z - 1)(z - 3)} = \frac{A}{z - 1} + \frac{B}{z - 3}, \text{ or } 1 = A(z - 3) + B(z - 1).$$

Setting $z = 1$, we get $1 = -2A$, i.e. $A = -1/2$; setting $z = 3$, we get $1 = 2B$, i.e. $B = 1/2$. Thus we have

$$f(z) = -\frac{1}{2} \cdot \frac{1}{z - 1} + \frac{1}{2} \cdot \frac{1}{z - 3}.$$

Taking into account that $1 < |z| < 3$, we can write

$$f(z) = -\frac{1}{2} \cdot \frac{1/z}{1 - 1/z} - \frac{1}{2} \cdot \frac{1/3}{1 - z/3}.$$

Consequently,

$$f(z) = -\frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=1}^{\infty} \frac{z^{n-1}}{3^n} \right).$$

1009. Expand the function $f(z) = \dfrac{z^4}{(z-2)^2}$ into the Laurent series in the powers of $z - 2$.

Solution. We set $z - 2 = z'$. Then we have

$$f(z) = \frac{z^4}{(z-2)^2} = \frac{(z'+2)^4}{z'^2} = \frac{z'^4 + 8z'^3 + 24z'^2 + 32z' + 16}{z'^2}$$
$$= \frac{16}{z'^2} + \frac{32}{z'} + 24 + 8z' + z'^2,$$

thas is,

$$f(z) = \frac{16}{(z-2)^2} + \frac{32}{z-2} + 24 + 8(z-2) + (z-2)^2.$$

The principal part here contains two terms and the regular part, three terms. Since the expansion contains a finite number of terms, it is valid for any point of the plane, except for $z = 2$. That point is a second-order pole of the function $f(z)$. The residue of this function about the pole $z = 2$ is the coefficient in $(z - 2)^{-1}$, i.e. 32.

1010. Investigate the convergence of the series

$$\ldots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \ldots.$$

1011. Investigate the convergence of the series

$$\ldots + \frac{4}{z^4} + \frac{3}{z^3} + \frac{2}{z^2} + \frac{1}{z} + 1 + 2z + (2z)^2 + (2z)^3 + \ldots.$$

1012. Expand the function $f(z) = \dfrac{z^2}{z-1}$ into the Laurent series in the powers of z in the neighbourhood of the point (1) $z = 0$; (2) $z = \infty$.

1013. Expand into the Laurent series the function

$$f(z) = \begin{cases} \dfrac{\sinh z - z}{z^5}, & \text{if } z \neq 0; \\ \infty, & \text{if } z = 0. \end{cases}$$

1014. Find the poles of the function $f(z) = \dfrac{z}{(z^2 - 1)(z^2 + 1)^2}$.

1015. Expand the function $f(z) = 1/z$ into the Taylor series in the powers of $z - 1$. Find the domain of convergence of the series.

1016. Expand into the Maclaurin series the function

$$f(z) = \begin{cases} \dfrac{1 - \cos z}{z^2}, & \text{if } z \neq 0; \\ 1/2, & \text{if } z = 0. \end{cases}$$

1017. Expand the function $f(z) = 2^z + 2^{1/z} - 2$ into the Laurent series, the function being defined throughout the plane except for the point $z = 0$.

7.6. Calculating the Residues of Functions. Using Residues in Calculation of Integrals

Assume that a is an nth-order pole of the function $f(z)$. The residue of the function $f(z)$ about its nth-order pole can be calculated by the formula

$$\operatorname*{res}_a f(z) = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}[(z-a)^n \cdot f(z)]}{dz^{n-1}}.$$

If a is a pole of the first order of the function $f(z)$, then

$$\operatorname*{res}_a f(z) = \lim_{z \to a} (z - a)f(z).$$

Suppose the functions $\varphi(z)$ and $\psi(z)$ are regular in the neighbourhood of the point $z = a$, $\varphi(a) \neq 0$ and $\psi(z)$ possesses a root of the first order at the point $z = a$. Then, in calculating the residue of the function $f(z) = \varphi(z)/\psi(z)$ at the simple pole $z = a$ it is convenient to use the formula

$$\operatorname*{res}_a f(z) = \frac{\varphi(a)}{\psi'(a)}.$$

Suppose $f(z)$ is an analytic function in the domain D, except for a finite number of the poles $a_1, a_2, \dots, a_k$. Let us denote by γ an arbitrary piecewise smooth closed

contour containing the points $a_1, a_2, \dots, a_k$ in its interior and lying entirely in the domain D. In that case, $\int_\gamma f(z)\,dz$ is equal to the *sum of the residues of the function $f(z)$ about the poles $a_1, a_2, \dots, a_k$ multiplied by $2\pi i$*, i.e.

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{res}_{a_j} f(z)$$

(the fundamental theorem on residues).

Let us consider a special case. Assume that $f(z)$ is an analytic function in the domain D, the point a belongs to the domain D, but $f(a) \neq 0$. In that case, the function $F(z) = \dfrac{f(z)}{z-a}$ possesses the first-order pole a in D. Let us find the residue of the function $F(z)$ about the pole a:

$$\operatorname*{res}_{a} F(z) = \lim_{z \to a} (z-a) \cdot F(z) = \lim_{z \to a} f(z) = f(a).$$

Applying the fundamental theorem on residues, we get

$$\int_\gamma F(z)\,dz = 2\pi i\, f(a),$$

or

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\,dz = f(a).$$

We have obtained an extremely important formula in the theory of functions of a complex variable known as *Cauchy's formula*.

However, it is necessary to note that the derivation of Cauchy's formula must precede the proof of the fundamental theorem on residues. We have made advantage of the situation here to acquaint the reader with this important formula.

Suppose $f(z)$ is an analytic function in the upper half-plane, including the real axis, except the finite number of poles a_k $(k = 1, 2, \dots, m)$, lying above the real axis. Assume, in addition, that the product $z^2 f(z)$ possesses a finite limit as $|z| \to +\infty$. In that case, to calculate the definite integral $\int_{-\infty}^{+\infty} f(x)\,dx$ of the function of the real variable, use is made of the formula

$$\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i\,(r_1 + r_2 + \dots + r_m),$$

where r_k $(k = 1, 2, \ldots, m)$ is the residue of the function $f(z)$ about the pole a_k.

1018. Find the residues of the function $f(z) = \dfrac{z}{(z-1)(z-3)}$.

Solution. The poles of the function are the points $z = 1$ and $z = 3$:

$$\operatorname*{res}_{1} f(z) = \lim_{z \to 1} (z-1) \cdot \frac{z}{(z-1)(z-3)} = \lim_{z \to 1} \frac{z}{z-3} = \frac{1}{2},$$

$$\operatorname*{res}_{3} f(z) = \lim_{z \to 3} (z-3) \cdot \frac{z}{(z-1)(z-3)} = \lim_{z \to 3} \frac{z}{z-1} = \frac{1}{2}.$$

1019. Find the residues of the function $f(z) = \dfrac{1}{z^2+4}$.

Solution. We have

$$f(z) = \frac{1}{(z-2i)(z+2i)}.$$

The poles of the function are the points $2i$ and $-2i$:

$$\operatorname*{res}_{2i} f(z) = \lim_{z \to 2i} (z-2i) \cdot \frac{1}{(z-2i)(z+2i)} = \lim_{z \to 2i} \frac{1}{z+2i} = \frac{4}{4i} = -\frac{i}{4},$$

$$\operatorname*{res}_{-2i} f(z) = \lim_{z \to -2i} (z+2i) \cdot \frac{1}{(z-2i)(z+2i)} = \lim_{z \to -2i} \frac{1}{z-2i} = -\frac{1}{4i} = \frac{i}{4}.$$

1020. Find the residues of the function $f(z) = \dfrac{1}{z^2-2z+5}$.

Solution. The poles of the function are the roots of the denominator: $z = 1 \pm 2i$. Consequently,

$$f(z) = \frac{1}{(z-1-2i)(z-1+2i)}.$$

We find

$$\operatorname*{res}_{1+2i} f(z) = \lim_{z \to 1+2i} \cdot \frac{z-1-2i}{(z-1-2i)(z-1+2i)} = \lim_{z \to 1+2i} \frac{1}{z-1+2i} = -\frac{i}{4},$$

$$\operatorname*{res}_{1-2i} f(z) = \lim_{z \to 1-2i} \cdot \frac{z-1+2i}{(z-1-2i)(z-1+2i)} = \lim_{z \to 1-2i} \frac{1}{z-1-2i} = \frac{i}{4}.$$

1021. Find the residue of the function $f(z) = \dfrac{z^2}{(z-2)^3}$.

Solution. Since $z = 2$ is the third-order pole, we have

$$\operatorname*{res}_{2} f(z) = \frac{1}{2!} \lim_{z\to 2} \frac{d^2\left[\dfrac{z^2}{(z-2)^3}\cdot(z-2)^3\right]}{dz^2} = \frac{1}{2!} \lim_{z\to 2} \frac{d^2(z^2)}{dz^2} = \frac{1}{2!}\cdot 2 = 1.$$

1022. Find the residue of the function $f(z) = \dfrac{1}{1-\cos z}$ about the pole $z = 0$.

Solution. The point $z = 0$ is the second-order pole. Indeed,

$$\lim_{z\to 0} \frac{z^2}{1-\cos z} = \lim_{z\to 0} \frac{2z}{\sin z} = 2$$

is a finite quantity. Then

$$\operatorname*{res}_{0} f(z) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{z^2}{1-\cos z}\right) = \lim_{z\to 0} \frac{2z\,(1-\cos z) - z^2 \sin z}{(1-\cos z)^2}$$

$$= \lim_{z\to 0} \frac{2z\left(\dfrac{z^2}{2!} - \dfrac{z^4}{4!} + \ldots\right) - z^2\left(\dfrac{z}{1!} - \dfrac{z^3}{3!} + \ldots\right)}{\left(\dfrac{z^2}{2!} - \dfrac{z^4}{4!} + \ldots\right)^2}$$

$$= \lim_{z\to 0} \frac{\dfrac{z^5}{12}}{\left(\dfrac{z^2}{2!} - \dfrac{z^4}{4!} + \ldots\right)^2} = 0.$$

1023. Find $\displaystyle\int_\gamma \frac{z+1}{(z-1)(z-2)(z-3)}\, dz$, where γ is a closed contour having the poles $z = 1$, $z = 2$, $z = 3$ in its interior.

Solution. Let us determine the residues of the integrand:

$$\operatorname*{res}_{1} f(z) = \lim_{z\to 1} \frac{z+1}{(z-2)(z-3)} = 1,$$

$$\operatorname*{res}_{2} f(z) = \lim_{z\to 2} \frac{z+1}{(z-1)(z-3)} = -3,$$

$$\operatorname*{res}_{3} f(z) = \lim_{z\to 3} \frac{z+1}{(z-1)(z-2)} = 2.$$

Consequently,

$$\int_\gamma \frac{z+1}{(z-1)(z-2)(z-3)}\, dz = 2\pi i\,(1 - 3 + 2) = 0.$$

1024. Find $\displaystyle\int_\gamma \frac{z^2}{(z^2+1)(z-2)}\, dz$, where γ is the circle $|z| = 3$.

Solution. We have $f(z) = \dfrac{z^2}{(z-i)(z+i)(z-2)}$. The poles i, $-i$, 2 are

inside the closed contour γ. Hence

$$\operatorname{res}_{i} f(z) = \lim_{z\to i} (z - i) f(z) = \lim_{z\to i} \frac{z^2}{(z + i)(z - 2)} = \frac{1}{2i(2 - i)},$$

$$\operatorname{res}_{-i} f(z) = \lim_{z\to -i} (z + i)\cdot f(z) = \lim_{z\to -i} \frac{z^2}{(z - i)(z - 2)} = -\frac{1}{2i(2 + i)},$$

$$\operatorname{res}_{2} f(z) = \lim_{z\to 2} (z - 2)\cdot f(z) = \lim_{z\to 2} \frac{z^2}{z^2 + 1} = \frac{4}{5};$$

$$\int_\gamma \frac{z^2}{(z^2 + 1)(z - 2)} dz = 2\pi i \left[\frac{1}{2i(2 - i)} - \frac{1}{2i(2 + i)} + \frac{4}{5} \right]$$

$$= \pi \left(\frac{1}{2 - i} - \frac{1}{2 + i} + \frac{8}{5} i \right) = \pi \left(\frac{2}{5} i + \frac{8}{5} i \right) = 2\pi i .$$

1025. Calculate the definite integral $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(x^2 + 4)^2}$.

Solution. The function $\dfrac{1}{(z^2 + 4)^2}$ is analytic in the upper half-plane, except for the pole $2i$. In addition,

$$\lim_{|z| \to +\infty} z^2 f(z) = \lim_{|z| \to +\infty} \frac{z^2}{(x^2 + 4)^2} = 0,$$

that is, is a finite quantity.

Let us determine the residue of the function $f(z) = 1/(z^2 + 4)^2$ about the second-order pole $2i$:

$$\operatorname{res}_{2i} f(z) = \lim_{z\to 2i} \frac{d}{dz} \left[\frac{(z - 2i)^2}{(z^2 + 4)^2} \right] = \lim_{z\to 2i} \frac{d}{dz} \left[\frac{1}{(z + 2i)^2} \right]$$

$$= \lim_{z\to 2i} \frac{2}{(z + 2i)^3} = \frac{2}{64i} = -\frac{1}{32} i.$$

Consequently,

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2 + 4)^2} = 2\pi i \left(-\frac{1}{32} i \right) = \frac{\pi}{16}.$$

1026. Find $\displaystyle\int_\gamma \frac{dz}{z(z + 2)(z + 4)}$, if γ is a circle: (1) $|z| = 1$; (2) $|z| = 3$; (3) $|z| = 5$.

Solution. We find the residues of the integrand about the poles $z = 0$, $z = -2$, $z = -4$:

$$\operatorname*{res}_{0} f(z) = \lim_{z\to 0} z\cdot f(z) = \lim_{z\to 0} \frac{1}{(z+2)(z+4)} = \frac{1}{8},$$

$$\operatorname*{res}_{-2} f(z) = \lim_{z\to -2} (z+2)\cdot f(z) = \lim_{z\to -2} \frac{1}{z(z+4)} = -\frac{1}{4},$$

$$\operatorname*{res}_{-4} f(z) = \lim_{z\to -4} (z+4)\cdot f(z) = \lim_{z\to -4} \frac{1}{z(z+2)} = \frac{1}{8}.$$

(1) The contour γ, i.e. the circle $|z| = 1$, contains only the pole $z = 0$ in its interior; then

$$\int_\gamma f(z)\,dz = 2\pi i\cdot\frac{1}{8} = \frac{\pi i}{4}.$$

(2) The contour γ, i.e. the circle $|z| = 3$, contains the poles $z = 0$ and $z = -2$ in its interior; then

$$\int_\gamma f(z)\,dz = 2\pi i\cdot\left(\frac{1}{8} - \frac{1}{4}\right) = -\frac{\pi i}{4}.$$

(3) the contour γ, i.e. the circle $|z| = 5$, contains the poles $z = 0$, $z = -2$, $z = -4$ in its interior; then

$$\int_\gamma f(z)\,dz = 2\pi i\cdot\left(\frac{1}{8} - \frac{1}{4} + \frac{1}{8}\right) = 0.$$

1027. Find the residue of the function $f(z) = \dfrac{z+i}{z-i}$.

1028. Find the residues of the function $f(z) = \dfrac{z^2+1}{z^2-1}$.

1029. Find the residue of the function $f(z) = 1/\sin z$ about the pole $z = \pi$.

1030. Find the residue of the function $f(z) = (z+1)/z^2$.

1031. Find the integral $\displaystyle\int_\gamma \frac{z^2}{z-a}\,dz$, γ being a circle $|z| = R > |a|$.

1032. Find the integral $\displaystyle\int_\gamma \frac{z}{(z-a)(z-b)}\,dz$, γ being a circle $|z| = R$, $R > |a|$, $R > |b|$, $a \neq b$.

1033. Find the integral $\displaystyle\int_\gamma \frac{dz}{z^2-2z+2}$, γ being a circle containing the poles of the denominator in its interior.

1034. Find the integral $\displaystyle\int_\gamma \frac{z}{(z-i)(z-3)}\,dz$, γ being a circle $|z| = 2$.

1035. Calculate the definite integral $\displaystyle\int_{+\infty}^{+\infty} \frac{dx}{(x^2+1)^3}$.

Chapter 8

Elements of Operational Calculus

8.1. Finding an Image of a Function

8.1.1. Main definitions. Let the function $f(t)$ possess the following properties:

1°. $f(t) = 0$ for $t < 0$.

2°. $|f(t)| < Me^{s_0 t}$ for $t > 0$, where $M > 0$ and s_0 are some real constants.

3°. On any finite closed interval $[a, b]$ of the positive semi-axis Ot the function $f(t)$ satisfies **Dirichlet's conditions,** that is: (a) *it is bounded*; (b) *it is either continuous or possesses only a finite number of points of discontinuity of the 1st kind*; (c) *it has a finite number of extrema.*

In operational calculus, functions of this kind are called the *originals of the Laplace transformation.*

Suppose $p = \varphi + \beta i$ is a complex parameter, with $\operatorname{Re} p = \alpha \geqslant s_1 > s_0$.

Under the conditions formulated above, the integral $\int\limits_0^\infty e^{-pt} f(t)\, dt$ converges and is a function of p:

$$\int\limits_0^{+} e^{-pt} f(t)\, dt = \bar{f}(p).$$

This integral is known as the *Laplace integral* and the function of the complex argument p it defines is called the *Laplace transform* of the function $f(t)$, or the *Laplace image* of $f(t)$, or simply the *image* of $f(t)$.

The fact that the function $\bar{f}(p)$ is the image of the original function $f(t)$ is written as follows:

$$\bar{f}(p) = L\,[f(t)], \quad \text{or} \quad \bar{f}(p) \div f(t).$$

It is assumed that the value of the original function $f(t)$ at its every point of discontinuity of the 1st kind t_0 is the half-sum of its limiting values on the left and on the right of that point:

$$f(t_0) = (1/2)[f(t_0 - 0) + f(t_0 + 0)] \quad \text{for} \quad t_0 \neq 0 \quad \text{and}$$
$$f(0) = f(+0) \quad \text{for} \quad t_0 = 0.$$

If this condition is satisfied, the correspondence between the original functions and their images possesses the following properties:

the correspondence is mutually one-to-one (that is, every original function is associated with only one image and vice versa),

any linear combination of a finite set of original functions is associated with a linear combination of their images.

Thus, if $\bar{f}_k(p) \doteqdot f_k(t)$ $(k = 1, 2, \ldots, n)$, then

$$\sum_{k=1}^{k=n} c_k \bar{f}_k(p) \doteqdot \sum_{k=1}^{k=n} c_k f_k(t)$$

8.1.2. Finding the image of a function. The table given below indicates only the value of $f(t)$ for $t > 0$ and the examples that follow deal only with that value (it is always assumed that $f(t) = 0$ if $t < 0$).

Table of Images of the Basic Elementary Functions

No.	$f(t)$ for $t > 0$	$\bar{f}(p)$	No.	$f(t)$ for $t > 0$	$\bar{f}(p)$
I	1	$\frac{1}{p}$	VI	$e^{\alpha t} \cdot \cos \beta t$	$\frac{p-\alpha}{(p-\alpha)^2+\beta^2}$
II	$\frac{t^n}{n!}$	$\frac{1}{p^{n+1}}$	VII	$e^{\alpha t} \cdot \sin \beta t$	$\frac{\beta}{(p-\alpha)^2+\beta^2}$
III	$e^{\alpha t}$	$\frac{1}{p-\alpha}$	VIII	$\frac{t^n}{n!} \cdot e^{\alpha t}$	$\frac{1}{(p-\alpha)^{n+1}}$
IV	$\cos \beta t$	$\frac{p}{p^2+\beta^2}$	IX	$t \cdot \cos \beta t$	$\frac{p^2-\beta^2}{(p^2+\beta^2)^2}$
V	$\sin \beta t$	$\frac{\beta}{p^2+\beta^2}$	X	$t \cdot \sin \beta t$	$\frac{2p\beta}{(p^2+\beta^2)^2}$

1036. Find the image of the function $f(t) = a^t$.

Solution. Since $a = e^{\ln a}$, it follows that $f(t) = e^{t \ln a}$. Using formula III given in the table, we obtain

$$\bar{f}(p) = \frac{1}{p - \ln a}.$$

1037. Find the image of the function $f(t) = \cos^3 t$.

Solution. We make use of Euler's formula $\cos t = (e^{ti} + e^{-ti})/2$. Then we have

$$\cos^3 t = \left(\frac{e^{ti} + e^{-ti}}{2}\right)^3 = \frac{1}{8}\,(e^{3ti} + 3e^{ti} + 3e^{-ti} + e^{-3ti})$$

$$= \frac{1}{4} \cdot \frac{e^{3ti} + e^{-3ti}}{2} + \frac{3}{4} \cdot \frac{e^{ti} + e^{-ti}}{2} = \frac{1}{4}\cos 3t + \frac{3}{4}\cos t.$$

Applying formula IV, we get

$$\bar{f}(p) = \frac{1}{4} \cdot \frac{p}{p^2+9} + \frac{3}{4} \cdot \frac{p}{p^2+1} = \frac{p(p^2+7)}{(p^2+1)(p^2+9)}.$$

1038. Find the image of the function $f(t) = \sinh bt$.

Solution. By the definition of the hyperbolic sine, we have $f(t) = (1/2)e^{bt} - (1/2)e^{-bt}$. Consequently,

$$\bar{f}(p) = \frac{1}{2(p-b)} - \frac{1}{2(p+b)} = \frac{b}{p^2 - b^2}.$$

1039. Find the image of the function $f(t) = \sinh at \sin bt$.

Solution. Since $\sinh at = (e^{at} - e^{-at})/2$, it follows that

$$f(t) = \frac{1}{2} e^{at} \sin bt - \frac{1}{2} e^{-at} \sin bt.$$

Applying formula VII, we get

$$\bar{f}(p) = \frac{1}{2} \frac{b}{(p-a)^2 + b^2} - \frac{1}{2} \frac{b}{(p+a)^2 + b^2}$$

$$= \frac{2pab}{[(p-a)^2 + b^2][(p+a)^2 + b^2]}.$$

1040. Find the image of the function $f(t) = t \cosh bt$.

Solution. Since

$$f(t) = t \cdot \frac{e^{bt} + e^{-bt}}{2} = \frac{1}{2} te^{bt} + \frac{1}{2} te^{-bt},$$

it follows that applying formula VIII for $n = 1$, $\alpha = \pm b$, we get

$$\bar{f}(p) = \frac{1}{2(p-b)^2} + \frac{1}{2(p+b)^2} = \frac{p^2 + b^2}{(p^2 - b^2)^2}.$$

Find the images of the following functions.

1041. $f(t) = \sin^2 t$. **1042.** $f(t) = e^t \cos^2 t$. **1043.** $f(t) = \cosh bt$. **1044.** $f(t) = \sinh at \cos bt$. **1045.** $f(t) = \cosh at \sin bt$. **1046.** $f(t) = \cosh at \cos bt$. **1047.** $f(t) = t \cosh bt$.

8.2. Recovering the Original Function from Its Image.

To recover the original function from its image in the simplest cases, use is made of the table of images of the basic elementary functions and of the expansion theorems (the first and the second).

The second expansion theorem makes it possible to recover the original from the image which is a fractional-rational function of p, i.e. $\bar{f}(p) = u(p)/v(p)$, where $u(p)$ and $v(p)$ are the polynomials of p of the degree m and n respectively, with $m < n$.

If the expansion of $v(p)$ into prime factors has the form

$$v(p) = (p - p_1)^{k_1}(p - p_2)^{k_2} \dots (p - p_r)^{k_r} (k_1 + k_2 + \dots + k_r = n),$$

then, as is known, the function $\bar{f}(p)$ can be decomposed into the sum of partial fractions of the form $\dfrac{A_{j,s}}{(p - p_j)^{k_j-s+1}}$, where j assumes all the values from 1 to r, and s assumes all the values from 1 to k_j. Thus it follows that

$$\bar{f}(p) = \sum_{j=1}^{j=r} \sum_{s=1}^{s=k_j} \frac{A_{j,s}}{(p-p_j)^{k_j-s+1}}. \tag{1}$$

All the coefficients in this expansion can be determined by the formula

$$A_{j,s} = \frac{1}{(s-1)!} \lim_{p \to p_j} \left\{ \frac{d^{s-1}}{dp^{s-1}} [(p-p_j)^{k_j} \cdot \bar{f}(p)] \right\}. \tag{2}$$

The coefficients $A_{j,s}$ can also be determined by the elementary techniques of integral calculus for rational fractions. In particular, it is of use in the case when all complex roots of the denominator $v(p)$ are simple and pairwise conjugate.

If all the roots of $v(p)$ are simple, that is,

$$v(p) = (p-p_1)(p-p_2) \dots (p-p_n) \quad (p_j \neq p_k \quad \text{for} \quad j \neq k),$$

the expansion is simplified:

$$\bar{f}(p) = \sum_{j=1}^{j=n} \frac{A_j}{p-p_j}, \quad \text{where} \quad A_j = \frac{u(p_j)}{v'(p_j)}. \tag{3}$$

When using one or another technique to decompose $\bar{f}(p)$ into partial fractions, the original $f(t)$ can be recovered by the following formulas:

(a) in the case of multiple roots of the denominator $v(p)$:

$$\bar{f}(t) = \sum_{j=1}^{j=r} \sum_{s=1}^{s=k_j} A_{j,s} \frac{t^{k_j-s}}{(k_j-s)!} \cdot e^{p_j t}; \tag{4}$$

(b) in the case of simple roots of the denominator $v(p)$:

$$f(t) = \sum_{j=1}^{j=n} \frac{u(p_j)}{v'(p_j)} \cdot e^{p_j t}. \tag{5}$$

If the image of the desired function can be expanded into a power series in the powers of $1/p$, that is,

$$\bar{f}(p) = \frac{a_0}{p} + \frac{a_1}{p^2} + \dots + \frac{a_n}{p^{n+1}} + \dots$$

(the series converging to $\bar{f}(p)$ for $|p| > R$, where $R = \lim\limits_{n \to \infty} |a_{n+1}/a_n| \neq \infty$),

then the original $f(t)$ can be recovered by the formula

$$f(t) = a_0 + a_1 \cdot \frac{t}{1!} + a_2 \cdot \frac{t^2}{2!} + \ldots + a_n \cdot \frac{t^n}{n!} + \ldots .$$

In this case, the series converges for all the values of t (**the first expansion theorem**).

1048. Recover the original of the function $\bar{f}(p) = \dfrac{p}{p^2 - 2p + 5}$.

Solution. We use the elementary techniques of decomposition of this fraction into the sum of the fractions whose originals are known:

$$\frac{p}{p^2 - 2p + 5} = \frac{p - 1 + 1}{(p - 1)^2 + 4} = \frac{p - 1}{(p - 1)^2 + 4} + \frac{1}{(p - 1)^2 + 4}.$$

But formulas VI and VII of the table (see 8.1.2) yield

$$\frac{p - 1}{(p - 1)^2 + 4} \div e^t \cdot \cos 2t; \quad \frac{1}{(p - 1)^2 + 4} = \frac{1}{2} \cdot \frac{2}{(p - 1)^2 + 4} \div \frac{1}{2} e^t \cdot \sin 2t.$$

Therefore,

$$\frac{p}{(p - 1)^2 + 4} \div e^t \left(\cos 2t + \frac{1}{2} \sin 2t \right).$$

1049. Recover the original of the function $\bar{f}(p) = \dfrac{1}{p^3 - 8}$.

Solution. In this example as well we make use of the elementary techniques of decomposition known from integral calculus. We decompose the given fraction into partial fractions:

$$\frac{1}{p^3 - 8} = \frac{A}{p - 2} + \frac{Bp + C}{p^2 + 2p + 4}.$$

To determine the coefficients, we have the identity

$$1 \equiv A(p^2 + 2p + 4) + (Bp + C)(p - 2).$$

Setting $p = 2$, we find $1 \equiv 12A$; $A = 1/12$. Equating the coefficient in p^2 to zero and the constant term to unity, we get $A + B = 0$, $4A - 2C = 1$. Hence $B = -A = -1/12$; $C = 2A - 1/2 = -1/3$. Consequently,

$$\frac{1}{p^3 - 8} = \frac{1}{12} \cdot \frac{1}{p - 2} - \frac{1}{12} \cdot \frac{p + 4}{p^2 + 2p + 4}$$

$$= \frac{1}{12} \cdot \frac{1}{p - 2} - \frac{1}{12} \cdot \frac{(p + 1) + 3}{(p + 1)^2 + (\sqrt{3})^2}.$$

It follows that

$$\bar{f}(p) = \frac{1}{12}\cdot\frac{1}{p-2} - \frac{1}{12}\cdot\frac{p+1}{(p+1)^2+(\sqrt{3})^2} - \frac{\sqrt{3}}{12}\cdot\frac{\sqrt{3}}{(p+1)^2+(\sqrt{3})^2}.$$

From this, using formulas III, VI and VII from the table of images, we find

$$f(t) = \frac{1}{12}e^{2t} - \frac{1}{12}e^{-t}(\cos t\sqrt{3} + \sqrt{3}\cdot\sin t\sqrt{3}).$$

1050. Recover the original of the function $\bar{f}(p) = \dfrac{p}{(p-1)^3(p+2)^2}$.

Solution. The decomposition of $\bar{f}(p)$ into partial fractions has the form

$$\bar{f}(p) = \frac{A_{1,1}}{(p-1)^3} + \frac{A_{1,2}}{(p-1)^2} + \frac{A_{1,3}}{p-1} + \frac{A_{2,1}}{(p+2)^2} + \frac{A_{2,2}}{p+2}.$$

We find the coefficients in this decomposition, using Eq. (2):

$$A_{1,1} = \frac{1}{0!}\lim_{p\to 1}[(p-1)^3\cdot\bar{f}(p)] = \lim_{p\to 1}\frac{p}{(p+2)^2} = \frac{1}{9};$$

$$A_{1,2} = \frac{1}{1!}\lim_{p\to 1}\frac{d}{dp}[(p-1)^3\cdot\bar{f}(p)] = \lim_{p\to 1}\frac{d}{dp}\left[\frac{p}{(p+2)^2}\right] = \lim_{p\to 1}\left[\frac{1}{(p+2)^2} - \frac{2p}{(p+2)^3}\right] = \frac{1}{27};$$

$$A_{1,3} = \frac{1}{2!}\lim_{p\to 1}\frac{d^2}{dp^2}[(p-1)^3\cdot\bar{f}(p)] = \frac{1}{2}\lim_{p\to 1}\frac{d^2}{dp^2}\left[\frac{p}{(p+2)^2}\right] = \frac{1}{2}\lim_{p\to 1}\left[-\frac{4}{(p+2)^3} + \frac{6p}{(p+2)^3}\right] = -\frac{1}{27};$$

$$A_{2,1} = \frac{1}{0!}\lim_{p\to -2}[(p+2)^2\cdot\bar{f}(p)] = \lim_{p\to -2}\frac{p}{(p-1)^3} = \frac{2}{27};$$

$$A_{2,2} = \frac{1}{1!}\lim_{p\to -2}\frac{d}{dp}[(p+2)^2\cdot\bar{f}(p)] = \lim_{p\to -2}\frac{d}{dp}\left[\frac{p}{(p-1)^3}\right] = \lim_{p\to -2}\left[\frac{1}{(p-1)^3} - \frac{3p}{(p-1)^4}\right] = \frac{1}{27}.$$

Thus we have

$$\bar{f}(p) = \frac{1}{27}\cdot\left\{\frac{3}{(p-1)^3} + \frac{1}{(p-1)^2} - \frac{1}{p-1} + \frac{2}{(p+2)^2} + \frac{1}{p+2}\right\}.$$

From this, using formulas III and VIII, we find

$$f(t) = \frac{1}{27}\left(\frac{3}{2}t^2e^t + te^t - e^t + 2te^{-2t} + e^{-2t}\right)$$

$$= \frac{3t^2 + 2t - 2}{54}\cdot e^t + \frac{2t+1}{27}\cdot e^{-2t}.$$

1051. Recover the original of the function $\bar{f}(p) = \dfrac{p+1}{p(p-1)(p-2)(p-3)}$.

Solution. Since in the given case all the roots in the denominator are real and simple, it is expedient to use Eq. (5). We have

$$u(p) = p + 1,\quad v(p) = p(p-1)(p-2)(p-3) = p^4 - 6p^3 + 11p^2 - 6p;$$

$$v'(p) = 4p^3 - 18p^2 + 22p - 6.$$

We find the roots of $v(p)$: $p_1 = 0, p_2 = 1, p_3 = 2, p_4 = 3$. Next we obtain

$$\frac{u(p_1)}{v'(p_1)} = \frac{1}{-6} = -\frac{1}{6};\quad \frac{u(p_2)}{v'(p_2)} = \frac{2}{2} = 1;$$

$$\frac{u(p_3)}{v'(p_3)} = \frac{3}{-2} = -\frac{3}{2};\quad \frac{u(p_4)}{v'(p_4)} = \frac{4}{6} = \frac{2}{3}.$$

From this, by Eq. (5), we find

$$f(t) = -\frac{1}{6} + e^t - \frac{3}{2}e^{2t} + \frac{2}{3}e^{3t}.$$

1052. Recover the original of $\bar{f}(p) = \dfrac{1}{p(1+p^4)}$, using the first expansion theorem.

Solution. We have

$$\bar{f}(p) = \frac{1}{p(1+p^4)} = \frac{1}{p^5}\cdot\frac{1}{1+\dfrac{1}{p^4}} = \frac{1}{p^5} - \frac{1}{p^9} + \frac{1}{p^{13}} - \dots$$

This series converges at $|p| > 1$. From this we find

$$f(t) = \frac{t^4}{4!} - \frac{t^8}{8!} + \frac{t^{12}}{12!} - \frac{t^{16}}{16!} + \dots$$

1053. Recover $f(t)$ if $\bar{f}(p) = \dfrac{1}{p(p^2+1)(p^2+4)}$.

Hint. Decompose $\bar{f}(p)$ into partial fractions.

Recover the original functions from the following images:

1054. $\bar{f}(p) = \dfrac{1}{(p-1)(p^2-4)}$. **1055.** $\bar{f}(p) = \dfrac{p+3}{p(p^2-4p+3)}$.

1056. $\bar{f}(p) = \dfrac{1}{p(p^4 - 5p^2 + 4)}$.

1057. With the first expansion theorem recover the original of the function $\bar{f}(p) = 1/(p^k + a^k)$, where k is a positive integer.

8.3 Convolution of Functions. Images of the Derivatives and of the Integral of the Original Function

The *convolution* of two functions $f_1(t)$ and $f_2(t)$ is the function

$$F(t) = \int_0^t f_1(t - \tau) f_2(\tau)\, d\tau.$$

The integral defining the convolution does not change its value upon the rearrangement of the functions f_1 and f_2, therefore, the convolution of two functions is symmetric about the functions being convoluted.

The image of the convolution of two original functions is equal to the product of their images (**theorem on convolution of original functions**): if $\bar{f}_1(p) \leftarrow f_1(t)$, $\bar{f}_2(p) \leftarrow f_2(t)$, then

$$\int_0^t f_1(t - \tau)\cdot f_2(\tau)\, d\tau \rightarrow \bar{f}_1(p)\cdot\bar{f}_2(p).$$

Suppose the original function $f(t)$ is differentiable n times and its derivatives up to the nth order are original functions in their turn. Then, the **theorem on differentiation** of the original function holds true: if $\bar{f}(p) \leftarrow f(t)$ $(k = 1, 2, \ldots, n)$, then

$$f^{(k)}(t) \rightarrow p^k\cdot\bar{f}(p) - [p^{k-1}\cdot f(0) + p^{k-2}\cdot f'(0) + \ldots + f^{(k-1)}\cdot(0)].$$

In particular,

$$f'(t) \rightarrow p\cdot\bar{f}(p) - f(0),\quad f''(t) \rightarrow p^2\cdot\bar{f}(p) - p\cdot f(0) - f'(0),$$

and so on.

Theorem on integration is valid for all original functions: if $\bar{f}(p) \leftarrow f(t)$, then

$$\frac{\bar{f}(p)}{p} \leftarrow \int_0^t f(\tau)\, d\tau,$$

It can be seen that the images of the derivative and of the integral are recovered from the image of the function $f(t)$ by means of algebraic operations performed on $\bar{f}(p)$. It should also be pointed out (see the table of images in 8.1.2) that the images of a considerable number of functions used in practical applications ($e^{\alpha t}$, $\cos\beta t$, $\sin\beta t$, etc.) are algebraic functions of p.

This makes it possible to reduce many operations of mathematical analysis (solution of differential and integral equations and the like) to algebraic operations on the images of the desired functions.

1058. Using the theorem on convolution, recover the original of the function $\bar{f}(p) = \dfrac{p}{p^4 - 1}$.

Solution. We write $\bar{f}(p)$ in the form $\dfrac{p}{p^2 - 1} \cdot \dfrac{1}{p^2 + 1}$. By virtue of the fact that $\dfrac{p}{p^2 - 1} \leftarrow \cosh t$, $\dfrac{1}{p^2 + 1} \leftarrow \sin t$, we have, by the theorem on convolution,

$$\frac{p}{p^4 - 1} \leftarrow \int_0^t \cosh(t - \tau) \sin\tau \, d\tau$$

$$= -\frac{1}{2} [\sinh(t - \tau) \sin\tau + \cosh(t - \tau) \cos\tau]\big|_0^t = \frac{1}{2} (\cosh t - \cos t).$$

1059. Find the image of $y''(t) - y'(t) - y(t)$, if $y(0) = y'(0) = 0$ and $\bar{y}(p) \leftarrow y(t)$.

Solution. By the theorem on differentiation of the original function we have

$$y'(t) \leftarrow p\bar{y}(p) - y(0) = p \cdot \bar{y}(p),$$

$$y''(t) \leftarrow p^2 \cdot \bar{y}(p) - p \cdot y(0) - y'(0) = p^2 \cdot \bar{y}(p).$$

From this we find

$$y''(t) - y'(t) - y(t) \leftarrow (p^2 - p - 1) \cdot \bar{y}(p)$$

(the image of the sum of the functions is the sum of their images).

1060. Find the image of $y'(t) + y(t) + \int_0^t y(\tau)\, d\tau$, if $y(0) = 1$ and $\bar{y}(p) \leftarrow y(t)$.

Solution. In accordance with the theorem on differentiation and integration of the original function we have

$$y'(t) \leftarrow p \cdot \bar{y}(p) - y(0) = p \cdot \bar{y}(p) - 1, \quad \int_0^t y(\tau)\, d\tau \leftarrow \frac{\bar{y}(p)}{p}.$$

From this we find

$$y'(t) + y(t) + \int_0^t y(\tau)\, d\tau \leftarrow p\bar{y}(p) - 1 + \bar{y}(p) + \frac{\bar{y}(p)}{p}$$

$$= \frac{p^2 + p + 1}{p} \cdot \bar{y}(p) - 1.$$

1061. Find the convolution of the functions t and $\cos t$ and its image.

1062. Using the convolution theorem, recover the original function of $\bar{f}(p) = \dfrac{p^2}{(p^2+1)^2}$.

1063. Find the image of $F(t) = y(t) - 2y'(t)$ if $y(0) = 0$, $y(t) \to \bar{y}(p)$.

1064. Find the image of $F(t) = y'''(t) - y''(t) + 2y'(t) - 2y(t)$ if $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$, $y(t) \to \bar{y}(p)$.

1065. Find the image of $F(t) = y'(t) - \int_0^t y(\tau)\,d\tau$, if $y(0) = 0$, $y(t) \to \bar{y}(p)$.

8.4. Application of Operational Calculus to Solution of Some Differential and Integral Equations

If we are given a linear differential equation of the nth order with constant coefficients

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = f(t),$$

whose right-hand side is the original function, then the solution of that equation satisfying the arbitrary initial conditions of the form $y(0) = y_0$, $y'(0) = y_0$, ..., $y^{(n-1)}(0) = y_0^{(n-1)}$ (that is, the solution of the arbitrary Cauchy problem posed for that equation, with the initial conditions for $t = 0$) is an original function as well. Designating the image of this solution by $\bar{y}(p)$, we find the image of the left-hand side of the original differential equation and, equating it to the image of the function $f(t)$, we arrive at the so-called *representing equation* which is always a linear algebraic equation with respect to $\bar{y}(p)$. Determining $\bar{y}(p)$ by this equation, we recover the original $y(t)$.

The same method of passing to the representing equation enables us to easily find the solution of integral equations of the form

$$\int_0^t K(t-\tau)\,y(\tau)\,d\tau = f(t); \quad y(t) = f(t) + \int_0^t K(t-\tau)\,y(\tau)\,d\tau,$$

in which the functions $K(t)$ and $f(t)$ are originals since the integral appearing in these equations is a convolution of the functions $y(t)$ and $K(t)$.

These integral equations is a special case of *Volterra's integral equations of the 1st and the 2nd kind* whose general form can be obtained by the substitution of a certain function of two arguments $K(t, \tau)$ for the function $K(t - \tau)$ known as the *kernel* of the integral equation.

1066. Solve the differential equation $y'' - 2y' - 3y = e^{3t}$ if $y(0) = 0$, $y'(0) = 0$.

Solution. We pass to the images:

$$p^2\bar{y} - p\cdot y(0) - y'(0) - 2(p\bar{y} - y(0)) - 3\bar{y} = \frac{1}{p-3},$$

or

$$p^2\cdot\bar{y} - 2p\bar{y} - 3\bar{y} = \frac{1}{p-3}; \quad \bar{y} = \frac{1}{(p+1)(p-3)^2}.$$

We decompose this rational fraction into partial fractions:

$$\frac{1}{(p+1)(p-3)^2} = \frac{A}{(p-3)^2} + \frac{B}{p-3} + \frac{C}{p+1},$$

$$1 = A(p+1) + B(p-3)(p+1) + C(p-3)^2.$$

Setting $p = -1$, we get $1 = 16C$, i.e. $C = 1/16$; for $p = 3$ we have $1 = 4A$, i.e. $A = 1/4$. Comparing the coefficients in p^2, we get $0 = B + C$, i.e. $B = -C = -1/16$. Consequently,

$$\bar{y} = \frac{1}{4(p-3)^2} - \frac{1}{16(p-3)} + \frac{1}{16(p+1)},$$

whence

$$y = \frac{1}{4}te^{3t} - \frac{1}{16}e^{3t} + \frac{1}{16}e^{-t}.$$

1067. Solve the equation $y'' + y' - 2y = e^{-t}$, if $y(0) = 0$, $y'(0) = 1$.

Solution. We pass to the images:

$$[p^2\bar{y} - p\cdot y(0) - y'(0)] + [p\cdot\bar{y} - y(0)] - 2\bar{y} = \frac{1}{p+1}.$$

After simple transformations, we get

$$\bar{y} = \frac{p+2}{(p+1)(p^2+p-2)} = \frac{1}{p^2-1}.$$

Hence we have $y = \sinh t$.

1068. Solve the integral equation $y = \int_0^t y\,dt + 1$.

Solution. We construct the representing equation:

$$\bar{y} = \frac{\bar{y}}{p} + \frac{1}{p}, \quad \bar{y}(p-1) = 1, \quad \bar{y} = \frac{1}{p-1}.$$

Consequently, $y = e^t$.

1069. Solve the integral equation $\int_0^t y(\tau)\sin(t-\tau)\,d\tau = 1 - \cos t$.

Solution. The left-hand side of the equation is the convolution of the functions $y(t)$ and $\sin t$. Passing to the images, we get

$$\bar{y}(p)\cdot\frac{1}{p^2+1} = \frac{1}{p} - \frac{p}{p^2+1} = \frac{1}{(p^2+1)\cdot p}.$$

Consequently, $\bar{y}(p) = \dfrac{1}{p}$ and $y(t) = 1$.

1070. Solve the integral equation

$$\int_0^t y(\tau)e^{t-\tau}\,d\tau = y(t) - e^t.$$

Solution. The left-hand side of the equation is the convolution of the functions $y(t)$ and e^t. We pass to the images:

$$\bar{y}(p)\cdot\frac{1}{p-1} = \bar{y}(p) - \frac{1}{p-1}, \quad \bar{y}(p)\cdot\frac{1}{p-1} - \bar{y}(p) = -\frac{1}{p-1}.$$

Consequently, $\bar{y}(p) = \dfrac{1}{p-2}$, i.e. $y(t) = e^{2t}$.

1071. Solve the system of equations

$$\begin{cases} \dfrac{dx}{dt} = x + 2y, \\ \dfrac{dy}{dt} = 2x + y + 1, \end{cases}$$

if $x(0) = 0$, $y(0) = 5$.

Solution. Passing to the images, we get

$$\begin{cases} p\cdot\bar{x}(p) = \bar{x}(p) + 2\bar{y}(p), \\ p\cdot\bar{y}(p) - 5 = 2\bar{x}(p) + \bar{y}(p) + \dfrac{1}{p}. \end{cases}$$

Solving the system with respect to $\bar{x}$ and $\bar{y}$, we obtain

$$\bar{x}(p) = \frac{10p+2}{p(p+1)(p-3)}, \quad \bar{y}(p) = \frac{5p^2-4p-1}{p(p+1)(p-3)}.$$

To determine x, we make use of the expansion theorem and Eq. (5) in 8.2:

$$u(p) = 10p + 2, \quad v(p) = p^3 - 2p^2 - 3p, \quad v'(p) = 3p^2 - 4p - 3,$$

$$p_1 = 0, \quad p_2 = -1, \quad p_3 = 3;$$

$$\frac{u(p_1)}{v'(p_1)} = \frac{u(0)}{v'(0)} = -\frac{2}{3}, \quad \frac{u(p_2)}{v'(p_2)} = \frac{u(-1)}{v'(-1)} = -2,$$

$$\frac{u(p_3)}{v'(p_3)} = \frac{u(3)}{v'(p_3)} = \frac{8}{3}.$$

Thus it follows that $x = -\dfrac{2}{3} - 2e^{-t} + \dfrac{8}{3}e^{3t}$. By analogy we find $y = \dfrac{1}{3} + 2e^{-t} + \dfrac{8}{3}e^{3t}$.

Solve the following differential equations.

1072. $y' - 2y = 0$; $y(0) = 1$.
1073. $y' + y = e^t$; $y(0) = 0$.
1074. $y'' - 9y = 0$; $y(0) = y'(0) = 0$.
1075. $y'' + y' - 2y = e^t$; $y(0) = -1$, $y'(0) = 0$.
1076. $y''' - 6y'' + 11y' - 6y = 0$; $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$.

Solve the following systems of equations.

1077. $\dfrac{dx}{dt} = 2y$, $\dfrac{dy}{dt} = 2x$; $x(0) = 2$, $y(0) = 2$.
1078. $\dfrac{dx}{dt} = 3x + 4y$, $\dfrac{dy}{dt} = 4x - 3y$; $x(0) = y(0) = 1$.

Solve the following integral equations.

1079. $\int\limits_0^t y(\tau)(t - \tau)^2\, d\tau = \dfrac{1}{3} t^3$.

1080. $\int\limits_0^t y(\tau) \cos(t - \tau)\, d\tau = 1 - \cos t$.

8.5. The General Inversion Formula

Suppose the function $f(t)$ possesses the following properties:
1°. $f(t) \equiv 0$ for $t < 0$.
2°. $|f(t)| < Me^{s_0 t}$ for $t > 0$, where $M > 0$ and s_0 are some real constants.
3°. On any finite closed interval $[a, b]$ $(0 < a < b)$ the function satisfies the Dirichlet conditions.

In that case, the function $\bar{f}(p)$ specified by the equality $\bar{f}(p) = \int\limits_0^\infty e^{-pt} f(t)\, dt$ is analytic in the half-plane $\operatorname{Re} p \geqslant s_1 > s_0$.

Then the *inversion formula* (*Riemann-Mellin's formula*) holds true here:

$$f(t) = \frac{1}{2\pi i} \cdot \lim_{\omega \to \infty} \int\limits_{\sigma - i\omega}^{\sigma + i\omega} e^{pt} \bar{f}(p)\, dp, \quad \text{or} \quad f(t) = \frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} e^{pt} \bar{f}(p)\, dp,$$

which allows the original function $f(t)$ to be recovered from its image $\bar{f}(p)$, σ being an arbitrary number such that $\sigma > s_0$.

If $|\bar{f}(p)| < CR^{-k}$, where $p = Re^{i\theta}$, $-\pi \leqslant \theta < \pi$, $R > R_0$, R_0, C and $k > 0$

are constants, then the integral $\int\limits_{\sigma - i\infty}^{\sigma + i\infty} e^{pt}\bar{f}(p)\,dp$ in the inversion formula can be replaced by the integral $\int\limits_{\gamma} e^{pt}\bar{f}(p)\,dp$, where γ is a circle, with centre at the origin, containing in its interior all the poles of the function $F(p) = e^{pt}\bar{f}(p)$. Consequently,

$$f(t) = \frac{1}{2\pi i}\int\limits_{\gamma} e^{pt}\bar{f}(p)\,dp.$$

Applying the fundamental theorem on residues, we get

$$f(t) = \frac{1}{2\pi i}\cdot 2\pi i(r_1 + r_2 + \ldots + r_m),$$

where $r_1, r_2, \ldots, r_m$ are the residues of the function $F(p)$ about the poles. Thus, $f(t) = \sum\limits_{j=1}^{m} r_j$. This formula written out for the fractional-rational image is nothing other than formulas (4) and (5) of 8.2.

1081. Recover the original function from the image $\bar{f}(p) = \dfrac{1}{(p-1)^3}$.

Solution. We find the residue of the function $F(p) = \dfrac{e^{pt}}{(p-1)^3}$:

$$r = \frac{1}{2!}\lim_{p\to 1}\frac{d^2}{dp^2}[(p-1)^3\cdot F(p)]$$

$$= \frac{1}{2!}\lim_{p\to 1}\frac{d^2}{dp^2}(e^{pt}) = \frac{1}{2!}\lim_{p\to 1} t^2 e^{pt} = \frac{t^2}{2!}e^t.$$

Consequently, $f(t) = \dfrac{t^2 e^t}{2!}$.

1082. Recover the original function from its image

$$\bar{f}(p) = \frac{p}{(p+1)(p+2)(p+3)(p+4)}.$$

Solution. We have

$$F(p) = \frac{pe^{pt}}{(p+1)(p+2)(p+3)(p+4)}, \quad r_1 = \lim_{p\to -1}(p+1)\cdot F(p) = -\frac{1}{6}e^{-t},$$

$$r_2 = \lim_{p\to -2}(p+2)\cdot F(p) = e^{-2t}, \quad r_3 = \lim_{p\to -3}(p+3)\cdot F(p) = -\frac{3}{2}\cdot e^{-3t},$$

$$r_4 = \lim_{p\to -4}(p+4)\cdot F(p) = \frac{2}{3}\cdot e^{-4t}.$$

Consequently, $f(t) = -\dfrac{1}{6}e^{-t} + e^{-2t} - \dfrac{3}{2}e^{-3t} + \dfrac{2}{3}e^{-4t}$.

Recover the original functions from the given images:

1083. $\bar{f}(p) = \dfrac{4 - p - p^2}{p^3 - p^2}$.

1084. $\bar{f}(p) = \dfrac{1}{p^4 - 6p^3 + 11p^2 - 6p}$.

1085. $\bar{f}(p) = \dfrac{1}{(p-1)^3(p^3+1)}$.

8.6. Application of Operational Calculus to Solutions of Some Equations of Mathematical Physics

Let us consider the solutions of certain equations of mathematical physics, wave equations and heat equations. To solve these equations, it is most expedient to use the methods of operational calculus based on the idea of using the Laplace transformation. We shall restrict our discussion to the case when the desired function u depends on two independent variables x and t, where x is a space coordinate and t is time. The problem is non-stationary because we seek the solution which depends essentially on the initial conditions and, therefore, the physical processes proceed under unsteady conditions.

Assume that the partial differential equation is of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial u}{\partial x} + Cu + A_1\frac{\partial^2 u}{\partial t^2} + B_1\frac{\partial u}{\partial t} = 0, \tag{1}$$

where A, B, C, A_1, B_1 are continuous functions of x given in the interval $0 \leqslant x \leqslant l$ (it can be assumed that $A > 0$).

Let us consider two principal cases: (1) $A_1 < 0$, which corresponds to the hyperbolic type of the equation; (2) $A_1 = 0$, $B_1 < 0$, which corresponds to the parabolic type. Under these conditions, the non-stationary problem we have posed can be formulated as follows: it is required to find the solution $u(x, t)$ of Eq. (1) for $0 \leqslant x \leqslant l$ and $t \geqslant 0$ satisfying the initial conditions $u(x, t)\Big|_{t=0} = \varphi(x)$, $\dfrac{\partial u}{\partial t}\Big|_{t=0} = \psi(x)$ (the second condition is given for $A_1 < 0$ and the boundary conditions $u(x, t)\Big|_{x=0} = f(t)$, $\alpha\dfrac{\partial u}{\partial x}\Big|_{x=l} + \beta\dfrac{\partial u}{\partial t}\Big|_{x=l} = \gamma u(x, t)\Big|_{x=l}$, where α, β, γ are constants. Note that the second boundary condition becomes unnecessary as $l \to \infty$.

It is also assumed that $u(x, t)$, $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial^2 u}{\partial x^2}$, which are the functions of t, can serve as the original functions and that the images of the desired function and of its derivatives have the form

$$\bar{u}(x, p) = \int_0^\infty e^{-pt} u(x, t)\, dt, \quad \int_0^\infty e^{-pt} \frac{\partial u}{\partial x}\, dt$$

$$= \frac{d}{dx} \int_0^\infty u e^{-pt}\, dt = \frac{d\bar{u}}{dx}, \quad \int_0^\infty e^{-pt} \frac{\partial^2 u}{\partial x^2}\, dt = \frac{d^2}{dx^2} \int_0^\infty u e^{-pt} dt = \frac{d^2 \bar{u}}{dx^2}.$$

Here p is considered only as a parameter. As to the images of $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial t^2}$, they are

$$\frac{\partial u}{\partial t} \div p\bar{u} - u(x, 0), \quad \frac{\partial^2 u}{\partial t^2} \div p^2 \bar{u} - pu(x, 0) - \frac{\partial u(x, 0)}{t},$$

or, the initial conditions being taken into account,

$$\frac{\partial u}{\partial t} \div p\bar{u} - \varphi(x), \quad \frac{\partial^2 u}{\partial t^2} \div p^2 \bar{u} - p\varphi(x) - \psi(x);$$

the boundary conditions are $\bar{u}\big|_{x=0} = \bar{f}(p)$, $\left[\alpha \frac{d\bar{u}}{dx} + \beta(p\bar{u} - \varphi(x))\right]_{x=l} = \gamma \bar{u}\big|_{x=l}$.

Thus, the solution of Eq. (1) reduces to the solution of the operator equation

$$A \frac{d^2 \bar{u}}{dx^2} + B \frac{d\bar{u}}{dx} + M\bar{u} + N = 0, \tag{2}$$

where $M = C - A_1 p^2 + B_1 p$, $N = -A_1 p\varphi - A_1 \psi - B_1 \varphi$ (p being the parameter), which is an ordinary second-order differential equation.

Having found the image of the desired function $u(x, t)$, we can recover the original function with the aid of the table in 8.1.2 or the Riemann-Mellin inversion formula.

1086. The ends of the string $x = 0$ and $x = l$ are rigidly fixed. The initial deviation is given by the equation $u(x, 0) = A \sin(\pi x/l)$, $0 \leqslant x \leqslant l$; the initial velocity is zero. Find the deviation $u(x, t)$ for $t > 0$.

Solution. The differential equation of the problem has the form $\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2} \cdot \frac{\partial^2 u}{\partial t^2} = 0$. The initial conditions are $u(x, 0) = A \sin \frac{\pi x}{l}$, $\frac{\partial u(x, 0)}{\partial t} = 0$; the boundary conditions are $u(0, t) = u(l, t) = 0$. We write the corresponding operator equation

$$\frac{d^2 \bar{u}}{dx^2} - \frac{p^2}{a^2} \cdot \bar{u} = -pA \cdot \frac{1}{a^2} \sin \frac{\pi x}{l};$$

the boundary conditions are $\bar{u}|_{x=0} = \bar{u}|_{x=l} = 0$.

The general solution of the homogeneous equation is of the form

$$\bar{u} = C_1 e^{(p/a)x} + C_2 e^{-(p/a)x},$$

and the particular solution of the nonhomogeneous equation will be sought in the form

$$\bar{v} = \tilde{C}_1 \cos\frac{\pi x}{l} + \tilde{C}_2 \sin\frac{\pi x}{l},$$

that is,

$$\begin{array}{c|lcl}
-\dfrac{p^2}{a^2} & \bar{v} & = & \tilde{C}_1 \dfrac{\pi x}{l} + \tilde{C}_2 \sin\dfrac{\pi x}{l} \\[2ex]
0 & \bar{v}' & = & -\tilde{C}_1 \cdot \dfrac{\pi}{l} \sin\dfrac{\pi x}{l} + \tilde{C}_2 \cdot \dfrac{\pi}{l} \cos\dfrac{\pi x}{l} \\[2ex]
1 & \bar{v}'' & = & -\tilde{C}_1 \cdot \dfrac{\pi^2}{l^2} \cos\dfrac{\pi x}{l} - \tilde{C}_2 \cdot \dfrac{\pi^2}{l^2} \sin\dfrac{\pi x}{l}
\end{array}$$

$$-\frac{pA}{a^2}\sin\frac{\pi x}{l} = -\tilde{C}_2\left(\frac{p^2}{a^2} + \frac{\pi^2}{l^2}\right)\sin\frac{\pi x}{l} - \tilde{C}_1\left(\frac{p^2}{a^2} + \frac{\pi^2}{l^2}\right)\cos\frac{\pi x}{l}.$$

Hence $\tilde{C}_1 = 0$, $\tilde{C}_2 = \dfrac{pA}{p^2 + \pi^2 a^2/l^2}$. Thus, the general solution of the operator equation is

$$\bar{u}(x, p) = C_1 e^{(p/a)x} + C_2 e^{-(p/a)x} + \frac{Ap}{p^2 + a^2\pi^2/l^2} \cdot \sin\frac{\pi x}{l}.$$

Taking the boundary conditions into account, we obtain $\bar{u}(x, p) = \dfrac{Ap}{p^2 + a^2\pi^2/l^2}\sin\dfrac{\pi x}{l}$. The original of this image is the function

$$u(x, t) = A\cos\frac{\pi a t}{l}\sin\frac{\pi x}{l}.$$

1087. Find the solution of the equation $\dfrac{\partial u}{\partial t} = a^2\dfrac{\partial^2 u}{\partial x^2}$ satisfying the following initial and boundary conditions: $u(x, 0) = 0$; $u(0, t) = u_0$, $0 < x < \infty$, $t > 0$.

Solution. Let us write the operator equation

$$\frac{d^2\bar{u}(x, p)}{dx^2} - \frac{p}{a^2}\cdot\bar{u}(x, p) = 0.$$

The general solution of this equation is

$$\bar{u}(x, p) = C_1 e^{-x\sqrt{p}/a} + C_2 e^{x\sqrt{p}/a}.$$

By the hypothesis, the functions $u(x, t)$ and $\bar{u}(x, p)$ are bounded as $x \to \infty$, and therefore $C_2 = 0$.

Using the boundary condition $\bar{u}(x, p)|_{x=0} = u_0/p$, we find the arbitrary constant $C_1 = u_0/p$. Then we have $\bar{u} = (u_0/p)e^{-x\sqrt{p}/a}$. Using the relation $\frac{1}{p} e^{-a\sqrt{p}} \leftarrow \mathrm{Erf}\left(\frac{a}{2\sqrt{t}}\right)$, we recover the original for the function $\bar{u}(x, p)$.

The solution of the given equation has the form

$$u(x, t) = u_0 \cdot \mathrm{Erf}\left(\frac{x}{2a\sqrt{t}}\right),$$

where, as is known,

$$\mathrm{Erf}\, t = \frac{2}{\sqrt{\pi}} \cdot \int_t^{\infty} e^{-\tau^2}\, d\tau = 1 - \frac{2}{\sqrt{\pi}} \int_0^t e^{-\tau^2}\, d\tau = 1 - \mathrm{erf}\, t.$$

Consequently,

$$u(x, t) = u_0 \cdot \mathrm{Erf}\left(\frac{x}{2a\sqrt{t}}\right) = u_0\left(1 - \frac{2}{\sqrt{\pi}} \cdot \int_0^{x/(2a\sqrt{t})} e^{-\tau^2}\, d\tau\right).$$

1088. Find the solution of the heat equation $\frac{\partial^2 u}{\partial x^2} = \alpha^2 \frac{\partial u}{\partial t}$ satisfying the following initial and boundary conditions: $u(x, 0) = A \sin(n\pi x/l)$, $0 \leqslant x \leqslant l$; $u(0, t) = u(l, t) = 0$.

Solution. The operator equation corresponding to the given partial differential equation has the form

$$\frac{d^2\bar{u}}{dx^2} - \alpha^2 p \cdot \bar{u} = -\alpha^2 A \sin\frac{n\pi x}{l},$$

and its general solution is

$$\bar{u}(x, p) = C_1 e^{-\alpha\sqrt{p}x} + C_2 e^{\alpha\sqrt{p}x} + \frac{A}{p + (n^2\pi^2)/(\alpha^2 l^2)} \sin\frac{n\pi x}{l}.$$

Taking the boundary conditions $\bar{u}|_{x=0} = \bar{u}|_{x=l} = 0$ into account, we get

$$\bar{u}(x, p) = \frac{A}{p + (n^2\pi^2)/(\alpha^2 l^2)} \cdot \sin\frac{n\pi x}{l}.$$

The original of this solution is $u(x, t) = Ae^{-n^2\pi^2 t/(\alpha^2 l^2)} \sin\frac{n\pi x}{l}$.

1089. Find the solution of the equation $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}$ satisfying the initial condition $u(x, 0) = 0$ $(x > 0)$ and the boundary conditions $u(0, t) = 0$, $u(h, t) = u_0$.

Solution. We write the operator equation

$$\frac{d^2\bar{u}}{dx^2} - \frac{p}{a}\cdot\bar{u} = 0,$$

which must be solved under the conditions $\bar{u}(0, t) = 0, \bar{u}(h, t) = u_0/p$. We write the general solution of the operator equation in the form

$$\bar{u}(x,p) = A\cosh\sqrt{p/a}\,x + B\sinh\sqrt{p/a}\,x.$$

Using the boundary conditions, we find the constants A and B. Then we have

$$A = 0, \quad \frac{u_0}{p} = B\sinh\sqrt{p/a}\,h, \quad \text{i.e.} \quad B = \frac{u_0}{p\cdot\sinh\sqrt{p/a}\,h}.$$

Substituting the values of A and B into equation ($*$), we get $\bar{u} = \frac{u_0}{p}\cdot\frac{\sinh\sqrt{p/a}\,x}{\sinh\sqrt{p/a}\,h}$. In accordance with the Riemann-Mellin inversion formula we have

$$u(x,t) = \frac{u_0}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{pt}\frac{\sinh\sqrt{p/a}\,x}{\sinh\sqrt{p/a}\,h}\cdot\frac{dp}{p}. \qquad (**)$$

To calculate the integral, we shall find the residues of the integrand. Equating the denominator to zero and taking into consideration that the roots of the hyperbolic sine are pure imaginary and equal to the number which is of multiplicity π, we find

$$\sinh\sqrt{p/a}\,h = 0, \quad \sqrt{p_k/a}\,h = ik\pi, \quad p_k = -k^2\pi^2 a/h^2 \quad (k \in N).$$

All the k poles are simple, nonzero; therefore, applying Cauchy's residue theorem, we obtain

$$u(x,t) = \sum_{(p_k)} \operatorname{res} F(p)e^{pt}, \quad \text{where} \quad F(p) = \frac{M(p)}{p\cdot N(p)} = \frac{\sinh\sqrt{p/a}\,x}{p\cdot\sinh\sqrt{p/a}\,h},$$

where the degree of $M(p)$ does not exceed the degree of $N(p)$. Then,

$$\frac{M(p)}{pN(p)} \div \frac{M(0)}{N(0)} + \sum_{k=1}^{\infty}\frac{M(p_k)}{p_k\cdot N'(p_k)}e^{p_k t},$$

where $\frac{M(0)}{N(0)} = \lim_{p\to 0}\frac{\sinh\sqrt{p/a}\;x}{\sinh\sqrt{p/a}\;h} = \frac{x}{h}$, and $\frac{M(p_k)}{p_k N'(p_k)} = -\frac{2i\sinh(ik\pi x/h)}{k\pi\cosh(ik\pi)}$.
Expressing the hyperbolic functions in terms of spherical functions, we get

$$\frac{2\sin(k\pi x/h)}{\pi k\cos(k\pi)} = (-1)^k\cdot\frac{2}{\pi}\cdot\frac{\sin(k\pi x/h)}{k}.$$

Thus, equation ($**$) assumes the form

$$u(x, t) = u_0\left[\frac{x}{h} + \frac{2}{\pi} \sum_{k=1}^{\infty} (-1)^k \cdot e^{-ak^2\pi^2t/h^2} \cdot \frac{\sin(k\pi x/h)}{k}\right].$$

1090. Find the solution of the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$ satisfying the initial conditions $u(x, 0) = A\cos\frac{n\pi x}{l}$, $\frac{\partial u(x, 0)}{\partial t} = 0$, $0 \leqslant x \leqslant l$ and the boundary conditions $\frac{\partial u(0, t)}{\partial x} = \frac{\partial u(l, t)}{\partial x} = 0$.

1091. Find the solution of the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$ satisfying the initial conditions $u(x, 0) = 0$, $\frac{\partial u(x, 0)}{\partial t} = B\sin\frac{n\pi x}{l}$, $0 \leqslant x \leqslant l$ and the boundary conditions $u(0, t) = u(l, t) = 0$.

1092. Find the solution of the heat equation $\frac{\partial^2 u}{\partial x^2} = \alpha^2\frac{\partial u}{\partial t}$, satisfying the conditions $u(x, 0) = 0$, $x \geqslant 0$; $u(0, t) = A$, $\lim_{x\to\infty} u(x, t) = 0$.

Chapter 9

Calculation Methods

9.1. Approximate Solution of Equations

A basic step in approximating real roots of the equation $f(x) = 0$ is *to isolate the root*, that is, find the interval which does not contain any other roots of the given equation. We shall assume that in the closed interval $[a, b]$ the function $f(x)$ is continuous together with its derivatives $f'(x)$ and $f''(x)$, the values $f(a)$ and $f(b)$ of the function are of unlike signs at the end points of the interval, i.e. $f(a) \cdot f(b) < 0$, and both derivatives $f'(x)$ and $f''(x)$ retain their signs throughout the interval $[a, b]$.

Since the real roots of the equation $f(x) = 0$ are the abscissas of the intersection points of the curve $y = f(x)$ with the x-axis, we can perform isolation of the root by graphical methods. Instead of the equation $y = f(x)$ we can take the equation $y = kf(x)$, where k is a constant nonzero quantity, the equations $f(x) = 0$ and $kf(x) = 0$ being equivalent.

The constant quantity k can be taken such that the ordinates of the points of the graph should not be too large or, conversely, that the graph should not lie too close to the x-axis. Sometimes it is useful to write the equation $f(x) = 0$ in the form $\varphi(x) = \psi(x)$. The real roots of the original equation are the abscissas of the intersection points of the graphs of the functions $y = \varphi(x)$ and $y = \psi(x)$.

9.1.1. The chord method. Suppose we have to calculte the real root of the equation $f'(x) = 0$ isolated on the interval $[a, b]$. Let us consider the graph of the function $y = f(x)$. Assume that $f(a) < 0$ and $f(b) > 0$. We connect by the chord the points $A[a; f(a)]$ and $B[b; f(b)]$ of the graph and take for the approximate value of the desired root the abscissa x_1 of the intersection point of the chord AB and the x-axis. This approximate value can be found from the formula

$$x_1 = a - \frac{(b - a)f(a)}{f(b) - f(a)},$$

where x_1 belongs to the interval (a, b). Suppose, for instance, that $f(x_1) < 0$; then we can assume $[x_1, b]$ to be the new (narrower) interval of isolation of the root. Connecting the points $A_1[x_1; f(x_1)]$ and $B[b; f(b)]$, we obtain at the point of intersection of the chord and the x-axis the second approximation x_2, which we shall calculate by the formula

$$x_2 = x_1 - \frac{(b - x_1)f(x_1)}{f(b) - f(x_1)} ,$$

and so on. The number sequence $a, x_1, x_2, \ldots$ tends to the desired root of the equation $f(x) = 0$. The process must be continued until the decimal digits we want to re-

tain in the answer cease varying (that is, until we reach the required degree of accuracy).

If $\bar{x}$ is an exact root of the equation $f(x) = 0$, isolated on the interval $[a, b]$, and ξ is an approximate value of the root found by the chord method, then the estimation of the approximation error is

$$|\bar{x} - \xi| < -\frac{f(a) \cdot f(b)}{2} \cdot \max_{[a,b]} \left| \frac{f''(x)}{[f'(x)]^3} \right| .$$

9.1.2. The tangent method (Newton's method). Suppose that the real root of the equation $f(x) = 0$ is isolated on the interval $[a, b]$. We shall assume that all the restrictions concerning $f(x)$, formulated above, are also valid here. Let us take on the interval $[a, b]$ a number x_0 such that $f(x_0)$ has the same sign as $f''(x_0)$, i.e. $f(x_0) \cdot f''(x_0) > 0$ (in particular, the end point of $[a, b]$ at which this condition is satisfied can be taken as x_0). Let us draw a tangent to the curve $y = f(x)$ at the point $M_0[x_0; f(x_0)]$ and assume the abscissa of the intersection point of that tangent and the x-axis to be the approximate value of the root. This approximate value can be found from the formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Let us repeat the procedure at the point $M_1[x_1; f(x_1)]$ and find

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

and so on. The sequence $x_0, x_1, x_2, \ldots$ obtained in this way has the sought-for root as its limit.

The following inequality can be used to estimate the approximation error in calculating the value of the root by the Newton method:

$$|\bar{x} - \xi| < \frac{[f(\xi)]^2}{2} \cdot \max_{[a,b]} \left| \frac{f''(x)}{[f'(x)]^3} \right| .$$

9.1.3. The combined method of chords and tangents. Suppose it is required to find the real root of the equation $f(x) = 0$ isolated on the interval $[a, b]$. The values $f(a)$ and $f(b)$ are supposed to have different signs and each of the derivatives retains a definite sign on the isolation interval. Let us take on the interval $[a, b]$ a point x_0 such that $f(x_0)$ and $f''(x_0)$ (with x belonging to the isolation interval) are of like signs.

We make use of the formulas of the methods of chords and tangents:

$$x_{11} = x_0 - \frac{f(x_0)}{f'(x_0)}; \quad x_{12} = a - \frac{(b-a)f(a)}{f(b) - f(a)}.$$

The quantities x_{11} and x_{12} belong to the isolation interval, with $f(x_{11})$ and $f(x_{12})$ being of unlike signs.

We construct a new pair of approximate values of the root:

$$x_{21} = x_{11} - \frac{f(x_{11})}{f'(x_{11})}; \quad x_{22} = x_{11} - \frac{(x_{12} - x_{11})f(x_{11})}{f(x_{12}) - f(x_{11})}.$$

The points x_{21} and x_{22} of the number axis lie between the points x_{11} and x_{12}, with $f(x_{21})$ and $f(x_{22})$ being of unlike signs.

Let us now calculate the values

$$x_{31} = x_{21} - \frac{f(x_{21})}{f'(x_{21})}; \quad x_{32} = x_{21} - \frac{(x_{22} - x_{21})f(x_{21})}{f(x_{22}) - f(x_{21})},$$

and so on.

Each of the sequences

$$x_1, x_{21}, x_{31}, \ldots, x_{n1}, \ldots; x_{12}, x_{22}, x_{23}, \ldots, x_{n2}, \ldots$$

tends to the sought-for root, one of them increasing monotonically, and the other decreasing monotonically. Assume, for example, that $x_{n1} < \bar{x} < x_{n2}$, then $0 < \bar{x} - x_{n-1} < x_{n2} - x_{n1}$. Taking first a sufficiently small ε, we can, by increasing n, satisfy the inequality $x_{n2} - x_{n1} < \varepsilon$; consequently, the inequality $\bar{x} - x_{n1} < \varepsilon$ will be satisfied for the same value of n. Thus, x_{n1} is an approximate value of the root $\bar{x}$ calculated with an error not exceeding ε.

Thus, for instance, to find the approximate value of the root $\bar{x}$, with an accuracy to within 0.001, we must define n such that the values x_{n1} and x_{n2}, calculated to within 0.001, coincide.

9.1.4. The iteration method. If the given equation has been reduced to the form $x = \varphi(x)$, where $|\varphi'(x)| \leqslant r < 1$ everywhere on the closed interval $[a, b]$ on which the original equation possesses a single root, then, proceeding from a certain initial value x_0, belonging to the interval $[a, b]$, we can construct the sequence

$$x_1 = \varphi(x_0), \quad x_2 = \varphi(x_1), \ \ldots, \quad x_n = \varphi(x_{n-1}).$$

The limit of this sequence is the only root of the equation $f(x) = 0$ on the interval $[a, b]$. The error of the approximate value x_n of the root $\bar{x}$, found by the iteration method, can be estimated by the inequality

$$|\bar{x} - x_n| < \frac{r}{1 - r} |x_n - x_{n-1}|.$$

To find the approximate value of the root with an error not exceeding ε, it is sufficient to define n such that the following inequality holds true:

$$|x_n - x_{n-1}| < \frac{r - 1}{r} \varepsilon.$$

9.1.5. The trial-and-error method. The isolation interval of a real root can always be decreased by dividing it, say, in half and determining the subinterval at whose end points the function $f(x)$ changes sign. Then, the subinterval obtained is again divided in half, and so on, the procedure being carried on until the decimal digits retained in the answer cease varying.

1093. Using the chord method, find a positive root of the equation $x^4 - 2x - 4 = 0$ with an accuracy to within 0.01.

Solution. The positive root belongs to the interval (1; 1.7) since $f(1) = -5 < 0$ and $f(1.7) = 0.952 > 0$.

The first approximate value of the root is

$$x_1 = 1 - \frac{(1.7 - 1) \cdot f(1)}{f(1.7) - f(1)} = 1.588.$$

Since $f(1.588) = -0.817 < 0$, we again apply the chord method to the interval (1.588; 1.7):

$$x_2 = 1.588 - \frac{(1.7 - 1.588) \cdot f(1.588)}{f(1.7) - f(1.588)} = 1.639; \quad f(1.639) = -0.051 < 0.$$

We find the third approximate value:

$$x_3 = 1.639 - \frac{(1.7 - 1.639) \cdot f(1.639)}{f(1.7) - f(1.639)} = 1.642; \quad f(1.642) = -0.016 < 0,$$

and then the fourth approximate value:

$$x_4 = 1.642 - \frac{(1.7 - 1.642) \cdot f(1.642)}{f(1.7) - f(1.642)} = 1.643; \quad f(1.643) = 0.004 > 0.$$

Consequently, with an accuracy to within 0.01, the desired root is equal to 1.64.

1094. Solve the preceding example by the tangent method.

Solution. Here $f(x) = x^4 - 2x - 4$, $f'(x) = 4x^3 - 2$, $f''(x) = 12x^2$. Since, at $x_0 = 1.7$, $f(x)$ and $f''(x)$ have the same sign, namely, $f(1.7) = 0.952 > 0$ and $f''(1.7) > 0$, we use the formula $x_1 = x_0 - f(x_0)/f'(x_0)$, where $f'(1.7) = 4 \cdot 1.7^{3'} - 2 = 17.652$. Then we have

$$x_1 = 1.7 - 0.952/17.652 = 1.646.$$

We again apply the method of tangents and have $x_2 = x_1 - f(x_1)/f'(x_1)$, where $f(x_1) = f(1.646) = 0.048$, $f'(1.646) = 15.838$; hence

$$x_2 = 1.646 - 0.048/15.838 = 1.643.$$

By a similar method we find $f(1.643) = 0.004$; $f'(1.643) = 15.740$, that is,

$$x_3 = x_2 - f(x_2)/f'(x_2) = 1.643 - 0.004/15.740 = 1.6427.$$

Consequently, with an accuracy to within 0.01, the desired root is equal to 1.64.

1095. Using the combined method of chords and tangents, find the approximate value of the root of the equation $x^3 + x^2 - 11 = 0$, isolated on the interval (1, 2), with an accuracy to within 0.001.

Solution. We have $f(x) = x^3 + x^2 - 11$, $f'(x) = 3x^2 + 2x$, $f''(x) = 6x + 2$. In the indicated interval $f''(x) > 0$, therefore, in the method of tangents we assume $x_0 = 2$ to be the first approximation since $f(2) = 1 > 0$;

$$x_{11} = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{1}{16} = 1.9375 \approx 1.94;$$

$$x_{12} = a - \frac{(b-a)f(a)}{f(b) - f(a)} = 1 - \frac{(2-1)(-9)}{1-(-9)} = 1 + \frac{9}{10} = 1.9.$$

The desired root belongs to the interval (1.9; 1.94); we have $f(1.9) = -0.531$; $f(1.94) = 0.065$; $f'(1.94) = 15.172$;

$$x_{21} = 1.94 - \frac{0.065}{15.172} = 1.936;$$

$$x_{22} = 1.9 - \frac{0.04 \cdot (-0.531)}{0.065 + 0.531} = 1.936.$$

Since the values x_{21} and x_{22} calculated with an accuracy to within 0.001 have coincided, the approximate value of the root $\bar{x}$, calculated to within 0.001, is 1.936.

1096. Use the method of iteration to find the approximate value of the root of the equation $2 - \log x - x = 0$ with an accuracy to within 0.001.

Solution. We find the isolation interval of the real root of the equation, represent the given equation in the form $\log x = -x + 2$ and construct the graphs of the functions $y = \log x$ and $y = -x + 2$. The abscissa of the point M of intersection of the graphs lies in the interval [1, 2], therefore we can take $x_0 = 1$ as the initial value of $\bar{x}$ (Fig. 69).

Let us write the original equation as $x = 2 - \log x$. Here $\varphi(x) = 2 - \log x$, $\varphi'(x) = -(\log e)/x$, i.e. $|\varphi'(x)| < 1$ in the interval [1, 2] and, therefore, the iteration method is applicable here. We find the first approximate value:

$$x_1 = 2 - \log x_0 = 2 - \log 1 = 2.$$

Then we find the second and the following approximations:

$$x_2 = 2 - \log x_1 = 2 - \log 2 = 2 - 0.3010 = 1.6990;$$

$$x_3 = 2 - \log 1.6990 = 2 - 0.2302 = 1.7698;$$

$$x_4 = 2 - \log 1.7698 = 2 - 0.2480 = 1.7520;$$

$$x_5 = 2 - \log 1.7520 = 2 - 0.2435 = 1.7565;$$

$$x_6 = 2 - \log 1.7565 = 2 - 0.2445 = 1.7555;$$

$$x_7 = 2 - \log 1.7555 = 2 - 0.2444 = 1.7556.$$

Thus, the sought-for root is $x \approx 1.755$.

1097. Use the trial-and-error method to solve the equation $x^3 + 2x - 7 = 0$ with an accuracy to within 0.01.

Solution. We can use graphical means to define the isolation interval of the real root, constructing the graphs of the functions $y = x^3$ and $y = -2x + 7$ (Fig. 70).

The only point of intersection of the graphs lies in the interval (1, 2). Consequently, the desired root belongs to that interval, i.e. we can assume $a = 1$, $b = 2$. Let us

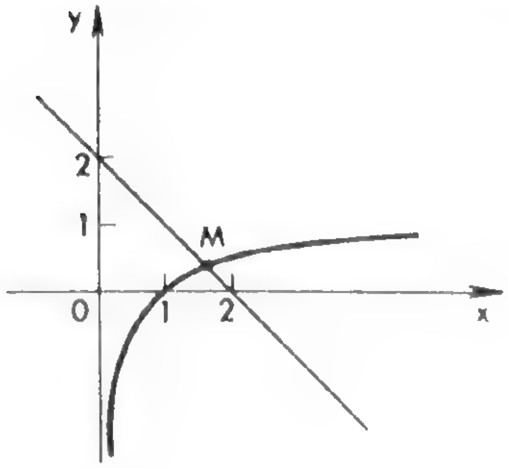


Fig. 69

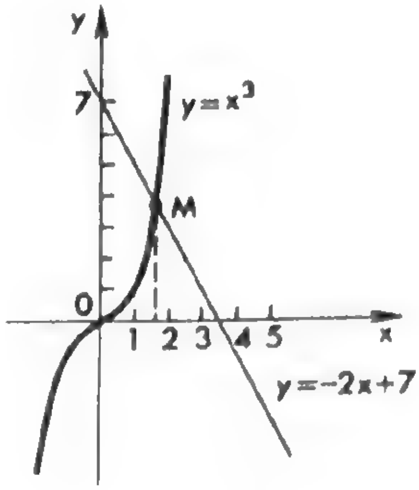


Fig. 70

find the values of the function at the end points of the interval: $f(1) = -4 < 0$; $f(2) = 5 > 0$. Dividing the interval $(1, 2)$ in half, we get $c_1 = (a + b)/2 = (1 + 2)/2 = 1.5$ and calculate $f(c_1) = f(1.5) = -0.625 < 0$. This means that the desired root lies in the interval (1.5; 2).

We assume $c_2 = 1.7; f(c_2) = f(1.7) = 1.313 > 0$, and see that the desired root is in the interval (1.5; 1.7). Now we assume $c_3 = 1.6; f(c_3) = f(1.6) = 0.296 > 0$. As a result, we have managed to make the isolation interval narrower, and the desired root is in the interval (1.5; 1.6).

Continuing the procedure, we obtain

$$c_4 = 1.55;\quad f(c_4) = f(1.55) = -0.176 < 0;$$

the isolation interval is (1.55; 1.6);

$$c_5 = 1.57;\quad f(c_5) = f(1.57) = 0.010 > 0;$$

the isolation interval is (1.55; 1.57);

$$c_6 = 1.56;\quad f(c_6) = f(1.56) = -0.084 < 0;$$

the isolation interval is (1.56; 1.57);

$$c_7 = 1.565;\quad f(c_7) = f(1.565) = -0.037 < 0;$$

the isolation interval is (1.565; 1.57);

$$c_8 = 1.568;\quad f(c_8) = f(1.568) = -0.009 < 0.$$

Thus we have obtained the interval (1.568; 1.57). Hence the desired root $x = 1.57$ with an accuracy to within 0.01.

1098. Use graphical means to determine the isolation intervals of the real roots of the equation $x^3 - 9x^2 + 18x - 1 = 0$.

1099. Use graphical means to determine the isolation intervals of the real roots of the equation $x^3 - 12x + 1 = 0$.

By the method of chords and tangents solve the following equations with an accuracy to within 0.01:

1100. $x^4 + 3x - 20 = 0$. **1101.** $x^3 - 2x - 5 = 0$. **1102.** $x^4 - 3x + 1 = 0$.
1103. $x^3 + 3x + 5 = 0$.
1104. $x^4 + 5x - 7 = 0$ (use the combined chords and tangents method).

By the iteration method solve the following equations with an accuracy to within 0.01:

1105. $x^3 - 12x - 5 = 0$. **1106.** $x^3 - 2x^2 - 4x - 7 = 0$.

Partitioning the isolation interval of the root into subintervals, solve the following equations by the trial-and-error method to within 0.01:

1107. $x + e^x = 0$. **1108.** $x^5 - x - 2 = 0$.

1109. Applying twice the method of chords, find the approximate value of the real root of the equation $x^3 - 10x + 5 = 0$ isolated on the interval $[0; 0.6]$. The approximations x_1 and x_2 must be calculated to the second decimal place. Estimate the error of the approximation x_2.

Solution. We find

$$f(x) = x^3 - 10x + 5; \quad f'(x) = 3x^2 - 10, \quad f''(x) = 6x;$$

$$f(0) = 5; \; f(0.6) = 0.216 - 6 + 5 = -0.784;$$

$$x_1 = 0 - \frac{0.6 \cdot 5}{-0.784 - 5} = \frac{3}{5.784} = 0.52;$$

$$f(0.52) = 0.141 - 5.2 + 5 = -0.059 < 0.$$

The new isolation interval is $(0; 0.52)$. We find the second approximation

$$x_2 = 0 - \frac{0.52 \cdot 5}{-0.059 - 5} = \frac{2.6}{5.059} = 0.51,$$

and estimate the error of this approximation by the formula

$$|\bar{x} - x_2| < -\frac{f(a)f(b)}{2} \cdot \max_{[a,\, b]} \left| \frac{f''(x)}{[f'(x)]^3} \right|,$$

assuming $a = 0$, $b = 0.52$; we have

$$|\bar{x} - x_2| < \frac{5 \cdot 0.059}{2} \cdot \max_{[0;\, 0.52]} \left| \frac{6x}{(3x^2 - 10)^3} \right|.$$

In the indicated interval, $\left| \dfrac{6x}{(3x^2 - 10)^3} \right| = \dfrac{6x}{(10 - 3x^2)^3}$. This function takes on the greatest value at $x = 0.52$. Thus, it follows that

$$|\bar{x} - x_2| < 0.1475 \cdot \frac{3.12}{(10 - 0.8112)^3}.$$

We have obtained the estimation of the approximate value of the root: $|\bar{x} - 0.5| <$

< 0.0006. It follows that both decimal digits are correct in the approximate value of the root $x_2 = 0.51$.

1110. Applying twice the method of tangents, find the approximate value of the real root of the equation $x^4 - 8x + 1 = 0$, isolated on the interval [1.6; 2]. The approximate values x_1 and x_2 must be calculated to the second decimal place. Estimate the error of the approximation x_2.

Solution. We find $f(x) = x^4 - 8x + 1$, $f'(x) = 4x^3 - 8$, $f''(x) = 12x^2$; $f(1.6) = -5.246$, $f(2) = 1$; $f''(x) > 0$, $f(2) = 1 > 0$, therefore, we assume $x_0 = 2$. We apply the formula

$$x_1 = x_0 - f(x_0)/f'(x_0), \quad \text{i.e.} \quad x_1 = 2 - 1/24 = 1.96.$$

Then we determine the second approximation. We find $f(x_1) = 1.96^4 - 8 \cdot 1.96 + 1 = 0.09$, $f'(x_1) = 4 \cdot 1.96^3 - 8 = 22.12$; hence

$$x_2 = x_1 - f(x_1)/f'(x_1), \quad \text{i.e.} \quad x_2 = 1.96 - 0.09/22.12 = 1.956 \approx 1.96.$$

The error of the approximate value of the root we have obtained is

$$|\bar{x} - x_2| < \frac{[f(x_2)]^2}{2} \cdot \max_{[1.6;\,2]} \left| \frac{f''(x)}{[f'(x)]^3} \right| .$$

In the interval (1.6; 2) we have

$$\left| \frac{f''(x)}{[f'(x)]^3} \right| = \left| \frac{12x^2}{(4x^3 - 8)^3} \right| = \frac{3x^2}{16(x^3 - 2)^3}.$$

Within the interval [1.6; 2] the function $\dfrac{x^2}{(x^3 - 2)^3}$ has no extrema. In attains the greatest value at $x = 1.6$:

$$|\bar{x} - 1.96| < \frac{0.09^2}{2} \cdot \frac{3.16^2}{16(1.6^3 - 2)^3};$$

i.e. $|\bar{x} - 1.96| < 0.0002$; consequently, all the digits are correct in the approximate value 1.96 of the root.

1111. Applying the iteration method five times, find the approximate root of the equation $2x - \cos x = 0$ isolated on the interval [0; 0.5] accurate to the third significant digit.

Solution. Let us write the given equation in the form $x = 0.5 \cos x$; consequently, $\varphi(x) = 0.5 \cos x$, $\varphi'(x) = -0.5 \sin x$. In the interval [0; 0.5] we have $|\varphi'(x)| < 0.5 = r < 1$. We assume $x_0 = 0.5$; $x_1 = 0.5 \cos x_0$, $x_2 = 0.5 \cos x_1$, etc. and perform the following calculations:

$$x_1 = 0.5 \cos 0.5 = 0.5 \cos 28°41' = 0.4386;$$

$$x_2 = 0.5 \cos 0.4386 = 0.5 \cos 25°08' = 0.4527;$$

$$x_3 = 0.5 \cos 0.4527 = 0.5 \cos 25°56' = 0.4496;$$

$$x_4 = 0.5 \cos 25°46' = 0.4503;$$

$$x_5 = 0.5 \cos 25°48' = 0.4502.$$

The error will be calculated by the formula

$$|\bar{x} - x_5| < \frac{r}{1 - r}\,|x_5 - x_4|.$$

We have

$$|\bar{x} - 0.4502| < |0.4502 - 0.4503|, \quad \text{i.e.} \quad |\bar{x} - 0.4502| < 0.0001,$$
$$\text{or} \quad 0.4501 < \bar{x} < 0.4503.$$

Consequently, with an accuracy to the third significant digit the approximate value is equal to 0.450.

9.1.6. Generalization of Newton's method of approximation (a). Chebyshev's method. Suppose it is required to find the real root of the equation $f(x) = 0$ isolated on the interval (a, b). The function $f(x)$ is assumed to be continuous together with its derivatives up to the nth order inclusive, with $f'(x) \neq 0$ in the interval (a, b). Let us consider the curve $x = \xi + A_1 y + A_2 y^2 + \ldots + A_n y^n$. We choose the parameters $\xi, A_1, A_2, \ldots, A_n$ such that the curves $y = f(x)$ and $x = \xi + \sum\limits_{k=1}^{n} A_k y^k$ have contact of order n at the point with the abscissa x_0 belonging to the interval (a, b). Recall (see 7.4 in Part 1) that the curves $y = f(x)$ and $y = \varphi(x)$ have *contact of order n* at the point with the abscissa x_0 if

$$f(x_0) = \varphi(x_0), \quad f'(x_0) = \varphi'(x_0), \quad f''(x_0) = \varphi''(x_0), \ldots, \quad f^{(n)}(x_0) = \varphi^{(n)}(x_0).$$

In terms of geometry, the point of contact of order n is the limiting position of $n + 1$ points of intersection of the curves $y = f(x)$ and $y = \varphi(x)$ as these intersection points tend to the point with the abscissa x_0. In the given case, the curve $y = \varphi(x)$ is implicitly defined by the equation $x = \xi + \sum\limits_{k=1}^{n} A_k y^k$.

With such a choice of the parameters $\xi, A_1, A_2, \ldots, A_n$, we can take as the approximate value of the sought-for root the abscissa of the point of intersection of the curve $x = \xi + \sum\limits_{k=1}^{n} A_k y^k$ and the x-axis, i.e. the number ξ.

If $n = 1$, then $\xi = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ (the formula of Newton's method).

If $n = 2$, then

$$\xi = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{[f(x_0)]^2 \cdot f''(x_0)}{2!\,[f'(x_0)]^3}. \tag{1}$$

If $n = 3$, then

$$\xi = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{[f(x_0)]^2 \cdot f''(x_0)}{2!\,[f'(x_0)]^3} - \frac{[f(x_0)]^3}{3!} \cdot \frac{3[f''(x_0)]^2 - f'(x_0) \cdot f'''(x_0)}{[f'(x_0)]^5}. \tag{2}$$

If $n = 4$, then

$$\xi = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{[f(x_0)]^2 \cdot f''(x_0)}{2![f'(x_0)]^3} - \frac{[f(x_0)]^3}{3!} \cdot \frac{3[f''(x_0)]^2 - f'(x_0) \cdot f'''(x_0)}{[f'(x_0)]^5}$$
$$- \frac{[f(x_0)]^4}{4!} \cdot \frac{[f'(x_0)]^2 \cdot f^{\mathrm{IV}}(x_0) - 10f'(x_0)f''(x_0)f'''(x_0) + 15[f''(x_0)]^3}{[f'(x_0)]^7}. \quad (3)$$

Here are the *estimations of the errors* of the values of the roots obtained by means of Eqs. (1) and (2).

For Eq. (1), at $n = 2$

$$|\bar{x} - \xi| < \frac{[f(x_0)]^3}{3!} \cdot \max_{[a,\, b]} \left| \frac{3[f''(x)]^2 - f'(x)f'''(x)}{[f'(x)]^5} \right| .$$

For Eq. (2), at $n = 3$

$$|\bar{x} - \xi| < \frac{[f(x_0)]^4}{4!} \cdot \max_{[a,\, b]} \left| \frac{[f'(x)]^2 f^{\mathrm{IV}}(x) - 10f'(x)f''(x)f'''(x) + 15[f''(x)]^3}{[f'(x)]^7} \right| .$$

(b) To find the real root of the equation $f(x) = 0$, isolated on the interval (a, b), we consider the curve

$$y = \frac{x - \xi_n}{A_0 + A_1(x - x_0) + A_2(x - x_0)^2 + \ldots + A_{n-1}(x - x_0)^{n-1}}, \quad (3)$$

having contact of order n with the curve $y = f(x)$ at the point with the abscissa $x_0 (a < x_0 < b)$. We take as the approximate value of the root the abscissa of the point of intersection of this curve and x-axis, i.e. ξ_n.

From the condition of contact we find this approximate value:

$$\xi_n = x_0 - b_0 \cdot \frac{D_{n-1}}{D_n}, \quad (4)$$

where

$$D_n = \begin{vmatrix} b_1 & b_0 & 0 & 0 & \ldots & 0 & 0 \\ b_2 & b_1 & b_0 & 0 & \ldots & 0 & 0 \\ b_3 & b_2 & b_1 & b_0 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n-1} & b_{n-2} & b_{n-3} & b_{n-4} & \ldots & b_1 & b_0 \\ b_n & b_{n-1} & b_{n-2} & b_{n-3} & \ldots & b_2 & b_1 \end{vmatrix},$$

$$b_k = \frac{f^{(k)}(x_0)}{k!} \quad (k = 1, 2, \ldots, n), \quad b_0 = f(x_0).$$

If $n = 1$, then Eq. (3) defines the straight line $y = \frac{1}{A_0}(x - \xi)$ and the approximate value of the root is specified by the formula of Newton's method.

Thus, Eq. (4) generalizes Newton's method for approximate solution of equations.

If $n = 2$, then

$$\xi_2 = x_0 - \frac{b_0 b_1}{b_1^2 - b_0 b_2}. \tag{5}$$

If $n = 3$, then

$$\xi_3 = x_0 - \frac{(b_1^2 - b_0 b_2) b_0}{\begin{vmatrix} b_1 & b_0 & 0 \\ b_2 & b_1 & b_0 \\ b_3 & b_2 & b_1 \end{vmatrix}}. \tag{6}$$

1112. Find the approximate value of $\sqrt{3}$ to within 0.0000001.

Solution. We apply to the equation $x^2 - 3 = 0$ Chebyshev's formula for $n = 4$. Assuming $x_0 = 1.7$, we have $f(x) = x^2 - 3$, $f'(x) = 2x$, $f''(x) = 2$, $f'''(x) = 0$, $f^{IV}(x) = 0$; $f(1.7) = -0.11$, $f'(1.7) = 3.4$, $f''(1.7) = 2$, $f'''(1.7) = 0$, $f^{IV}(1.7) = 0$. Consequently,

$$\xi = 1.7 + \frac{0.11}{3.4} - \frac{0.11^2 \cdot 2}{2!3.4^3} \cdot \frac{0.11^3}{3!} \cdot \frac{12}{3.4^5} - \frac{0.11^4}{4!} \cdot \frac{120}{3.4^7}$$

$$= 1.7 + 0.323529 - 0.0003078 + 0.0000058 - 0.0000001 = 1.7320508.$$

Since $f(1.7320508) < 0$, but $f(1.7320509) > 0$, all the digits are correct in the approximate value of the root $\xi = 1.7320508$.

1113. Find the approximate value of the real root of the equation $2x^3 + 2x - 7 = 0$ with an accuracy to within 0.000001.

Solution. We have $f(x) = 2x^3 + 2x - 7$, $f'(x) = 6x^2 + 2 > 0$; $f(x)$ is an increasing function; $f(1.3) = 4.394 + 2.6 - 7 = -0.006 < 0$, $f(1.4) = 5.488 + 2.8 - 7 = 1.288 > 0$. Consequently, the interval (1.3; 1.4) contains the only real root of the given equation.

We assume $x_0 = 1.3$. From Chebyshev's formula $n = 2$, we find $f(x) = 2x^3 + 2x - 7$, $f'(x) = 6x^2 + 2$, $f''(x) = 12x$; $f(1.3) = -0.006$, $f'(1.3) = 12.14$, $f''(1.3) = 15.6$; hence

$$\xi = 1.3 + \frac{0.006}{12.14} - \frac{0.000036}{2} \cdot \frac{15.6}{1789.1883}$$

$$= 1.3 + 0.0004942 - 0.0000002 = 1.300494;$$

$$f(1.300494) = 4.399009 + 2.600988 - 7 = -0.000003 < 0,$$

$$f(1.300495) = 4.399021 + 2.600990 - 7 = 0.000011 > 0.$$

Consequently, all the digits in the approximate value of the root $\xi = 1.300494$ are correct.

1114. Find the approximate value of $\sqrt[3]{5}$ with an accuracy to within 0.00001 with the aid of Chebyshev's formula. Assume $n = 3$.

1115. Assuming $n = 2$ in Chebyshev's formula, calculate the real root of the equation $3x^5 + 6x + 16 = 0$ to within 0.00001.

1116. Find the approximate value of $\sqrt{2}$ to within 0.00001.

Solution. We set $f(x) = x^2 - 2$. Then we apply Eq. (6), taking $x_0 = 1.4$. We have $f(x) = x^2 - 2$, $f'(x) = 2x$, $f''(x) = 2$, $f'''(x) = 0$, $b_0 = f(1.4) = -0.04$, $b_1 = f'(1.4) = 2.8$, $b_2 = (1.2) \cdot f''(1.4) = (1.2) \cdot 2 = 1$, $b_3 = f'''(1.4) = 0$. Consequently,

$$\xi_3 = 1.4 + \frac{(7.84 + 0.04) \cdot 0.04}{\begin{vmatrix} 2.8 & -0.04 & 1 \\ 1 & 2.8 & -0.4 \\ 0 & 1 & 2.8 \end{vmatrix}} = 1.4 + \frac{7.88 \cdot 0.04}{21.952 + 0.224}$$

$$= 1.4 + \frac{0.3152}{22.176} = 1.4 + 0.01421 = 1.41421.$$

All the decimal digits in the approximate value of the root are correct.

1117. Assuming $n = 2$, find the approximate value of the positive root of the equation $x^3 + x^2 - 4 = 0$ with an accuracy to within 0.0001.

Solution. Setting $x_0 = 1.3$, we have $f(x) = x^3 + x^2 - 4$, $f'(x) = 3x^2 + 2x$, $f''(x) = 6x + 2$, $b_0 = f(1.3) = -0.113$, $b_1 = f'(1.3) = 7.67$, $b_2 = (1.2) \cdot f''(1.3) = (1.2) \cdot 9.8 = 4.9$. Then it follows that

$$\xi_2 = 1.3 + \frac{7.67 \cdot 0.113}{7.67^2 + 0.113 \cdot 4.9} = 1.3 + \frac{0.86671}{59.3826} = 1.3146.$$

All the decimal digits are correct.

1118. Assuming $n = 2$, find the approximate value of the root of the equation $x + \ln x = 3$ with an accuracy to within 0.001.

1119. With the aid of Eq. (6) find the approximate value of $\sqrt[5]{5}$ with an accuracy to within 0.00001.

1120. With the aid of Eq. (5) calculate the negative root of the equation $5x^6 - 5x - 47.071 = 0$ with an accuracy to within 0.0001.

9.2. Interpolation

9.2.1. Lagrange's interpolation polynomial. Suppose we are given the following table of values:

x	x_1	x_2	x_3	...	x_n
y	y_1	y_2	y_3	...	y_n

It is required to derive the polynomial $y = f(x)$ of degree $m \leqslant n - 1$, which would assume the assigned values y_i for the corresponding values x_i, i.e. $y_i = f(x_i)$ $(i = 1, 2, \dots, n)$. In other words, the graph of this polynomial must pass through the given n points $M_i(x_i; y_i)$.

Let us designate as

$$\varphi(x) = (x - x_1)(x - x_2)(x - x_3) \dots (x - x_n)$$

the auxiliary polynomial of degree n, in which x_i are the given tabular values of the argument. Then, there holds the equality

$$f(x) = \frac{y_1 \cdot \varphi(x)}{(x-x_1)(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)}$$

$$+ \frac{y_2 \cdot \varphi(x)}{(x-x_2)(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)}$$

$$\dots + \frac{y_n \cdot \varphi(x)}{(x-x_n)(x_n-x_1)(x_n-x_2)\dots(x_n-x_{n-1})},$$

or

$$f(x) = \sum_{k=1}^{n} \frac{y_k \cdot \varphi(x)}{\varphi'(x_k)(x-x_k)}.$$

This is precisely the *Lagrange interpolation polynomial.*

1121. Derive the Lagrange polynomial for the following tabular values:

x	1	2	3	4
y	2	3	4	5

Solution. The auxiliary polynomial is of the form $\varphi(x) = (x - 1)(x - 2)(x - 3)(x - 4)$. Let us calculate $\varphi'(x)$ consecutively for the given values of x:

$$\varphi'(x) = (x - 2)(x - 3)(x - 4) + (x - 1)(x - 3)(x - 4) + (x - 1)(x - 2)(x - 4) + (x - 1)(x - 2)(x - 3);$$

$$\varphi'(1) = -6; \quad \varphi'(2) = 2, \quad \varphi'(3) = -2, \quad \varphi'(4) = 6.$$

Then we have

$$f(x) = \frac{2}{-6}(x - 2)(x - 3)(x - 4) + \frac{3}{2}(x - 1)(x - 3)(x - 4) + \frac{4}{-2}(x - 1)(x - 2)(x - 4) + \frac{5}{6}(x - 1)(x - 2)(x - 3) = x + 1.$$

Thus we see that in the given case the interpolation polynomial is the linear function $f(x) = x + 1$.

1122. Find the equation of the parabola passing through the points (2; 0), (4; 3), (6; 5), (8; 4), (10; 1).

Solution. The auxiliary polynomial has the form $\varphi(x) = (x - 2)(x - 4)(x - 6)(x - 8)(x - 10)$. We find

$$\varphi'(x) = (x-4)(x-6)(x-8)(x-10) + (x-2)(x-6)(x-8)(x-10) + (x-2)(x-4)(x-8)(x-10)$$
$$+ (x-2)(x-4)(x-6)(x-10) + (x-2)(x-4)(x-6)(x-8);$$

$$\varphi'(2) = 384, \quad \varphi'(4) = -96, \quad \varphi'(6) = 64, \quad \varphi'(8) = -96, \quad \varphi'(10) = 384.$$

Then we have

$$f(x) = \frac{0}{384}(x-4)(x-6)(x-8)(x-10) + \frac{3}{-96}(x-2)(x-6)(x-8)(x-10)$$
$$+ \frac{8}{64}(x-2)(x-4)(x-8)(x-10) - \frac{6}{96}(x-2)(x-4)(x-6)(x-10)$$
$$+ \frac{0}{384}(x-2)(x-4)(x-6)(x-8) = \frac{1}{32}(x^4 - 26x^3 + 220x^2 - 664x + 640).$$

Consequently, the desired parabola is of the fourth order:

$$y = \frac{1}{32}(x^4 - 26x^3 + 220x^2 - 664x + 640).$$

1123. Given the points (0; 3), (2; 1), (3; 5), (4; 7). Derive the equation of the polynomial assuming the indicated values for the given values of the argument.

1124. Construct the polynomial assuming the values given by the table

x	1	3	4	6
y	-7	5	8	14

1125. Construct the polynomial whose graph passes through the points (2; 3), (4; 7), (5; 9), (10; 19).

9.2.2. Newton's interpolation formula. Suppose $y_0, y_1, y_2, \ldots$ are the values of a certain function $y = f(x)$ corresponding to the equidistant values $x_0, x_1, x_2, \ldots$ of the argument (i.e. $x_{k+1} - x_k = \Delta x_k = \text{const}$).

We introduce the following designations:

$$y_1 - y_0 = \Delta y_0, \quad y_2 - y_1 = \Delta y_1, \ldots, \quad y_n - y_{n-1} = \Delta y_{n-1}$$

are the first differences of the given function;

$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$, $\Delta y_2 - \Delta y_1 = \Delta^2 y_1, \ldots$ are the second differences;

$$\Delta^n y_1 - \Delta^n y_0 = \Delta^{n+1} y_0,\ \Delta^n y_2 - \Delta^n y_1 + \Delta^{n+1} y_1,\ \ldots$$

are the $(n + 1)$th differences.

Performing successive substitutions, we get

$$\Delta^2 y_0 = y_2 - 2y_1 + y_0,\ \Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0,\ \cdots,$$

$$\Delta^n y_0 = \sum_{k=0}^{n} (-1)^k \cdot C_n^k \cdot y_{n-k}.$$

In the same way we obtain

$$y_1 = y_0 + \Delta y_0,\ y_2 = y_0 + 2\Delta y_0 + \Delta^2 y_0,\quad y_3 = y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0,\ \ldots,$$

$$y_n = \sum_{k=0}^{n} C_n^k \, \Delta^k y_0 = (1 + \Delta)^n \cdot y_0$$

$$= y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \ldots + \Delta^n y_0. \quad (1)$$

We write the table of differences:

$$\begin{array}{llllll}
x_0 & y_0 & & & & \\
x_1 & y_1 & \Delta y_0 & & & \\
 & & & \Delta^2 y_0 & & \\
x_2 & y_2 & \Delta y_1 & & \Delta^3 y_0 & \\
 & & & \Delta^2 y_1 & & \Delta^4 y_0 \\
x_3 & y_3 & \Delta y_2 & & \Delta^3 y_1 & \\
 & & & \Delta^2 y_2 & & \\
x_4 & y_4 & \Delta y_3 & & &
\end{array}$$

If, in Eq. (1), we assume that n is not only a positive integer, but can be arbitrary $(n = t)$, we obtain *Newton's interpolation formula:*

$$y_t = y_0 + \frac{t}{1!}\Delta y_0 + \frac{t(t-1)}{2!}\Delta^2 y_0 + \frac{t(t-1)(t-2)}{3!}\Delta^3 y_0 + \ldots + \Delta^t y_0. \quad (2)$$

We have obtained such a function of t which turns into y_0 for $t = 0$, into y_1 for $t = 1$, into y_2 for $t = 2$, and so on. Since with the constant step h the successive values of the argument x are specified by the formula $x_n = x_0 + nh$, it follows that $n = (x_n - x_0)/h$. Then, setting $x = x_0 + th$, i.e. $t = (x - x_0)/h$, we reduce Eq. (1) to the form

$$y_n = y_0 + \frac{x - x_0}{h}\Delta y_0 + \frac{(x - x_0)(x - x_0 - h)}{2!h^2}\Delta^2 y_0 + \dots . \qquad (3)$$

1126. From the table find the value of y for $x = 3.1$, using Newton's interpolation formula.

x	1	2	3	4	5	6	7
y	3	7	13	21	31	43	57

Solution. We compile a table of differences:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	3			
		4		
2	7		2	
		6		0
3	13		2	
		8		0
4	21		2	
		10		0
5	31		2	
		12		0
6	43		2	
		14		
7	57			

Here $x_0 = 3$, $x = 3.1$, $h = 1$. Then $t = (x - x_0)/h = (3.1 - 3)/1 = 0.1$. Let us write Newton's interpolation polynomial for this case:

$$y = y_0 + t \cdot \Delta y_0 + \frac{t(t-1)}{1 \cdot 2}\Delta^2 y_0.$$

or

$$y = 13 + 0.1 \cdot 8 + \frac{0.1(0.1 - 1)}{2} \cdot 2 = 13.71.$$

Consequently, for $x = 3.1$ and $y = 13.71$ the interpolation polynomial for this table has the form

$$y = 3 + (x - 1) \cdot 4 + \frac{(x-1)(x-2)}{2} \cdot 2 = x^2 + x + 1.$$

1127. Given the following decimal logarithms: $\log 2.0 = 0.30103$, $\log 2.1 = 0.32222$, $\log 2.2 = 0.34242$, $\log 2.3 = 0.36173$, $\log 2.4 = 0.38021$, $\log 2.5 = 0.39794$. Using Newton's interpolation formula, find $\log 2.03$.

Solution. We compile a table of differences:

x	$\log x$	Δy	$\Delta^2 y$	$\delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2.0	0.30103					
		2119				
2.1	0.32222		−99			
		2020		10		
2.2	0.34242		−89		−4	
		1931		6		6
2.3	0.36173		−83		2	
		1848		8		
2.4	0.38021		−75			
		1773				
2.5	0.39794					

Here $x_0 = 2.0$, $x = 2.03$, $h = 0.1$. Then we have $t = (x - x_0)/h = (2.03 - 2.0)/0.1 = 0.3$. Hence

$$y = y_0 + t \cdot \Delta y_0 + \frac{t(t-1)}{2!}\Delta^2 y_0 + \frac{t(t-1)(t-2)}{3!}\Delta^3 y_0$$

$$+ \frac{t(t-1)(t-2)(t-3)}{4!}\Delta^4 y_0 + \frac{t(t-1)(t-2)(t-3)(t-4)}{5!}\Delta^5 y_0$$

$$= 0.30103 + 0.3 \cdot 0.02119 + \frac{1}{2} \cdot 0.3 \cdot 0.7 \cdot 0.00099$$

$$+ \frac{1}{6} \cdot 0.3 \cdot 0.7 \cdot 1.7 \cdot 0.00010 + \frac{1}{24} \cdot 0.3 \cdot 0.7 \cdot 1.7 \cdot 2.7 \cdot 0.00004$$

$$+ \frac{1}{120} \cdot 0.3 \cdot 0.7 \cdot 1.7 \cdot 2.7 \cdot 3.7 \cdot 0.00006 = 0.30750.$$

Thus, $\log 2.03 = 0.30750$. The five-digit table of logarithms gives the same result.

1128. Given every other number from 4 to 10 of the five-digit logarithms. Using Newton's interpolation formula, calculate four-digit logarithms of the numbers 6.5 through 7.0 with $h = 0.1$.

1129. Knowing the squares of the numbers 5, 6, 7, 8, find the square of the number 6.25.

1130. Derive the Newton interpolation polynomial for the function given by the table

x	0	1	2	3	4
y	1	4	15	40	85

9.3. Approximation of Definite Integrals

If $f(x)$ is a continuous function differentiable a sufficient number of times on the interval $[a, b]$ and $h = (b - a)/n$, $x_k = x_0 + kh$ $(k = 0, 1, 2, \ldots, n)$, $y_k = f(x_k)$, then the following formulas for approximating definite integrals hold true.

The *rectangle rule* is

$$\int_a^b (x)dx = h(y_0 + y_1 + y_2 + \ldots + y_{n-1}) + R_n \qquad (1)$$

or

$$\int_a^b f(x)dx = h(y_1 + y_2 + \ldots + y_n) + R_n; \tag{2}$$

the limiting absolute error is

$$R_n \leqslant \frac{h}{2}(b-a)M_1, \quad \text{where } M_1 = \max_{[a,b]} |f'(x)|. \tag{3}$$

The *trapezoidal rule* is

$$\int_a^b f(x)dx = h\left(\frac{y_0 + y_n}{2} + y_1 + y_2 + \ldots + y_{n-1}\right) + R_n \tag{4}$$

the limiting absolute error is

$$R_n \leqslant \frac{h^2}{12}(b-a)M_2, \quad \text{where} \quad M_2 = \max_{[a,b]} |f''(x)| \tag{5}$$

(the exact value of the error is $\delta_t = -(h^2/12)(b-a)f''(c)$, where $a < c < b$);

Simpson's rule is

$$\int_a^b f(x)dx = \frac{h}{3}[(y_0 + y_{2k}) + 4(y_1 + y_3 + \ldots + y_{2k-1})$$
$$+ 2(y_2 + y_4 + \ldots + y_{2k-2})] + R_n; \tag{6}$$

the limiting absolute error is

$$R_n \leqslant \frac{h^4}{180}(b-a)M_3, \quad \text{where} \quad M_3 = \max_{[a,b]} |f^{\mathrm{IV}}(x)| \tag{7}$$

(the exact value of the error is $\delta_S = -(h^4/180)(b-a)f^{\mathrm{IV}}(c)$, where $a < c < b$).

When it is difficult fo find the fourth derivative of the integrand function, the following means may be used to estimate the error of the calculation of the integral $\int_a^b f(x)dx$ by Simpson's rule.

Setting $n = 4k$, we calculate the approximate value of the given integral by Simpson's rule for the step $h = (b-a)/4k$; suppose the value of the integral obtained is I_1; then we double the step h and perform calculations by Simpson's rule for the step $h_1 = (b-a)/2k$; suppose the value of the integral obtained is I_2; the error of the second approximation is nearly 16 times that of the first approximation and they both have the same sign. Therefore, the error of the first calculation (for the step $h = (b-a)/4k$) is specified by the following formula (taking the sign of the error into account):

$$\delta_S \approx (I_1 - I_2)/15$$

(this method may be called the estimation of the error of Simpson's rule by doubling the calculation step).

1131. Calculate $I = \int_1^2 \sqrt{x}dx$ by the rectangle rule, partitioning the integration interval into 10 subintervals. Estimate the error.

Solution. Here $y = \sqrt{x}$; for $n = 10$ we have $h = (2 - 1)/10 = 0.1$. The division points are $x_0 = 1$, $x_1 = x_0 + h = 1.1$, $x_2 = 1, 2, \ldots, x_9 = 1.9$. We find the corresponding values of the integrand: $y_0 = \sqrt{x_0} = \sqrt{1} = 1$, $y_1 = \sqrt{1.1} = 1.049$, $y_2 = 1.095$, $y_3 = 1.140$, $y_4 = 1.183$, $y_5 = 1.225$, $y_6 = 1.265$, $y_7 = 1.304$, $y_8 = 1.342$, $y_9 = 1.378$.

Using the rectangle rule, we get

$$I = 0.1(1.000 + 1.049 + 1.095 + 1.140 + 1.183 + 1.225 + 1.265 + 1.304 + 1.342 + 1.378) = 0.1 \cdot 11.981 \approx 1.20$$

Let us estimate the error. In the given case, $f'(x) = 1/(2\sqrt{x})$ attains its greatest value equal to 0.5 on the interval [1, 2] for $x = 1$. Thus, $|f'(x)| \leqslant M_1 = 1/2$. From this we find by Eq. (3) that

$$R_n \leqslant \frac{0.1}{2} \cdot 1 \cdot \frac{1}{2} = 0.025.$$

Consequently, $I \approx 1.20 \pm 0.025$.

For the sake of comparison, let us calculate the same integral by the formula of Newton-Leibniz:

$$I = \int_1^2 \sqrt{x}dx = \int_1^2 x^{1/2}dx = \frac{2}{3}(2\sqrt{2} - 1) \approx 1.219.$$

Thus we see that the actual value of the integral indeed lies in the interval we have found.

1132. Calculate the same integral by the trapezoidal rule, assuming $n = 10$; estimate the error.

Solution. With the same notation using the trapezoidal rule, we get

$$I = 0.1 \cdot \left(\frac{1 + 1.414}{2} + 1.049 + 1.095 + 1.140 + 1.183 + 1.225 + 1.265 + 1.304 + 1.342 + 1.378\right) = 1.218.$$

Next we have $f''(x) = -1/(4\sqrt{x^3})$; $|f''(x)| \leqslant 1/4$ on the interval [1, 2]. Thus we find by Eq. (5)

$$R_n \leqslant \frac{0.1}{12} \cdot 1 \cdot \frac{1}{4} \approx 0.002$$

If follows that $I \approx 1.218 \pm 0.002$.

1133. Approximate $I = \int_0^1 \sqrt{1 + x^2}dx$ by Simpson's formula with an accuracy to within 0.001.

Solution. First, using Eq. (7), we determine what step h we must take to attain the assigned degree of accuracy. We have

$$f(x) = \sqrt{1+x^2};\ f'(x) = x/\sqrt{1+x^2};\ f''(x) = 1/\sqrt{(1+x^2)^3};$$
$$f'''(x) = -3x/\sqrt{(1+x^2)^5};\ f^{IV}(x) = (12x^2-3)/\sqrt{(1+x^2)^7}.$$

The greatest value of $|f^{IV}(x)|$ on the interval [0, 1] is attained at the point $x = 0$, it is $|f^{IV}(0)| = 3$. Hence,

$$R_n \leqslant \frac{h^4}{180}(b-a)|f^{IV}(x)| = \frac{h^4}{180}\cdot 1\cdot 3.$$

Since the error must not exceed 0.001, it follows that $h^4/60 \leqslant 0.001$, i.e. $h^4 \leqslant 0.06$. We can take $h = 0.5$ (if $h = 0.5$, then $h^4 = 0.0625$). that is, somewhat larger than the desired value, but this will not affect the accuracy of calculations since when estimating we have used the limiting absolute error, the quantity which is obviously larger than the actual error. Thus, to reach the desired degree of accuracy, it is sufficient to divide the integration interval in half.

Let us calculate the values of the function $f(x) = \sqrt{1+x^2}$ for $x = 0$; 0.5 and 1: $f(0) = 1.0000$; $f(0.5) = 1.1180$; $f(1.0) = 1.4142$ (we shall carry out the calculations with one extra digit). Therefore

$$I \approx \frac{0.5}{3}\cdot[1.0000 + 4\cdot 1.1180 + 1.4142] = 1.1477.$$

Thus, rounding off the last digit, we find $I \approx 1.148$.

Let us calculate, for the sake of comparison, the exact value of this integral by the Newton-Leibniz formula:

$$I = \int_0^1 \sqrt{1+x^2}\,dx = \left[\frac{x}{2}\cdot\sqrt{1+x^2} + \frac{1}{2}\ln(x+\sqrt{1+x^2})\right]_0^1$$
$$= \frac{1}{2}\left[\sqrt{2} + \ln(1+\sqrt{2})\right] \approx \frac{1}{2}(1.4142 + 0.8814) \approx 1.1478$$

Thus we see that calculated by Simpson's rule, the value of this integral has not three, but even four correct decimal digits.

1134. Calculate $I = \int_0^1 \frac{dx}{1+x^2}$ by Simpson's rule, assuming $n = 8$. Proceed up to the sixth decimal digit. Estimate the error of the result obtained making use of the method of doubing the calculation step; compare the result with the actual value of the integral, taking this value with one extra (seventh) digit.

Solution. We must determine the values of the integrand for the following values of the argument ($h_1 = 0.125$): $x_0 = 0$; $x_1 = 0.125$; ... ; $x_8 = 1.0$. We find the corresponding values, $f(x) = 1/(1 + x^2)$: $y_0 = 1.000000$; $y_1 = 0.984625$; $y_2 = 0.941176$; $y_3 = 0.876712$; $y_4 = 0.800000$; $y_5 = 0.719101$; $y_6 = 0.640000$; $y_7 = 0.566389$; $y_8 = 0.500000$ and substitute the data obtained into Simpson's formula for $h_1 = 0.125$ and $h_2 = 0.25$:

$$I_1 = \frac{h_1}{3}[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{0.125}{3} \cdot [1.000000 + 0.500000 + 4(0.984615 + 0.876712 + 0.719101$$

$$+ 0.566389) + 2 \cdot (0.941176 + 0.800000 + 0.640000) = \frac{1}{24} \cdot 18.849548$$

$$\approx 0.785398;$$

$$I_2 = \frac{h_2}{3}[y_0 + y_8 + 4(y_2 + y_6) + 2y_4] = \frac{0.25}{3}$$

$$\times [1.000000 + 0.500000 + 4(0.941176 + 0.640000) + 2 \cdot 0.800000]$$

$$= \frac{1}{12} \cdot 9.424704 = 0.785392.$$

Hence we have

$$\delta_{I_1} \approx \frac{I_1 - I_2}{15} = \frac{0.000006}{15} \approx 0.0000004.$$

Thus, all the six digits in I_1 must be correct. The actual value of the integral is

$$I = \int_0^1 \frac{dx}{1 + x^2} = \arctan x \Big|_1^0 = \frac{\pi}{4} \approx 0.7853979, \dots ,$$

and this confirms the result obtained.

1135. Calculate $\int_1^2 \frac{dx}{x^2}$ by Simpson's rule with an accuracy to within 0.0001, assuming $n = 10$.

1136. Calculate $\int_1^8 \frac{dx}{\sqrt{x} + 1}$ by Simpson's rule, assuming $n = 8$. Estimate the error by the method of doubling the step; compare the result obtained with the exact value of the integral. The calculations must be carried out with five decimal digits.

1137. Calculate $I = \int_0^{\pi/2} \sqrt{1 - 0.5 \sin^2 x}\, dx$ by the trapezoidal rule, assuming $n = 6$; estimate the error beforehand, in order to determine with how many digits (with one extra digit) the calculations must be performed.

1138. Calculate $\ln 2 = \int_2^1 \frac{dx}{x}$ by the trapezoidal rule with an accuracy to within 0.01, assuming $n = 5$.

1139. Calculate $\int_1^2 \frac{\ln x}{x}\, dx$ by Simpson's rule with an accuracy to within 0.01, assuming $n = 4$.

1140. Calculate $\int_0^1 e^{-x^2} dx$ by the trapezoidal rule with an accuracy to within 0.01, assuming $n = 4$.

1141. Calculate $\int_0 \frac{\cos x}{1 + x}\, dx$ by the trapezoidal rule with an accuracy to within 0.01, assuming $n = 6$.

9.4. Approximation of Multiple Integrals

9.4.1. Analogue of rectangle rules. (a) Let us consider a closed domain D bounded by the curves $x = a$, $x = b$, $y = \varphi(x)$, $y = \psi(x)$, where $\varphi(x)$ and $\psi(x)$ are functions continuous on the interval $[a, b]$, with $\varphi(x) \leqslant \psi(x)$* (Fig. 71). Let us partition the domain D into n subdomains by the curves

$$y = \varphi(x) + \frac{j}{n}\,[\psi(x) - \varphi(x)] \quad (j = 0, 1, 2, \dots, n). \tag{1}$$

Next we partition the interval $[a, b]$ into m equal subintervals by the points $a = x_0 < x_1 < x_2 < \dots < x_{m-1} < x_m = b$ and draw straight lines, parallel to the y-axis, through these points:

$$x = x_i \quad (i = 0, 1, 2, \dots, m). \tag{2}$$

Two families of curves, (1) and (2), divide the domain D into mn curvilinear rectangles with vertices at the points

* Note that this condition does not violate the generality of the discussion.

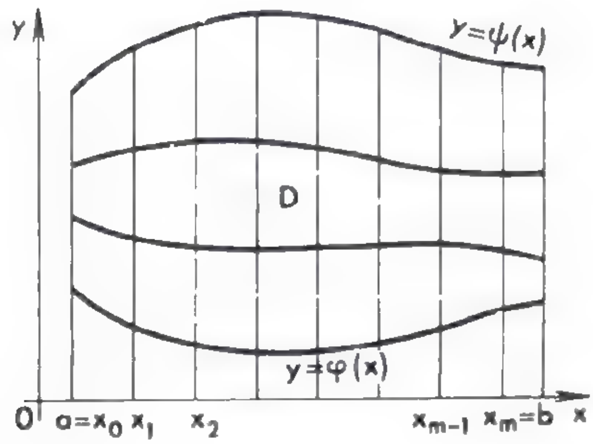


Fig. 71

$P_{ij}(x_i; y_{ij}), P_{i+1,j}(x_{i+1}; y_{i+1,j}), P_{i,j+1}(x_i; y_{i,j+1}), P_{i+1,j+1}(x_{i+1}; y_{i+1,j+1})$; $i = 0, 1, 2, \ldots, m$; $j = 0, 1, 2, \ldots, n$. For a fixed i $(0 \leqslant i \leqslant m)$ the length of the vertical side of the rectangle does not depend on j and constitutes

$$|P_{ij}P_{i,j+1}| = \frac{\psi(x_i) - \varphi(x_i)}{n}; j = 0, 1, 2, \ldots, n.$$

Let us designate the area of the curvilinear rectangle shown in Fig. 72 as $\Delta\omega_{ij}$. It can be calculated by the formula

$$\Delta\omega_{ij} = \frac{1}{n} \int\limits_{x_i}^{x_{i+1}} [\psi(x) - \varphi(x)]\, dx. \tag{3}$$

It follows from Eq. (3) that the value of $\Delta\omega_{ij}$ does not depend on j. Taking this into account, we introduce the designations $\Delta\omega_{ij} = \Delta\omega_i$; $0 \leqslant i \leqslant m - 1$, $0 \leqslant j \leqslant n - 1$. We replace the double integral $\iint\limits_D f(x, y)\, dx\, dy$, where the function $f(x, y)$ is continuous in the domain D, by a two-dimensional integral sum, taking the points P_{ij} as the nodes:

$$\iint\limits_D f(x, y)\, dx\, dy \approx \sum_{i=0}^{m-1} \Delta\omega_i \sum_{i=0}^{n-1} z_{ij}, \tag{4}$$

where

$$z_{ij} = f(x_i, y_{ij}), y_{ij} = \varphi(x_i) + \frac{j}{n} [\psi(x_i) - \varphi(x_i)]. \tag{5}$$

Taking consecutively the points $P_{i+1,j}, P_{i,j+1}, P_{i+1,j+1}$ as the nodes, we obtain three more formulas, respectively, for approximating the double integral:

$$\iint\limits_D f(x, y)\, dx\, dy \simeq \sum_{i=0}^{m-1} \Delta\omega_i \sum_{j=1}^{n-1} z_{i+1,j}; \tag{6}$$

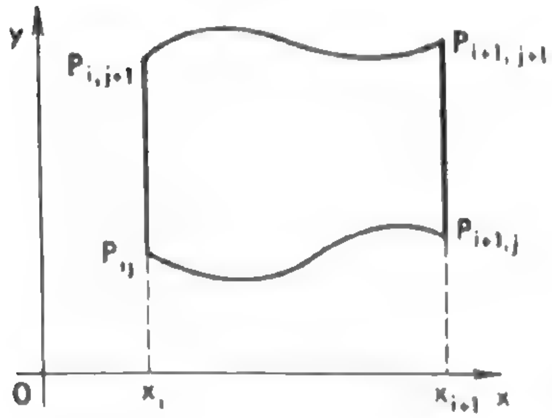


Fig. 72

$$\iint_D f(x, y)\,dx\,dy \approx \sum_{i=0}^{m-1} \Delta\omega_i \sum_{i=0}^{n-1} z_{i,j+1}; \tag{7}$$

$$\iint_D f(x, y)\,dx\,dy = \sum_{i=0}^{m-1} \Delta\omega_i \sum_{j=0}^{n-1} z_{i+1,j+1}. \tag{8}$$

Formulas (4), (6), (7) and (8) are the analogues of the formulas of rectangles for approximating definite integrals. It is evident that these formulas are the more exact the larger are the numbers m and n, that is, the smaller is the length of each of the subintervals.

(b) In a special case, when the domain D is a rectangle defined by the inequalities $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$, the elementary areas $\Delta\omega_i$ are equal to one another and can be calculated by the formula $\Delta\omega = (b - a)(d - c)/(mn)$. Formulas (4), (6), (7) and (8) will assume the following forms, respectively:

$$\iint_D f(x, y)\,dx\,dy \approx \frac{(b-a)(d-c)}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} z_{ij}, \tag{9}$$

$$\iint_D f(x, y)\,dx\,dy \approx \frac{(b-a)(d-c)}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} z_{i+1,j}, \tag{10}$$

$$\iint_D f(x, y)\,dx\,dy \approx \frac{(b-a)(d-c)}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} z_{i,j+1}, \tag{11}$$

$$\iint_D f(x, y)\,dx\,dy \approx \frac{(b-a)(d-c)}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} z_{i+1,j+1}. \tag{12}$$

Formulas (9)-(12) can be called the *formulas of parallelepipeds*.

(c) If the function $f(x, y)$ is monotonic with respect to each of the variables x and y, then there holds the estimation

$$\frac{(b - a)(d - c)}{mn}\mu \leqslant \iint\limits_D f(x, y)\, dx\, dy \leqslant \frac{(b - a)(d - c)}{mn}M, \tag{13}$$

where M and μ are, respectively, the greatest and the least sum of the sums $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} z_{ij}$, $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} z_{i+1,j}$, $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} z_{i,j+1}$, $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} z_{i+1,j+1}$.

(d) Assume that the function $f(x, y)$ and its partial derivatives $f'_x(x, y)$ and $f'_y(x, y)$ are continuous in the domain D which is a rectangle $a \leqslant x \leqslant b, c \leqslant y \leqslant d$. Then, the error of the approximate formulas (9)-(12) can be estimated from the inequality

$$|R| < \frac{(b - a)(d - c)}{2}\left[\frac{M_1(b - a)}{m} + \frac{M_2(d - c)}{n}\right], \tag{14}$$

where

$$M_1 = \max_{\substack{a \leqslant x \leqslant b \\ c \leqslant y \leqslant d}} |f'_x(x, y)|, \; M_2 = \max_{\substack{a \leqslant x \leqslant b \\ c \leqslant y \leqslant d}} |f_y{}'(x, y)|.$$

9.4.2. Analogue of the tangent formula. (a) Let us consider the double integral $I = \iint\limits_D f(x, y)\, dx\, dy$. Assume that the domain D is a rectangle $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$, at whose all points the following conditions are satisfied:

$$AC - B^2 > 0, A < 0, C < 0, \tag{15}$$

where $A = f''_{x^2}$, $C = f''_{y^2}$, $B = f''_{xy}$. These conditions ensure the convexity of the surface $z = f(x, y)$ at all points of the domain D (similarly, the conditions $AC - B^2 > 0$, $A > 0$, $C > 0$ ensure the concavity of that surface).

Then, the following approximation is valid for calculating the double integral

$$\iint\limits_D f(x, y)\, dx\, dy \approx (b - a)(d - c) f(\bar{x}, \bar{y}), \tag{16}$$

where $\bar{x} = (a + b)/2, \bar{y} = (c + d)/2$.

(b) Let us partition the domain D by the straight lines $x = x_i$ $(i = 0, 1, 2, \dots, m)$ and $y = y_j$ $(j = 0, 1, 2, \dots, n)$ into mn equal rectangles. Calculating the double integral over each elementary rectangle with the aid of formula (16) and summing up the results, we arrive at the following approximation:

$$\iint\limits_D f(x, y)\, dx\, dy \approx \frac{(b - a)(d - c)}{mn} \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} f(\bar{x}_i, \bar{y}_j), \tag{17}$$

where $\bar{x}_i = (x_{i+1} + x_i)/2, \bar{y}_j = (y_{j+1} + y_j)/2$.

Formula (17) yields the approximate value of the double integral with an excess, provided conditions (15) are satisfied. Note that we can also use (17) in the case when the first condition (15) is not satisfied. In that case we cannot say, however, whether we have found the approximate value of the double integral with an excess or with a deficiency.

9.4.3. Analogue of the trapezoidal formula. (a) Let us consider the double integral $I = \int\int_D f(x, y)\,dx\,dy$ if the domain D is a rectangle $a \leqslant x \leqslant b, c \leqslant y \leqslant d$.

Then, the following formula holds for approximating the double integral:

$$\int\int_D f(x, y)\,dx\,dy \approx \frac{(b - a)(d - c)}{4}(z_1 + z_2 + z_3 + z_4), \tag{18}$$

where $z_1 = f(a, c), z_2 = f(b, c), z_3 = f(a, d), z_4 = f(b, d)$.

This formula yields the approximate value of the double integral with an excess, provided conditions (15) is satisfied.

The error of (18) is estimated with the aid of the inequality

$$(b - a)(d - c)f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) < \int\int_D f(x, y)\,dx\,dy$$
$$< (b - a)(d - c)\frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{4}. \tag{19}$$

(b) Let us partition the domain D by straight lines, parallel to the coordinate axes, into mn equal rectangles. Calculating the double integral over each elementary rectangle with the aid of (18) and summing up the results, we arrive at the following formula for approximating the double integral:

$$\int\int_D f(x, y)\,dx\,dy \approx \frac{(b - a)(d - c)}{4mn}(S_0 + 2S_1 + 4S_2), \tag{20}$$

where $S_0 = z_{00} + z_{m0} + z_{0n} + z_{mn}$ is the sum of the values of the function at the vertices of the rectangle;

$S_1 = \sum_{i=1}^{m-1}(z_{i0} + z_{in}) + \sum_{j=1}^{n-1}(z_{0j} + z_{mj})$ is the sum of the values of the function at the nodes lying on the sides of the rectangle, not counting the vertices;

$S_2 = \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} z_{ij}$ is the sum of the values of the function at the nodes lying in the interior of the rectangle.

When satisfying conditions (15) by analogy with inequality (19) we get the estimation

$$\frac{(b-a)(d-c)}{4mn}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} f(\bar{x}_i, \bar{y}_j) < \iint_D f(x, y)\,dx\,dy$$
$$< \frac{(b-a)(d-c)}{4mn}(S_0 + 2S_1 + 4S_2), \tag{21}$$

where $\bar{x}_i = (x_{i+1} + x_i)/2$, $\bar{y}_j = (y_{j+1} + y_j)/2$.

The error of approximation (20) can be also estimated with the aid of (14).

(c) If the domain D is bounded by the curves $x = a$, $x = b$, $y = \varphi(x)$, $y = \psi(x)$, then we can take as an approximate value of the double integral $\iint_D f(x, y)\,dx\,dy$ the arithmetic mean of the results of approximating the double integral by (4), (6), (7) and (8):

$$\iint_D f(x, y)\,dx\,dy$$
$$\approx \frac{1}{4}\sum_{i=0}^{m-1} \Delta\omega_i \sum_{j=0}^{n-1} (z_{ij} + z_{i+1,j} + z_{i,j+1} + z_{i+1,j+1}), \tag{22}$$

where $\Delta\omega_i$ $(i = 0, 1, 2, \dots, m-1)$ are calculated by (3), and the values z_{ij} by (5). It is expedient to use (4), (6), (7), (8) and (22) in the cases when it is sufficiently easy to find the exact or approximate values of the areas $\Delta\omega_i$.

9.4.4. Analogue of Simpson's formula. (a) Let us consider the case of a rectangular domain D, defined by the inequalities $-h \leqslant x \leqslant h$, $-l \leqslant y \leqslant l$. We choose the coefficients of the third-degree polynomial

$$P_3(x, y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}$$

such that in a specially selected five points (nodes) the values of the function $f(x, y)$ and of the polynomial $P_3(x, y)$ coincide. Then we have

$$\int_{-h}^{h}\int_{-l}^{l} f(x, y)\,dx\,dy \approx \int_{-h}^{h}\int_{-l}^{l} P_3(x, y)\,dx\,dy.$$

Taking into account that $\int_{-a}^{a} \varphi(t)\,dt = 0$ if $\varphi(-t) = -\varphi(t)$ on $(-a, a)$, we get

$$\int_{-h}^{h}\int_{-l}^{l} f(x, y)\,dx\,dy \approx \frac{4hl}{3}(a_{20}h^2 + a_{02}l^2 + 3a_{00}). \tag{23}$$

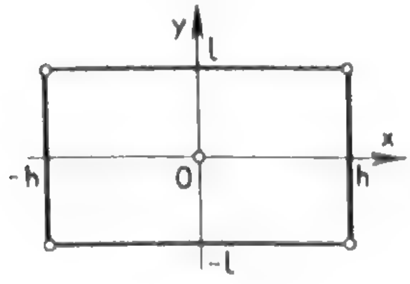
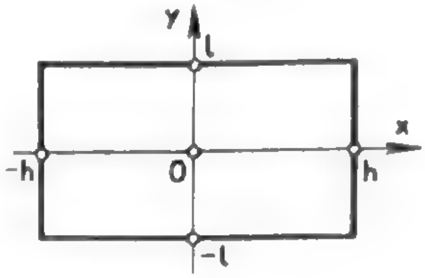


Fig. 73

Fig. 74



If the nodes are chosen as shown in Figs. 73 and 74, formula (23) can be written in the form

$$\int_{-l}^{l}\int_{-h}^{h} f(x, y)\,dx\,dy \approx \frac{hl}{3}[f(h, l) + f(-h, l) + f(h, -l) + f(-h, -l) + 8f(0, 0)] \quad (24)$$

or

$$\int_{-h}^{h}\int_{-l}^{l} f(x, y)\,dx\,dy \approx \frac{2}{3}\,hl\,[f(h, 0) + f(-h, 0) + f(0, l) + f(0, -l) + 2f(0, 0)] \quad (25)$$

For the rectangle $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ formulas (24) and (25) assume the forms

$$\int_a^b\int_c^d f(x, y)\,dx\,dy \approx \frac{(b-a)(d-c)}{12} \times \left[f(a, c) + f(a, d) + f(b, c) + f(b, d) + 8f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right], \quad (26)$$

$$\int_a^b\int_c^d f(x, y)\,dx\,dy \approx \frac{(b-a)(d-c)}{6}\left[f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+b}{2}\right) + f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) + 2f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right], \quad (27)$$

respectively. Approximations (26) and (27) are the more exact the smaller are the dimensions of the rectangle; as follows from the above-said, they are exact for third-degree polynomials.

(b) Partitioning the rectangle by straight lines, parallel to the coordinate axes, into $4mn$ equal rectangles, applying formula (26) to each of these rectangles and summing up the results, we arrive at the formula

$$\int_a^b\int_c^d f(x, y)\,dx\,dy \approx \frac{(b-a)(d-c)}{12mn}(S_0 + 2S_1 + 4S_2 + 8S_3), \quad (28)$$

where

$$S_0 = f(a, c) + f(a, d) + f(b, c) + f(b, d),$$

$$S_1 = \sum_{i=1}^{m-1} [f(x_{2i}, c) + f(x_{2i}, d)] + \sum_{j=1}^{n-1} [f(a, y_{2j}) + f(b, y_{2j})],$$

$$S_2 = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} f(x_{2i}, y_{2j}),$$

$$S_3 = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f\left(\frac{x_{2i} + x_{2(i+1)}}{2}, \frac{y_{2j} + y_{2(j+1)}}{2}\right).$$

If we use (27) in the preceding argument, then

$$\int_a^b\int_c^d f(x, y)\, dx\, dy \approx \frac{(b-a)(d-c)}{6mn}(S_1 + S_2 + 4S_3), \tag{29}$$

where

$$S_1 = \sum_{j=0}^{n-1} \left[f\left(a, \frac{y_{2j} + y_{2(j+1)}}{2}\right) + f\left(b, \frac{y_{2j} + y_{2(j+1)}}{2}\right)\right]$$

$$+ \sum_{i=0}^{m-1} \left[f\left(\frac{x_{2i} + x_{2(i+1)}}{2}, c\right) + f\left(\frac{x_{2i} + x_{2(i+1)}}{2}, d\right)\right],$$

$$S_2 = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f\left(\frac{x_{2i} + x_{2(i+1)}}{2}, \frac{y_{2j} + y_{2(j+1)}}{2}\right),$$

$$S_3 = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left[f\left(\frac{x_{2i} + x_{2(i+1)}}{2}, y_{2j}\right) + f\left(x_{2i}, \frac{y_{2j} + y_{2(j+1)}}{2}\right)\right].$$

Approximations (25)-(28) yield the exact result if the integrand is a polynomial of degree not higher than the third in the variables x, y, i.e. $f(x, y) \equiv P_3(x, y)$.

(c) Suppose the domain D is defined by the inequalities $x_0 \leqslant x \leqslant x_2$, $y_0(x) \leqslant y \leqslant y_2(x)$. We partition the domain D into four subdomains by the straight line $x_1 = (x_0 + x_2)/2$ and the curve $y_1(x) = [y_0(x) + y_2(x)]/2$ (Fig. 75). We denote $y_j(x_i) = y_{ij}$. As before, $f(x_i, y_{ij}) = z_{ij}$ $(i, j = 0, 1, 2)$. Let us consider

$$I = \iint_D f(x, y)\, dx\, dy = \int_{x_0}^{x_2} dx \int_{y_0(x)}^{y_2(x)} f(x, y)\, dy.$$

Applying the Simpson formula for $n = 2$ several times, we obtain the approximation

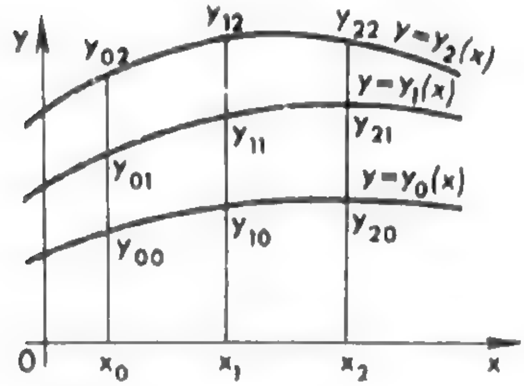


Fig. 75

$$\int_D\int f(x, y)\,dx\,dy \approx \frac{x_2 - x_0}{36}[(y_{02} - y_{00})(z_{00} + 4z_{01} + z_{02})$$

$$+ 4(y_{12} - y_{10})(z_{10} + 4z_{11} + z_{12}) + (y_{22} - y_{20})(z_{20} + 4z_{21} + z_{22})] \qquad (30)$$

Note that if $y_2(x) - y_0(x) = k = \text{const}$, (30) assumes the form

$$\int_D\int f(x, y)\,dx\,dy \approx k \cdot \frac{x_2 - x_0}{36}$$

$$\times\ [z_{00} + z_{02} + z_{20} + z_{22} + 4(z_{01} + z_{10} + z_{12} + z_{21}) + 16\,z_{11}]. \qquad (31)$$

In particular, (31) is valid if the domain of integration D is a rectangle $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ with sides parallel to the coordinate axes. In that case

$$\int_D\int f(x, y)\,dx\,dy \approx \frac{(b - a)(d - c)}{36}\Big\{f(a, c) + f(a, d)$$

$$+ f(b, c) + f(b, d) + 4\left[f\left(a, \frac{c + d}{2}\right) + f\left(b, \frac{c + d}{2}\right)\right.$$

$$\left. + f\left(\frac{a + b}{2}, c\right) + f\left(\frac{a + b}{2}, d\right)\right] + 16f\left(\frac{a + b}{2}, \frac{c + d}{2}\right)\Big\}. \qquad (32)$$

Formula (30) yields the exact result if the integrand is a third-degree polynomial in y for a fixed x and the result of the approximation of the inner integral is a polynomial of degree not higher than the third in x for a fixed y (or in y for a fixed x).

(d) If the integration domain D is a circle with centre at the origin and radius r (Fig. 76), then to approximate a double integral, it is expedient to pass to polar coordinates:

$$\int_D\int f(x, y)\,dx\,dy = \int_0^{2\pi} d\varphi \int_0^r f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho.$$

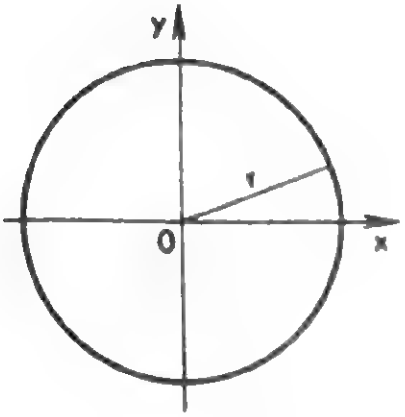


Fig. 76

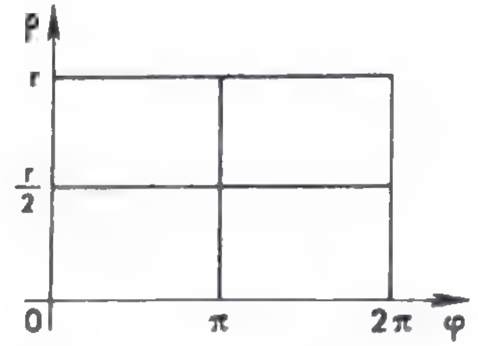


Fig. 77

Let us partition the rectangle in the plane $\varphi O \rho$ (Fig. 77) by the straight lines $\varphi = \pi$ and $\rho = r/2$ into four congruent parts. Calculating the values of the integrand at the nodes and applying successively formulas (24) and (25), we get, respectively,

$$\int\limits_D\int f(x, y)\, dx\, dy \approx \frac{S}{3}\left[f(r, 0) + 2f\left(-\frac{r}{2}, 0\right)\right], \tag{33}$$

$$\int\limits_D\int f(x, y)\, dx\, dy \approx \frac{S}{3}\left[f\left(\frac{r}{2}, 0\right) + f(-r, 0) + f\left(-\frac{r}{2}, 0\right)\right], \tag{34}$$

where $S = \pi r^2$ is the area of the circle. Approximations (33) and (34) are exact when $F(\rho, \varphi)$ is a polynomial of degree not higher than the third in ρ and φ.

Using (32), we get

$$\int\limits_D\int f(x,y)\, dx\, dy \approx \frac{S}{9}\left[f(r, 0) + 2f\left(\frac{r}{2}, 0\right) + 2f(-r, 0) + 4f\left(-\frac{r}{2}, 0\right)\right] \tag{35}$$

This formula is exact if the function $F(\rho, \varphi)$ is a polynomial of degree not higher than the third in ρ for a fixed φ (or in φ for a fixed ρ).

(e) If the integration domain is bounded by an ellipse $x^2/a^2 + y^2/b^2 = 1$, then with the aid of a transformation of the coordinates by the formulas $x = a\rho \cos \varphi$, $y = a\rho \sin \varphi$ the double integral can be rewritten in the form

$$I = \int\limits_D\int f(x, y)\, dx\, dy \approx \int\limits_0^{2\pi}\int\limits_0^{1} ab\rho \cdot f(a\rho \cos \varphi, b\rho \sin \varphi)\, d\rho d\varphi.$$

For such a domain, (24), (25) and (32) assume the respective forms

$$I \approx \frac{S}{3}\left[f(a, 0) + 2f\left(-\frac{a}{2}, 0\right)\right], \tag{36}$$

$$I \approx \frac{S}{3}\left[f\left(\frac{a}{2}, 0\right) + f(-a, 0) + f\left(-\frac{a}{2}, 0\right)\right], \tag{37}$$

$$I \approx \frac{S}{3}\left[f(a, 0) + 2f\left(\frac{a}{2}, 0\right) + 2f(-a, 0) + 4f\left(-\frac{a}{2}, 0\right)\right. \tag{38}$$

where $S = \pi ab$ is the area of the ellipse.

1142. Calculate the double integral $I = \iint\limits_D (x + y)^2 dx\, dy$, if the domain D is bounded by the curves $x = 1$, $x = 3$, $y = x^2$, $y = x + x^2$.

Solution. We first find the exact value of the integral:

$$I = \int_1^3 \int_{x^2}^{x+x^2} (x + y)^2 dx\, dy = \frac{1}{3}\int_1^3 (x + y)^3 \Big|_{x^2}^{x + x^2} dx$$

$$= \frac{1}{3}\int_1^3 [(2x + x^2)^3 - (x + x^2)^3]\, dx$$

$$= \frac{1}{3}\int_1^3 (7x^3 + 9x^4 + 5x^5)\, dx = \frac{1}{3}\left[\frac{7}{4}x^4 + \frac{9}{5}x^5 + \frac{1}{2}x^6\right]_1^3$$

$$= \frac{1}{3}\cdot\left(\frac{567}{4} + \frac{2187}{5} + \frac{729}{2} - \frac{7}{4} - \frac{9}{5} - \frac{1}{2}\right) = 313.2.,$$

Then we approximate the double integral using (4), (6), (7), (8) and (22) and compare the values obtained with the exact values.

We set $m = 4$, $n = 4$ and calculate the values y_{ij} ($i = 0, 1, 2, 3$; $j = 0, 1, 2, 3$) using (5):

$$y_{ij} = x_i^2 + \frac{x_i}{4} j = x_i\left(x_i + \frac{1}{4} j\right).$$

Since $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, $x_4 = 3$, it follows that $y_{00} = 1$; $y_{01} = 1.25$; $y_{02} = 1.5$; $y_{03} = 1.75$; $y_{04} = 2$; $y_{10} = 2.25$; $y_{11} = 2.625$; $y_{12} = 3$; $y_{13} = 3.375$; $y_{14} = 3.75$; $y_{20} = 3.75$; $y_{21} = 4.5$; $y_{22} = 5$; $y_{23} = 5.5$; $y_{24} = 6$; $y_{30} = 6.25$; $y_{31} = 6.875$; $y_{32} = 7.5$; $y_{33} = 8.125$; $y_{34} = 8.75$; $y_{40} = 9$; $y_{41} = 9.75$; $y_{42} = 10.5$; $y_{43} = 11.25$; $y_{44} = 12$. In accordance with (3), we have

$$\Delta\omega_i = \frac{1}{4}\int_{x_i}^{x_{i+1}} x\,dx = \frac{1}{8}x^2\Bigg|_{x_i}^{x_{i+1}} = \frac{1}{8}(x_{i+1}^2 - x_i^2).$$

For $i = 0, 1, 2, 3$, we get $\Delta\omega_0 = 0.156$; $\Delta\omega_1 = 0.219$; $\Delta\omega_2 = 0.281$; $\Delta\omega_3 = 0.344$.

Furthermore, since

$$z_{ij} = (x_i + y_{ij})^2 \quad (i = 0, 1, 2, 3; j = 0, 1, 2, 3),$$

it follows that $z_{00} = 4$, $z_{01} = 5.063$; $z_{02} = 6.25$; $z_{03} = 7.563$; $z_{04} = 9$; $z_{10} \approx 14.063$; $z_{11} \approx 17.016$; $z_{12} = 20.25$; $z_{13} \approx 23.766$; $z_{14} \approx 27.563$; $z_{20} = 36$; $z_{21} = 42.25$; $z_{22} = 49$; $z_{23} = 56.25$; $z_{24} = 64$; $z_{30} \approx 76.563$; $z_{31} \approx 87.891$; $z_{32} = 100$; $z_{33} = 112.891$; $z_{34} \approx 126.563$; $z_{40} \approx 144$; $z_{41} \approx 162.563$; $z_{42} = 182.250$; $z_{43} = 199.516$; $z_{44} = 225$.

Using now (4), (6), (7) and (8), we find, respectively,

$$\sum_{i=0}^{3}\Delta\omega_i\sum_{j=0}^{3} z_{ij} = \sum_{i=0}^{3}\Delta\omega_i(z_{i0} + z_{i1} + z_{i2} + z_{i3})$$
$$\approx 22.876\cdot 0.156 + 75.095\cdot 0.219 + 183.5\cdot 0.281 + 377.345\cdot 0.344 \approx 201.386;$$

$$\sum_{i=0}^{3}\Delta\omega_i\sum_{j=0}^{3} z_{i+1,j} = \sum_{i=0}^{3}\Delta\omega_i(z_{i+1,0} + z_{i+1,1}$$
$$+ z_{i+1,2} + z_{i+1,3}) \approx 75.095\cdot 0.156 + 183.5\cdot 0.219$$
$$+ 377.345\cdot 0.281 + 688.329\cdot 0.344 \approx 394.721;$$

$$\sum_{i=0}^{3}\Delta\omega_i\sum_{j=0}^{3} z_{i,j+1} = \sum_{i=0}^{3}\Delta\omega_i(z_{i1} + z_{i2} + z_{i3} + z_{i4})$$
$$\approx 27.876\cdot 0.156 + 88.595\cdot 0.219 + 211.5\cdot 0.281 + 427.345\cdot 0.344 \approx 230.190;$$

$$\sum_{i=0}^{3}\Delta\omega_i\sum_{j=0}^{3} z_{i+1,j+1} = \sum_{i=0}^{3}\Delta\omega_i(z_{i+1,1} + z_{i+1,2} + z_{i+1,3} + z_{i+1,4})$$
$$\approx 88.595\cdot 0.156 + 211.5\cdot 0.219 + 427.345\cdot 0.281 + 769.329\cdot 0.344 \approx 444.873.$$

The absolute and relative errors of the values obtained are rather large which can be explained by the smallness of the numbers m and n.

Applying approximation (22), we get

$$I \approx \frac{201.386 + 230.190 + 394.721 + 444.873}{4} = \frac{1271.17}{4} \approx 317.793$$

Then the relative error is

$$\delta = \frac{317.793 - 313.2}{313.2}\cdot 100\% \approx 1.3\%.$$

1143. Using inequality (21), estimate the double integral $I = \iint_D (x^2 + y^2)\,dx\,dy$, if the domain D is a rectangle bounded by the straight lines $x = 0$, $x = 4$, $y = 1$, $y = 5$.

Solution. Here $f(x, y) = x^2 + y^2$, $f'_x(x, y) = 2x$, $f'_y(x, y) = 2y$, $f''_{x^2}(x, y) = 2$, $f''_{y^2}(x, y) = 2$, $f''_{xy}(x, y) = 0$ and, therefore, the conditions $AC - B^2 > 0$, $A > 0$, $C > 0$ are satisfied. We set $m = 4$, $n = 4$. The values of x and y corresponding to the partition points are the following: $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$; $y_0 = 1$, $y_1 = 2$, $y_2 = 3$, $y_3 = 4$, $y_4 = 5$. Since $z_{ij} = x_i^2 + y_j^2$, it follows that $z_{00} = 1$, $z_{01} = 4$, $z_{02} = 9$, $z_{03} = 16$, $z_{04} = 25$, $z_{10} = 2$, $z_{11} = 5$, $z_{12} = 10$, $z_{13} = 17$, $z_{14} = 26$, $z_{20} = 5$, $z_{21} = 8$, $z_{22} = 13$; $z_{23} = 20$, $z_{24} = 29$, $z_{30} = 10$, $z_{31} = 13$, $z_{32} = 18$, $z_{33} = 25$, $z_{34} = 34$, $z_{40} = 17$, $z_{41} = 20$, $z_{42} = 25$, $z_{43} = 32$, $z_{44} = 41$. In accordance with (20) we have

$$I \approx \frac{1}{4}(S_0 + 2S_1 + 4S_2),$$

where

$$S_0 = 1 + 25 + 17 + 41 = 84,\ S_1 = 2 + 5 + 10 + 20 + 25 + 32 + 34 + 29 + 26 + 4 + 9 + 16 = 212, S_2 = 5 + 10 + 17 + 8 + 13 + 20 + 13 + 18 + 25 = 129.$$

Consequently,

$$I \approx \frac{1}{4}(84 + 2 \cdot 212 + 4 \cdot 129) = \frac{1}{4} \cdot 1024 = 256.$$

To approximate the double integral by (17), we first find $\bar{x}_i = (x_{i+1} + x_i)/2$, $\bar{y}_j = (y_{j+1} + y_j)/2$ $(i = 0, 1, 2, 3; j = 0, 1, 2, 3)$; we have $\bar{x}_0 = 0.5$; $\bar{x}_1 = 1.5$; $\bar{x}_2 = 2.5$; $\bar{x}_3 = 3.5$; $\bar{y}_0 = 1.5$; $\bar{y}_1 = 2.5$; $\bar{y}_2 = 3.5$; $\bar{y}_3 = 4.5$. We denote $\bar{z}_{ij} = \bar{x}_i^2 + \bar{y}_j^2$ and calculate

$$\begin{aligned}
&\bar{z}_{00} = 0.5^2 + 1.5^2 = 2.5; &&\bar{z}_{10} = 1.5^2 + 1.5^2 = 4.5;\\
&\bar{z}_{01} = 0.5^2 + 2.5^2 = 6.5; &&\bar{z}_{11} = 1.5^2 + 2.5^2 = 8.5;\\
&\bar{z}_{02} = 0.5^2 + 3.5^2 = 12.5; &&\bar{z}_{12} = 1.5^2 + 3.5^2 = 14.5;\\
&\bar{z}_{03} = 0.5^2 + 4.5^2 = 20.5; &&\bar{z}_{13} = 1.5^2 + 4.5^2 = 22.5;\\
&\bar{z}_{20} = 2.5^2 + 1.5^2 = 8.5; &&\bar{z}_{30} = 3.5^2 + 1.5^2 = 14.5;\\
&\bar{z}_{21} = 2.5^2 + 2.5^2 = 12.5; &&\bar{z}_{31} = 3.5^2 + 2.5^2 = 18.5;\\
&\bar{z}_{22} = 2.5^2 + 3.5^2 = 18.5; &&\bar{z}_{32} = 3.5^2 + 3.5^2 = 24.5;\\
&\bar{z}_{23} = 2.5^2 + 4.5^2 = 26.5^2; &&\bar{z}_{33} = 3.5^2 + 4.5^2 = 32.5.
\end{aligned}$$

Then we have

$$I \approx \frac{4\cdot 4}{4\cdot 4}(2.5 + 6.5 + 12.5 + 20.5 + 4.5 + 8.5 + 14.5$$

$$+ 22.5 + 8.5 + 12.5 + 18.5 + 26.5 + 14.5 + 18.5 + 24.5 + 32.5) = 248.$$

Thus, $248 < I < 256$.

We find the exact value of the integral:

$$I = \int_0^4 \int_1^5 (x^2 + y^2) \quad dx\,dy = \int_0^4 \left[x^2y + \frac{1}{3}\,y^3\right]_1^5 dx$$

$$= \int_0^4 \left(5x^2 + \frac{125}{3} - x^2 - \frac{1}{3}\right) dx = \left[\frac{4}{3}\,x^3 + \frac{124}{3}\,x\right]_0^4 = 250\frac{2}{3} \approx 250.667.$$

Thus, approximations (20) and (17) have the respective relative errors

$$\delta_1 = \frac{256 - 250.667}{250.667}\cdot 100\% \approx 2.1\%;\ \delta_2 = \frac{250.667 - 248}{250.667}\cdot 100\% \approx 1.1\%.$$

1144. Using (32), approximate the double integral $I = \int\limits_D (x^2 + 2y)\,dx\,dy$, if the domain D is a rectangle $0 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 6$.

Solution. Let us calculate the exact value of the integral:

$$I = \int_0^4 dx \int_0^6 (x^2 + 2y)\,dy = \int_0^4 \quad [x^2y + y^2]_0^6 dx$$

$$= \int_0^4 (6x^2 + 36)\,dx = [2x^3 + 36x]_0^4 = 272.$$

Here $a = 0$, $b = 4$, $c = 0$, $d = 6$; $f(x, y) = x^2 + 2y$; $f(a, c) = 0$; $f(a, d) = 12$; $f(b, c) = 16$; $f(b, d) = 28$; $f\left(a, \frac{c+d}{2}\right) = 6$; $f\left(b, \frac{c+d}{2}\right) = 22$, $f\left(\frac{a+b}{2}, c\right) = 4$;

$f\left(\frac{a+b}{2}, d\right) = 16$; $f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = 10$. Applying (32), we find

$$I = \int_0^4 \int_0^6 (x^2 + 2y)\,dx\,dy = \frac{4\cdot 6}{36}[0 + 12 + 16 + 28$$

$$+ 4(6 + 22 + 4 + 16) + 16 \cdot 10] = \frac{2}{3}(56 + 4 \cdot 48 + 160) = 272.$$

We have obtained the exact result since the integrand $f(x, y) = x^2 + 2y$ is a polynomial in x and y of degree lower than the third.

1145. Calculate the double integral $I = \iint\limits_D \left(\frac{x^2}{9} + \frac{y^2}{4}\right)^{3/2} dx\, dy$, if the integration domain D is defined by the inequalities $-3 \leqslant x \leqslant 3$, $-2 \leqslant y \leqslant 2$.

Solution. We pass to polar coordinates, setting $x = 3\rho \cos \varphi$, $y = 2\rho \sin \varphi$. Then $x^2/9 + y^2/4 = \rho^2$, $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant \rho \leqslant 1$. Using (38), we find the approximate value of the integral:

$$I \approx \frac{2}{3}\pi\left(1 + 2 \cdot \frac{1}{8} + 2 + 4 \cdot \frac{1}{8}\right) = \frac{2}{3}\pi\left(3 + \frac{1}{2} + \frac{1}{4}\right) = 2.5\pi.$$

The exact value of the integral is

$$I = 6 \int_0^{2\pi}\int_0^1 \rho^4\, d\rho\, d\varphi = \frac{6}{5}\int_0^{2\pi} \rho^5 \Big|_0^1 d\varphi = \frac{6}{5}\varphi \Big|_0^{2\pi} = 2.4\pi.$$

The relative error is

$$\delta = (2.5\pi - 2.4\pi)/2.4\pi \cdot 100\% \approx 4.2\%$$

1146. Approximate the double integral $I = \iint\limits_D \left(\frac{x}{2} + y\right) dx\, dy$ by formula (30), if the domain D is bounded by the curves $x = 2$, $x = 4$, $y = x^2/2$, $y = 2x$.

Solution. We find the exact value of the integral:

$$I = \int_2^4 dx \int_{x^2/2}^{2x} \left(\frac{x}{2} + 2\right) dy = \int_2^4 \left[\frac{x}{2}y + \frac{y^2}{2}\right]_{x^2/2}^{2x} dx$$

$$= \int_2^4 \left(x^2 + 2x^2 - \frac{x^3}{4} - \frac{x^4}{8}\right) dx = \left[x^3 - \frac{x^4}{16} - \frac{x^5}{40}\right]_2^4$$

$$= 64 - 16 - 25.6 - 8 + 1 + 0.8 = 16.2$$

Here

$$x_0 = 2,\ \ x_2 = 4,\ \ x_1 = (x_0 + x_2)/2 = 3,\ y_0 = x^2/2,\ \ y_2 = 2x,$$
$$y_1 = (y_0 + y_2)/2 = x^2/4 + x;$$
$$y_{ij} = y_j(x_i);\ \ y_{00} = y_0(x_0) = 2,\ \ y_{01} = y_1(x_0) = 3,$$
$$y_{02} = y_2(x_0) = 4,\ \ y_{10} = y_0(x_1) = 4.5,\ \ y_{11} = y_1(x_1) = 5.25,$$

$y_{12} = y_2(x_1) = 6,\quad y_{20} = y_0(x_2) = 8,\quad y_{21} = y_1(x_2) = 8,$
$y_{22} = y_2(x_2) = 8.\ z_{ij} = f(x_i, y_{ij}) = x_i/2 + y_{ij}\quad (i, j = 0, 1, 2);$
$z_{00}=3;\ z_{01} = 4,\ z_{02}=5,\ z_{10}=6,\ z_{11}=6.75,\ z_{12}=7.5,\ z_{20}=10,\ z_{21}=10,\ z_{22}=10,$

By (30) we get

$$I = \int_2^4 \int_{x^2/2}^{2x} \left(\frac{x}{2} + y\right) dx\, dy$$

$$= \frac{4-2}{36}[(4-2)(3 + 4\cdot 4 + 5) + 4(6 - 4.5)(6 + 4\cdot 6.75 + 7.5)$$
$$+ (8-8)(10 + 4\cdot 10 + 10)] = 16\frac{1}{6} = 16.167.$$

1147. Using (26) and (32), approximate the double integral $I = \iint_D (xy + 3\sqrt{y}),\, dx$ dy, if the domain D is a rectangle $0 \leqslant x \leqslant 2,\ 1 \leqslant y \leqslant 9$.

9.5. Using the Monte Carlo Method to Calculate Definite and Multiple Integrals

9.5.1. Calculating definite integrals by the Monte Carlo Method. (a) It is required to calculate the integral $\int_0^1 \varphi(t)\, dt$. Assume that t is a uniformly distributed random variable, $p(t)$ is the density function of that random variable:

$$p(t) = \begin{cases} 0 & \text{if } t < 0; \\ 1, & \text{if } 0 \leqslant t \leqslant 1; \\ 0, & \text{if } t > 1. \end{cases}$$

Then, the mathematical expectation of the stochastic function $\varphi(t)$ is specified by the equation

$$M\left[\varphi(t)\right] = \int_0^1 \varphi(t)\, p(t)\, dt.$$

Taking into account the values of $p(t)$, we get

$$M\left[\varphi(t)\right] = \int_0^1 \varphi(t)\, dt. \tag{1}$$

Let us find the approximate value of the mathematical expectation. Suppose that as a result of N trials we have obtained N values of the random variable: $t_1, t_2, \ldots, t_N$ These values can be taken from the table of random numbers (see Table VI in Appendix). Then, in accordance with Chebyshev's theorem, the approximate value of

$M + \varphi(t)$] will be given by the equation

$$M[\varphi(t)] \approx \frac{1}{N}\sum_{i=1}^{N} \varphi(t_i) \tag{2}$$

Approximations (1) and (2) yield

$$\int_0^1 \varphi(t)\,dt \approx \frac{1}{N}\sum_{i=1}^{N} \varphi(t_i) \tag{3}$$

(b) Let us now consider a general case: suppose we have to find $\int_a^b f(x)\,dx$. We pass to a new variable t with the aid of the equality $x = a + (b - a)t$. Then we have

$$\int_a^b f(x)\,dx = (b-a)\int_0^1 \varphi(t)\,dt, \tag{4}$$

where $\varphi(t) = f[a + (b - a)t]$. Using (3) for approximating the integral on the right-hand side of (4), we get

$$\int_a^b f(x)\,dx \approx \frac{b-a}{N}\sum_{i=1}^{N} \varphi(t_i), \quad \text{or} \quad \int_a^b f(x)dx \approx \frac{b-a}{N}\sum_{i=1}^{N} f(x_i), \tag{5}$$

where $x_i = a + (b - a)t_i$ $(i = 1, 2, \ldots, N)$.

The table for calculating the definite integral by (5) has the form

i	t_i	$x_i = a + (b-a)t_i$	$f(x_i)$
1	t_1	x_1	$f(x_1)$
2	t_2	x_2	$f(x_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	t_N	x_N	$f(x_N)$
			$\sum_{i=1}^{N} f(x_i)$

The method of approximating definite integrals with the aid of formula (5) just discussed is one of the special cases of the method of statistical sampling (Monte Carlo method).

(c) We indicate another technique of calculating definite integrals based on the application of the Monte Carlo method. It follows from the geometrical meaning of a definite integral that the integral $I = \int_a^b f(x)dx$ expresses the area of a curvilinear

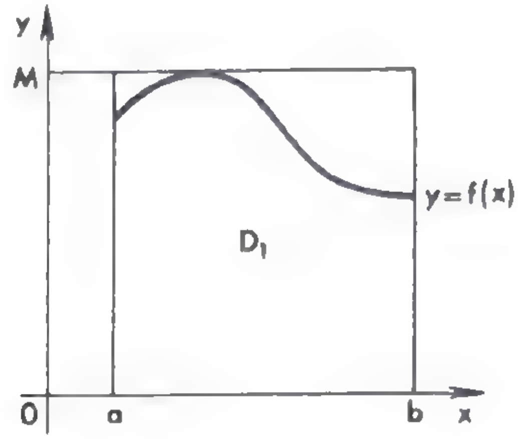


Fig. 78

trapezoid bounded by the curves $x = a, x = b, y = 0, y = f(x)$, if the function $f(x)$ is continuous and nonnegative on the interval $[a, b]$. Let us consider a rectangle bounded by the straight lines $x = a, x = b, y = 0, y = M$, where $M \geqslant \max\limits_{a \leqslant x \leqslant b} f(x)$ (Fig. 78). If $f(x)$ satisfies the inequality $f(x) \geqslant 0$ not at all points of the interval $[a, b]$, we shall use the identity

$$\int_a^b f(x)\,dx = \int_a^b [f(x) + h]\,dx - h(b - a),$$

where the number $h > 0$ is chosen such that $f(x) + h \geqslant 0$ for $x \in [a, b]$.

This method, as well as the preceding one, is based on the use of the table of random numbers belonging to the interval [0, 1]. Therefore, it is necessary to pass from the variables x, y to the variables ξ, η so that the domain D_1 would transform into a certain domain D lying in the interior of the unit square $0 \leqslant \xi \leqslant 1, 0 \leqslant \eta \leqslant 1$ (Fig. 79). For that purpose, we set $x = a + (b - a)\xi, y = M\eta$. Then, $dx = (b - a)\,d\xi$ and as x varies in the range from a to b the variable ξ assumes the values from 0 to 1. The given definite integral is transformed as

$$I = (b - a)M \int_0^1 \varphi(\xi)\,d\xi, \tag{6}$$

where

$$\varphi(\xi) = \frac{1}{M} f[a + (b - a)\xi] \tag{7}$$

It follows from (7) that $f(x) = M\varphi(\xi)$. Let us consider the set of random points $(\xi_1; \eta_1), (\xi_2; \eta_2), \ldots, (\xi_N; \eta_N)$, uniformly distributed over a unit square. Suppose n points fall in the domain D. Since the random points are distributed uniformly, it follows that

$$\frac{n}{N} \xrightarrow{\text{in probability}} \frac{\int_0^1 \varphi(\xi)\,d\xi}{1},$$

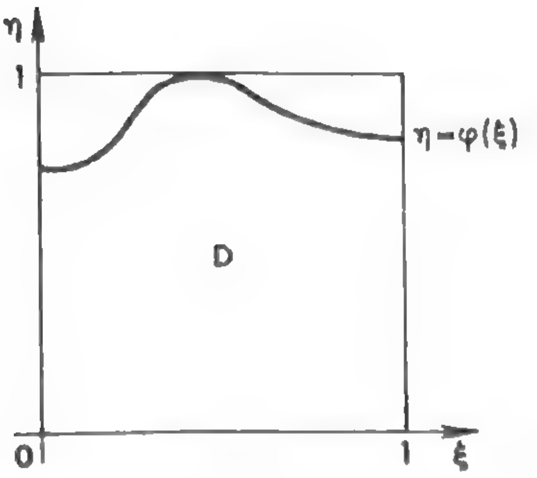


Fig. 79

where the number 1 expresses the area of the unit square. Then we have

$$\int_0^1 \varphi(\xi)\, d\xi \approx \frac{n}{N}. \tag{8}$$

We can infer from (6) and (8) that

$$\int_a^b f(x)\, dx \approx \frac{(b-a)\, n \cdot M}{N}. \tag{9}$$

This is the approximation for calculating definite integrals by the Monte Carlo method.

Approximation (9) can be rewritten in the form

$$\frac{\int_a^b f(x)dx}{M(b-a)} \approx \frac{n}{N}, \tag{10}$$

from which it follows that the ratio between the area of the curvilinear trapezoid D_1 and the area of the rectangle (see Fig. 78) is approximately equal to the ratio between the number of the random points lying in the interior of the curvilinear trapezoid and the number of the random points lying inside the rectangle.

The table for calculating definite integrals by (9) is of the form

i	ξ_i	η_i	$x_i = a + (b-a)\xi_i$	$y_i = M\eta_i$	$Y_i = f(x_i)$
1	ξ_1	η_1	x_1	y_1	$f(x_1)$
2	ξ_2	η_2	x_2	y_2	$f(x_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	ξ_N	η_N	x_N	y_N	$f(x_N)$

Among the values y_i ($i = 1, 2, \ldots, N$) we must choose those for which the inequality $y_i < Y_i$ is satisfied. The number of such values is n.

9.5.2. Calculating multiple integrals by the Monte Carlo method. (a) It is required to calculate $\iint\limits_D f(x, y)\,dx\,dy$, where the domain D is defined by the inequalities $a \leqslant x \leqslant b$, $\varphi_1(x) \leqslant y \leqslant \varphi_2(x)$. We shall assume that the functions $\varphi_1(x)$ and $\varphi_2(x)$ continuous on the interval $[a, b]$ satisfy the inequalities $\varphi_1(x) \geqslant c$, $\varphi_2(x) \leqslant d$ (Fig. 80).

We perform a change of variables by the formulas $x = a + (b - a)\xi$, $y = c + (d - c)\eta$. Under this transformation, the domain D passes into a domain Δ contained in the unit square $0 \leqslant \xi \leqslant 1$, $0 \leqslant \eta \leqslant 1$ (Fig. 81). Suppose n is the number of random points $(\xi_i; \eta_i)$ ($i = 1, 2, \ldots, n$) which have fallen in the domain Δ and N is the number of random points lying in the interior of the unit square. It is evident that the domain D will contain n points $(x_i; y_i)$, where $x_i = a + (b - a)\,\xi_i$, $y_i = c + (d - c)\eta_i$ ($i = 1, 2, \ldots, n$). By the mean-value theorem we have

$$\iint\limits_D f(x, y)\,dx\,dy \approx f(\bar{x}, \bar{y}) \cdot S, \tag{11}$$

where $(\bar{x}; \bar{y}) \in D$ and S is the area of the domain D. We shall take for the approximate value of $f(\bar{x}, \bar{y})$ the arithmetic mean of the values of the function $f(x, y)$ at n random points which have fallen in the domain D:

$$f(\bar{x}, \bar{y}) \approx \frac{1}{n} \cdot \sum_{i=1}^{n} f(x_i, y_i). \tag{12}$$

Taking (11) and (12) into account, we obtain

$$\iint\limits_D f(x, y)\,dx\,dy \approx \frac{S}{n} \cdot \sum_{i=1}^{n} f(x_i, y_i). \tag{13}$$

It is convenient to use (13) if the area S is easy to compute.

By analogy with (10) we can write

$$\frac{S}{(d - c)(b - a)} \approx \frac{n}{N},$$

where S is the area of the domain D. Then we have

$$S \approx \frac{n(b - a)(d - c)}{N}. \tag{14}$$

Formulas (13) and (14) yield the approximation of a double integral:

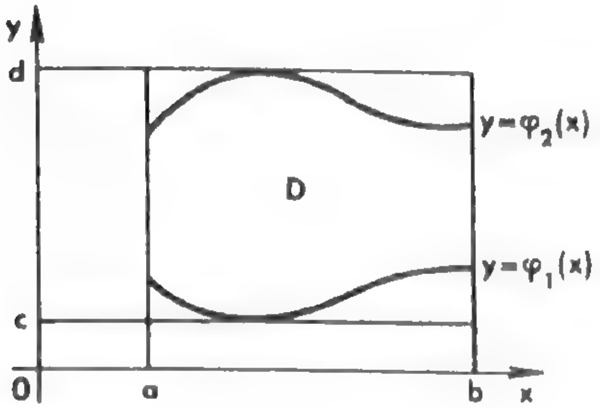


Fig. 80

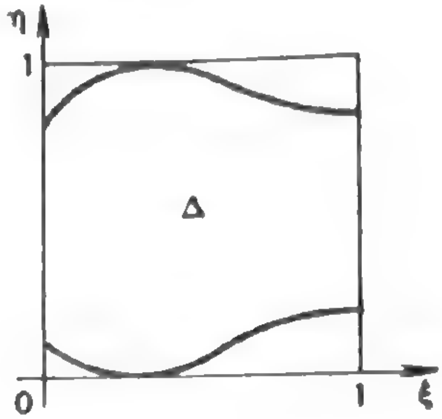


Fig. 81

$$\int_D f(x, y)\, dx\, dy \approx \frac{(b-a)(d-c)}{N} \cdot \sum_{i=1}^{n} f(x_i, y_i). \tag{15}$$

The following table is convenient for approximating double integrals by (15):

i	ξ_i	η_i	$x_i = a + (b-a)\xi_i$	$y_i = c + (d-c)\eta_i$	$\underline{y}_i = \varphi_1(x_i)$	$\bar{y}_i = \varphi_2(x_i)$	$f(x_i, y_i)$
1	ξ_1	η_1	x_1	y_1	$\varphi_1(x_1)$	$\varphi_2(x_1)$	$f(x_1, y_1)$
2	ξ_2	η_2	x_2	y_2	$\varphi_1(x_2)$	$\varphi_2(x_2)$	$f(x_2, y_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	ξ_N	η_N	x_N	y_N	$\varphi_1(x_N)$	$\varphi_2(x_N)$	$f(x_N, y_N)$

Among the values $y_i (1 \leqslant i \leqslant N)$ we must choose those for which the condition $\underline{y}_i \leqslant y_i \leqslant \bar{y}_i$ is satisfied. They are n in number.

(b) Let us generalize (9) for the case of the double integral $\iint_D f(x, y)\, dx\, dy$, if the domain D is defined by the inequalities $a \leqslant x \leqslant b$, $\varphi_1(x) \leqslant y \leqslant \varphi_2(x)$. We denote by M the number such that $M \geqslant \max\limits_{\substack{a \leqslant x \leqslant b \\ c \leqslant y \leqslant d}} f(x, y)$. As is known, the double integral $\iint_D f(x, y)\, dx\, dy$ expresses the volume of the cylindrical body V defined by the inequalities $a \leqslant x \leqslant b$, $\varphi_1(x) \leqslant y \leqslant \varphi_2(x)$, $0 \leqslant z \leqslant f(x, y)$. This cylindrical body lies within the parallelepiped $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$, $0 \leqslant z \leqslant M$.

We pass to the new variables ξ, η, ζ by the formulas $x = a + (b-a)\xi$, $y = c + (d-c)\eta$, $z = M\zeta$. Then the domain V is transformed into the domain Ω defined by the inequalities

$$0 \leqslant \xi \leqslant 1, \quad \frac{\varphi_1(x) - c}{d - c} \leqslant \eta \leqslant \frac{\varphi_2(x) - c}{d - c}, \quad 0 \leqslant \zeta \leqslant 1.$$

The domain Ω lies inside the unit cube bounded by the planes $\xi = 0$, $\xi = 1$, $\eta = 0$,

$\eta = 1, \zeta = 0, \zeta = 1$. Hence

$$I = (b - a)(d - c) \cdot M \iint_\Delta \varphi(\xi, \eta)\, d\xi\, d\eta,$$

where $\varphi(\xi, \eta) = \frac{1}{M} f\,[a + (b - a)\xi, c + (d - c)\eta]$, and Δ is the domain obtained from D as a result of the change of variables.

Let us consider the set of stochastic points $(\xi_1; \eta_1; \zeta_1), (\xi_2; \eta_2; \zeta_2), \ldots, (\xi_N; \eta_N; \zeta_N)$ uniformly distributed in the interior of the unit cube. The number of such points falling in the domain Δ will be denoted by n. Since the stochastic points are distributed uniformly, we have

$$\frac{n}{N} \xrightarrow{\text{in probability}} \iint_\Delta \varphi(\xi, \eta)\, d\xi\, d\eta, \quad \text{or} \quad \iint_\Delta \varphi(\zeta, \eta) d\xi\, d\eta \approx \frac{n}{N}.$$

Returning to the variables x and y, we obtain an approximation for double integrals by the Monte Carlo method:

$$\iint_\Delta f(x, y)\, dx\, dy \approx \frac{(b - a)(d - c) \cdot n \cdot M}{N}. \tag{16}$$

The table for calculations performed by (16) has the form

i	ξ_i	η_i	ζ_i	$x_i = a + (b-a)\xi_i$	$y_i = c + (d-c)\eta_i$	$z_i = M\xi_i$	$\underline{y}_i = \varphi_1(x_i)$	$\overline{y}_i = \varphi_2(x_i)$	$z_i = f(x_i, y_i)$
1	ξ_1	η_1	ζ_1	x_1	y_1	z_1	$\varphi_1(x_1)$	$\varphi_2(x_1)$	$f(x_1, y_1)$
2	ξ_2	η_2	ζ_2	x_2	y_2	z_2	$\varphi_2(x_2)$	$\varphi_2(x_2)$	$f(x_2, y_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	ξ_N	η_N	ζ_N	x_N	y_N	z_N	$\varphi_1(x_N)$	$\varphi_2(x_N)$	$f(x_N, y_N)$

We can find the number n as follows: among the values y_i $(i = 1, 2, \ldots, N)$ we must choose those for which there holds the inequality

$$\underline{y}_i < y_i < \overline{y}_i. \tag{17}$$

Correspondingly, among the values z_i we must choose those for which there holds the inequality

$$z_i < Z_i. \tag{18}$$

Note that it is expedient to seek not all the values $Z_i = f(x_i, y_i)$ but only the values corresponding to those y_i for which condition (17) is satisfied.

(c) A formula similar to (9) and (15) holds true for integrals of multiplicity k as well:

$$\iint_V \ldots \int f(x_1, x_2, \ldots, x_k)\, dx_1\, dx_2 \ldots dx_k \approx \frac{n \cdot M}{N} \cdot \prod_{i=1}^{k} (b_i - a_i). \tag{19}$$

where the domain V belongs to the k-dimensional parallelepiped the coordinates of

whose points satisfy the k inequalities $a_i \leqslant x_i \leqslant b_i$ ($i = 1, 2, \ldots, k$) and the function $f(x_1, x_2, \ldots, x_k)$ is continuous in the domain V and satisfies the condition $0 \leqslant f(x_1, x_2, \ldots, x_k) \leqslant M$.

The derivation of formulas (9), (15) and (19) is based on the use of the notion of convergence in probability. Therefore, the ratio n/N is the more stable the larger is N. This means that for any arbitrary small number $\varepsilon > 0$ the probability of the inequality $|I - \tilde{I}| < \varepsilon$, where I is the exact value of the integral and $\tilde{I}$ is its approximate value obtained by the Monte Carlo method, increases with an increase in N. Nevertheless, it may happen so that we shall get $|I - \tilde{I}| > \varepsilon$ for very large N as well, but we come across it in practice very seldom.

As to the Monte Carlo method, the examples we have given are of illustrative nature and are presented to acquaint the reader with the essence of the method.

By virtue of the remarks made above, electronic computers should be used for approximate calculations of integrals by the Monte Carlo method, with the requisite program prepared beforehand.

1148. Using (3), approximate the integral $I = \int_0^1 (1 - t^2)\, dt$, having taken 30 consecutive values from the table of random numbers (see Appendix) and restricting yourself to three digits.

Solution. The table has the form

i	t_i	t_i^2	i	t_i	t_i^2	i	t_i	t_i^2
1	0.857	0.734	11	0.609	0.371	21	0.070	0.005
2	0.457	0.209	12	0.179	0.032	22	0.692	0.478
3	0.499	0.249	13	0.974	0.949	23	0.696	0.484
4	0.762	0.581	14	0.011	0.0001	24	0.203	0.041
5	0.431	0.186	15	0.098	0.010	25	0.350	0.122
6	0.698	0.487	16	0.805	0.648	26	0.900	0.810
7	0.038	0.001	17	0.516	0.266	27	0.451	0.203
8	0.558	0.311	18	0.296	0.088	28	0.318	0.101
9	0.653	0.426	19	0.149	0.022	29	0.798	0.637
10	0.573	0.328	20	0.815	0.664	30	0.111	0.012

Thus we have

$$\sum_{i=1}^{30} (1 - t_i^2) = 30 - \sum_{i=1}^{30} t_i^2 = 30 - 9.455 = 20.545,$$

whence, by (3), we get

$$\int_0^1 (1 - t^2)\, dt = \frac{1}{30} \cdot 20.545 \approx 0.685.$$

The exact value of the integral is

$$I = \left(t - \frac{t^3}{3}\right)\Bigg|_0^1 = \frac{2}{3} \approx 0.667.$$

This means that the absolute error is $|0.667 - 0.685| = 0.018$, and the relative error is $\delta = (0.018/0.667) \cdot 100\% \approx 2.7\%$.

1149. Compute the definite integral $I = \int_2^3 (x^2 + x^3)\,dx$, using approximation (5).

Solution. We take 20 values from the table of random numbers, beginning with the third. The table has the form

i	t_i	$x_i = 2 + t_i$	x_i^2	x_i^3	$f(x_i) = x_i^2 + x_i^3$
1	0.499	2.499	6.245	15.606	21.851
2	0.762	2.762	7.629	21.070	28.699
3	0.431	2.431	5.910	14.367	20.277
4	0.698	2.698	7.279	19.639	26.918
5	0.038	2.038	4.153	8.464	12.617
6	0.558	2.558	6.543	16.738	23.281
7	0.653	2.653	7.038	18.672	25.710
8	0.573	2.573	6.620	17.034	23.654
9	0.609	2.609	6.807	17.759	24.566
10	0.179	2.179	4.748	10.346	15.094
11	0.974	2.974	8.645	26.305	35.150
12	0.011	2.011	4.044	8.133	12.177
13	0.098	2.098	4.402	9.235	13.637
14	0.805	2.805	7.868	22.07	29.938
15	0.516	2.516	6.330	15.926	22.256
16	0.296	2.296	5.276	12.104	17.380
17	0.149	2.149	4.618	9.924	14.542
18	0.815	2.815	7.924	22.307	30.231
19	0.070	2.070	4.285	8.870	13.155
20	0.692	2.692	7.247	19.508	26.755

Using (5) for $a = 2, b = 3, N = 20$, $\sum_{i=1}^{20} f(x_i) = 437.888$, we find

$$I \approx 437.888/20 = 21.894.$$

The exact value of the integral is

$$I = \int_2^3 (x^2 + x^3)\,dx = \left(\frac{x^3}{3} + \frac{x^4}{4}\right)\Bigg|_2^3 = 22\frac{7}{12} \approx 22.583.$$

The relative error is

$$\delta = (22.583 - 21.894)/22.583 \cdot 100\% \approx 3.1\%.$$

1150. Compute the definite integral $I = \int_2^3 (x^2 + x^3)dx$, using approximation (9).

Solution. Here $a = 2$, $b = 3$, $\max\limits_{2 \leqslant x \leqslant 3} (x^2 + x^3) = 36$. We put $x = 2 + \xi$, $y = 36\eta$ and take 40 values from the table of random numbers ($N = 20$). The calculation table has the form

i	ξ_i	η_i	$x_i = 2 + \xi_i$	$y_i = 36\eta_i$	x_i^2	x_i^3	$Y_i = x_i^2 + x_i^3$
1	0.857	0.457	2.857	16.452	8.162	23.319	31.481
2	0.499	0.762	2.499	27.432	6.245	15.606	21.851
3	0.431	0.698	2.431	25.128	5.910	14.367	20.277
4	0.038	0.558	2.038	20.088	4.153	8.464	12.617
5	0.653	0.573	2.653	20.628	7.038	18.672	25.710
6	0.609	0.179	2.609	6.444	6.807	17.759	24.566
7	0.974	0.011	2.974	0.396	8.845	26.305	35.150
8	0.098	0.805	2.098	28.980	4.402	9.235	13.637
9	0.516	0.296	2.516	10.656	6.330	15.926	22.256
10	0.149	0.815	2.149	29.340	4.618	9.924	14.542
11	0.070	0.692	2.070	24.912	4.285	8.870	13.155
12	0.696	0.203	2.696	7.308	7.268	15.595	26.863
13	0.350	0.900	2.350	32.400	5.523	12.979	18.502
14	0.451	0.318	0.451	11.448	6.007	14.723	20.730
15	0.798	0.111	2.798	3.996	7.829	21.906	29.735
16	0.933	0.199	2.933	7.164	8.602	25.230	33.832
17	0.183	0.421	2.183	15.156	4.765	10.402	15.167
18	0.338	0.104	2.338	3.744	5.466	12.780	18.246
19	0.190	0.150	2.190	5.400	4.796	10.503	15.299
20	0.449	0.320	2.449	11.520	5.998	14.689	20.687

As is seen from the table, $n = 13$. Consequently, we find by (9) that

$$I \approx (36 \cdot 13)/20 = 23.4; \ \delta = (23.4 - 22.583)/33.583 \cdot 100\% \approx 3.6\%.$$

1151. Applying (15), approximate the double integral $I = \iint\limits_D (x + 2y)dxdy$, if the domain D is defined by the inequalities $0 \leqslant x \leqslant 1$, $x/2 \leqslant y \leqslant x$ (Fig. 82).

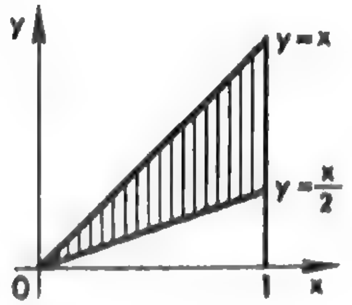


Fig. 82

Solution. Here $a = 0$, $b = 1$. Since the domain D lies in a unit square, it is not necessary to pass to new variables. We take 20 consecutive values from the table of random numbers. The calculation table has the form

i	x_i	y_i	$\bar{y}_i = x_i/2$	$\bar{y}_i = x_i$	$2y_i$	$f(x_i, y_i) = x_i + 2y_i$
1	0.857	0.457	0.428	0.857	0.914	1.771
2	0.499	0.762	0.249	0.499		
3	0.431	0.698	0.215	0.431		
4	0.038	0.558	0.019	0.038		
5	0.653	0.573	0.326	0.653	1.146	1.799
6	0.609	0.179	0.304	0.609		
7	0.974	0.011	0.487	0.974		
8	0.098	0.805	0.049	0.098		
9	0.516	0.296	0.258	0.516	0.592	1.108
10	0.149	0.815	0.074	0.149		

For $N = 10$ and $n = 3$, we get from (15)

$$I \approx (1.771 + 1.799 + 1.108)/10 = 4.578/10 \approx 0.458.$$

Next we find the exact value of the integral:

$$I = \iint\limits_D (x + 2y)dxdy = \int\limits_0^1 \int\limits_{x/2}^{x} (x + 2y)dxdy = \frac{1}{4}\int\limits_0^1 (x + 2y)^2 \Big|_{x/2}^{x} dx$$

$$= \frac{1}{4}\int\limits_0^1 (9x^2 - 4x^2)dx = \frac{5}{4} \cdot \frac{x^3}{3}\Big|_0^1 = \frac{5}{12} \approx 0.417.$$

Then $\delta = (0.458 - 0.417)/0.417 \cdot 100\% \approx 9.82\%$.

Here, as in the other examples, the number $n = 3$ is insufficient for statistical regularities to manifest themselves in full measure. Nevertheless, the result obtained is sufficient for rough orientation.

1152. Compute the double integral $I = \iint\limits_D \sqrt{x + y}\,dxdy$ by (16), the domain D being bounded by the curves $x = 0$, $x = 4$, $y = 3x$, $y = 8x$ (Fig. 83).

Solution. Having written the given double integral as an iterated integral, we get $I = \int\limits_0^4 dx \int\limits_{3x}^{8x} \sqrt{x + y}\,dy$. Here $a = 0$, $b = 4$, $\varphi_1(x) = 3x$, $\varphi_2(x) = 8x$; next we have $\varphi_1(x) \geqslant 0$, $\varphi_2(x) \leqslant 32$; therefore, $c = 0$, $d = 32$. Since $\max\limits_{\substack{0 \leqslant x \leqslant 4 \\ 0 \leqslant y \leqslant 32}} \sqrt{x + y} = 6$, we perform a change of variables by the formulas $x = 4\xi$, $y = 32\eta$, $z = 6\zeta$. The straight lines $y = 3x$ and $y = 8x$ are transformed, respectively, into the straight lines $\eta = (3/8)\xi$, $\eta = \xi$ (Fig. 84). We take 60 values ($N = 20$) from the table of random numbers. The calculation table has the form

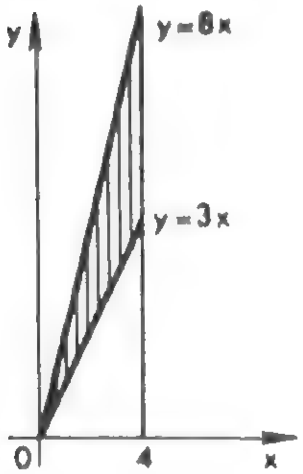
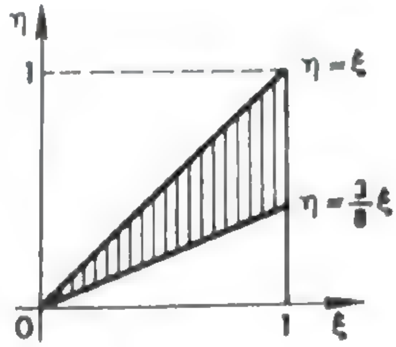


Fig. 83

Fig. 84



i	ξ_i	η_i	ζ_i	$x_i = 4\xi_i$	$y_i = 32\eta_i$	$z_i = 6\zeta_i$	$y_i = 3x_i$	$\bar{y}_i = 8x_i$	$x_i + y_i$	$Z_i = \sqrt{x_i + y_i}$
1	0.857	0.457	0.499	3.428	14.624	2.994	10.284	27.424	18.052	4.249
2	0.762	0.431	0.698	3.048	13.792	4.188	9.144	24.384	16.840	4.104
3	0.038	0.558	0.653	0.152	17.856		0.456	1.216		
4	0.573	0.609	0.179	2.292	19.488		6.876	18.336		
5	0.974	0.011	0.098	3.896	0.352		11.688	31.168		
6	0.805	0.516	0.296	3.220	16.512	1.776	9.660	25.760	19.732	4.441
7	0.149	0.815	0.070	0.596	26.080		1.788	4.768		
8	0.692	0.696	0.203	2.768	22.272		8.304	22.144		
9	0.350	0.900	0.451	1.400	28.800		4.200	11.200		
10	0.318	0.798	0.111	1.272	25.536		3.816	10.176		
11	0.933	0.199	0.183	3.732	6.368		11.196	26.976		
12	0.421	0.338	0.104	1.684	10.816	0.624	5.052	13.472	12.500	3.536
13	0.190	0.150	0.449	0.760	4.800	2.694	2.280	6.080	5.560	2.358
14	0.320	0.165	0.617	1.280	5.280	3.702	3.840	10.240	6.560	2.561
15	0.369	0.069	0.248	1.476	2.208		4.428	11.808		
16	0.960	0.652	0.367	3.840	20.864	2.202	11.520	30.720	24.704	4.970
17	0.168	0.261	0.189	0.672	8.352		2.016	5.376		
18	0.703	0.142	0.486	2.812	4.544		8.436	22.496		
19	0.233	0.424	0.291	0.932	13.568		2.796	7.456		
20	0.473	0.645	0.514	1.892	20.640		5.676	15.136		

As follows from the table, $n = 4$. Thus we find from (16) that

$$I \approx \frac{(4-0)(32-0)\cdot 6 \cdot 4}{20} = 153.6.$$

The exact value of the integral is

$$I = \frac{2}{3}\int_0^4 (x+y)^{3/2}\Big|_{3x}^{8x} dx = \frac{76}{15}x^{5/2}\Big|_0^4 = 162\,\frac{2}{15} \approx 162.1,$$

and the relative error is

$$\delta = (162.1 - 153.6)/162.1 \cdot 100\% \approx 5.2\%.$$

1153. Calculate the double integral $I = \iint\limits_D \sqrt{x + y + 1}\,dx\,dy$, where D is a rectangle $0 \leqslant x \leqslant 4$, $1 \leqslant y \leqslant 7$, using three methods: (1) by approximation (20) in 9.4.3; (2) by (28) in 9.4.3; (3) by (16), having taken 60 values from the table of random numbers. Estimate the relative error in each case.

1154. Use two methods to calculate the double integral $I = \iint\limits_D \frac{\cos y}{x}\,dx\,dy$, where the domain D is defined by the inequalities $0.2 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant x$: (1) by approximation (30) in 9.4.3; (2) by (16) having taken 90 values from the table of random numbers. Estimate the relative error.

1155. Approximate the triple integral $I = \iiint\limits_V (x + y + 2z)\,dx\,dy\,dz$, using approximation (19) for $k = 3$, if the domain D is defined by the inequalities $1 \leqslant x \leqslant 3$, $0 \leqslant y \leqslant x$, $x + y \leqslant z \leqslant x + 2y$.

Solution. For the triple integral, (19) assumes the form

$$I \approx \frac{(b - a)(d - c)(h - g) \cdot M \cdot n}{N}.$$

Here $a = 1, b = 3, c = 0, d = 3, g = 1, h = 9, M = \max\limits_{\substack{1 \leqslant x \leqslant 3 \\ 0 \leqslant y \leqslant 3 \\ 1 \leqslant z \leqslant 9}} (x + y + 2z) = 24.$

Let us perform a change of variables by the formulas $x = 1 + 2\xi$, $y = 3\eta$, $z = 1 + 8\zeta$, $u = 24\sigma$. We take 80 values from the table of random numbers ($N = 20$). The calculation table has the form

i	ξ_i	η_i	ζ_i	σ_i	x_i	y_i	z_i	u_i						
1	0.165	0.617	0.369	0.069	1.330	1.851	3.952	1.656	0	1.330				
2	0.248	0.960	0.652	0.367	1.496	2.880	6.216	8.08	0	1.496				
3	0.168	0.261	0.189	0.703	1.336	0.783	2.512	16.872	0	1.336	2.119	2.902	5.024	7.143
4	0.142	0.486	0.233	0.424	1.284	1.458	2.864	10.176	0	1.284				
5	0.291	0.473	0.645	0.514	1.582	1.419	6.160	12.336	0	1.582	3.001	4.420		
6	0.819	0.064	0.870	0.256	2.638	0.192	7.960	6.144	0	2.638	2.830	3.022		
7	0.347	0.151	0.912	0.191	1.694	0.453	8.296	4.584	0	1.694	2.147	2.600		
8	0.259	0.096	0.019	0.854	1.518	0.288	1.152	20.496	0	1.518	1.806	2.094	2.304	4.110
9	0.193	0.732	0.253	0.352	1.386	2.196	3.024	8.448	0	1.386				
10	0.729	0.102	0.222	0.088	2.458	0.306	2.776	2.112	0	2.458	2.764	3.070	5.552	8.316
11	0.205	0.562	0.851	0.647	1.410	1.686	7.808	15.528	0	1.410				
12	0.568	0.020	0.051	0.649	2.136	0.060	1.408	15.576	0	2.136	2.196	2.256		
13	0.179	0.896	0.453	0.546	1.358	2.688	4.624	13.104	0	1.358				
14	0.919	0.691	0.155	0.181	2.838	2.073	2.240	4.344	0	2.838	4.911	6.984		
15	0.273	0.876	0.690	0.494	1.546	2.628	6.520	11.856	0	1.546				
16	0.339	0.910	0.789	0.908	1.678	2.730	7.312	21.792	0	1.678				
17	0.263	0.131	0.389	0.438	1.526	0.393	4.112	10.512	0	1.526	1.919	2.312		
18	0.161	0.485	0.535	0.090	1.322	1.455	5.280	2.160	0	1.322				
19	0.142	0.321	0.969	0.091	1.284	0.963	8.752	2.184	0	1.284	2.247	3.210		
20	0.436	0.251	0.595	0.784	1.926	0.753	5.768	18.816	0	1.926	2.679	3.432		

We first find the values y_i ($1 \leqslant i \leqslant 20$) for which the condition $\underline{y}_i \leqslant y_i \leqslant \bar{y}_i$ is satisfied; they are 11 in number. Next, among the corresponding 11 values z_i we find those for which $\underline{z}_i \leqslant z_i \leqslant \bar{z}_i$; there are 3 such values. Finally, among the corresponding three values u_i we find those which satisfy the inequality $u_i < U_i$; their number is $n = 1$. Thus we have

$$I \approx (48 \cdot 24)/20 = 1152/20 = 57.6.$$

We find the exact value of the integral:

$$I = \int_1^3 dx \int_0^x dy \int_{x+y}^{x+2y} (x + y + 2z)dz = \frac{1}{4}\int_1^3 dx \int_0^x (x + y + 2z)^2 \Big|_{x+y}^{x+2y} dy$$

$$= \frac{1}{4}\int_1^3 dx \int_0^x [(3x + 5y)^2 - (3x + 3y)^2]dy$$

$$= \frac{1}{4}\int_1^3 \left[\frac{1}{15}(3x + 5y)^3 - \frac{1}{9}(3x + 3y)^3\right]_0^x dx$$

$$= \frac{1}{4}\int_1^3 \left[\frac{(8x)^3}{15} - \frac{(6x)^3}{9} - \frac{(3x)^3}{15} + \frac{(3x)^3}{9}\right] dx$$

$$= \frac{1}{4}\left(\frac{485}{15} - \frac{189}{9}\right) \cdot \frac{x^4}{4}\Big|_1^3 = 56\frac{2}{3} \approx 56.667;$$

$$\delta = (57.6 - 56.667)/56.667 \cdot 100\% \approx 1.6\%.$$

9.6. Numerical Integration of Differential Equations

9.6.1. Euler's method. The differential equation $y' = f(x, y)$ defines on the plane the so-called *direction field*, that is, it specifies, at each point of the plane at which there exists a function $f(x, y)$, the direction of the integral curve of the equation passing through that point. Suppose we have to solve the Cauchy problem, that is, to solve the equation $y' = f(x, y)$ satisfying the initial condition $y(x_0) = y_0$. Let us divide the interval $[x_0, X]$ into n equal subintervals and put $(X - x_0)/n = h$ (h is the step of variation of the argument). We assume that inside the subinterval from x_0 to $x_0 + h$ the function y' retains constant value $f(x_0, y_0)$. Then, $y_1 - y_0 \approx h \cdot f(x_0, y_0)$, where y_1 is the value of the sought-for function corresponding to the value $x_1 = x_0 + h$. Hence we get $y_1 \approx y_0 + h \cdot f(x_0, y_0)$. Repeating the procedure, we get the successive values of the function

$$y_2 \approx y_1 + h \cdot f(x_1, y_1), y_3 \approx y_2 + h \cdot f(x_2, y_2), \ldots, y_{k+1} \approx y_k + h \cdot f(x_k, y_k).$$

Thus we can construct the approximate integral curve as a polygonal line with ver-

tices $M_k(x_k; y_k)$, where $x_{k+1} = x_k + \Delta x_k$, $y_{k+1} = y_k + h \cdot f(x_k, y_k)$. This method is known as *Euler's method of polygons*, or simply *Euler's method.*

1156. Setting $h = 0.1$ find by Euler's method the values of the function y specified by the differential equation $y' = \dfrac{y - x}{y + x}$ subject to the initial condition $y(0) = 1$. You may restrict yourself to finding the first four values of y.

Solution. We find the consecutive values of the argument: $x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3$ and compute the corresponding values of the desired function:

$$y_1 = y_0 + h \cdot f(x_0, y_0) = 1 + 0.1 \cdot (1 - 0)/(1 + 0) = 1.1,$$

$$y_2 = y_1 + h \cdot f(x_1, y_1) = 1.1 + 0.1 \cdot (1.1 - 0.1)/(1.1 + 0.1) = 1.183,$$

$$y_3 = y_2 + h \cdot f(x_2, y_2) = 1.183 + 0.1 \cdot (1.183 - 0.2)/(1.183 + 0.2) = 1.254,$$

$$y_4 = y_3 + h \cdot f(x_3, y_3) = 1.254 + 0.1 \cdot (1.254 - 0.3)/(1.254 + 0.3) = 1.315.$$

Thus we obtain the following table:

x	0	0.1	0.2	0.3	0.4
y	1	1.1	1.18	1.25	1.31

1157. Setting $h = 0.1$ find by Euler's method four values of the function y, specified by the equation $y' = x + y$ subject to the initial condition $y(0) = 1$.

Solution. The values of the argument are $x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3$. We find the corresponding values of y

$$y_1 = y_0 + h \cdot f(x_0, y_0) = 1 + 0.1 \cdot (0 + 1) = 1.1,$$

$$y_2 = y_1 + h \cdot f(x_1, y_1) = 1.1 + 0.1 \cdot (0.1 + 1.1) = 1.22,$$

$$y_3 = y_2 + h \cdot f(x_2, y_2) = 1.22 + 0.1 \cdot (0.2 + 1.22) = 1.36,$$

$$y_4 = y_3 + h \cdot f(x_3, y_3) = 1.36 + 0.1 \cdot (0.3 + 1.36) = 1.52$$

and get the table

x	0	0.1	0.2	0.3	0.4
y	1	1.1	1.22	1.36	1.52

1158. Setting $h = 0.1$ find by Euler's method three values of the function y, specified by the equation $y' = 1 + x + y^2$ subject to the initial condition $y(0) = 1$.

1159. Setting $h = 0.1$ find by Euler's method four values of the function y, specified by the equation $y' = x^2 + y^3$ subject to the initial condition $y(0) = 0$.

1160. Setting $h = 0.1$ (four values) find by Euler's method the numerical solution of the equation $y' = y^2 + \dfrac{y}{x}$ subject to the initial condition $y(2) = 4$.

1161. Setting $h = 0.2$ find by Euler's method the numerical solution of the equation $y' = \dfrac{(x + y)(1 - xy)}{x + 2y}$ on the interval [0, 1] subject the initial condition $y(0) = 1$.

1162. Setting $h = 0.2$ find by Euler's method the numerical solution of the system of equations $\dfrac{dx}{dt} = \dfrac{y - x}{t}$, $\dfrac{dy}{dt} = \dfrac{x + y}{t}$ subject to the initial conditions $x(1) = 1$, $y(1) = 1$, $1 \leqslant t \leqslant 2$.

9.6.2. Runge-Kutta method. Assume that the function y is specified by the differential equation $y' = f(x, y)$ under the initial condition $y(x_0) = y_0$. When performing numerical integration of such an equation by the *Runge-Kutta method*, four numbers are determined:

$$k_1 = h \cdot f(x, y), \qquad k_2 = h \cdot f\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right),$$

$$k_3 = h \cdot f\left(x + \frac{h}{2}, y + \frac{k_2}{2}\right), \qquad k_4 = h \cdot f(x + h, y + k_3).$$

If we put $y(x + h) = y(x) + \Delta y$, we can prove that $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$. The calculation scheme has the form

x	y	$k_j = h \cdot f(x, y)$	Increment
x_0	y_0	k_1	$\Delta y_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
$x_0 + \frac{1}{2}h$	$y_0 + \frac{1}{2}k_1$	k_2	
$x_0 + \frac{1}{2}h$	$y_0 + \frac{1}{2}k_2$	k_3	
$x_0 + h$	$y_0 + k_3$	k_4	
$x_1 = x_0 + h$	$y_1 = y_0 + k$		

1163. Compile a table of values of the function y specified by the equation $y' = y - \dfrac{2x}{y}$ subject to the initial condition $y(0) = 1$ in the interval [0, 1]; $h = 0.2$ (the exact solution is $y = \sqrt{2x + 1}$).

Solution. We find the following numbers:

$$k_1 = h \cdot f(x, y) = 0.2 \cdot \left(1 - \frac{2 \cdot 0}{1}\right) = 0.2,$$

$$k_2 = h \cdot f\left(x + \frac{h}{2}, y + \frac{K_1}{2}\right) = 0.2 \cdot f(0.1; 1.1) = 0.2 \cdot \left(1.1 - \frac{0.2}{1.1}\right) = 0.1836,$$

$$k_3 = h \cdot f\left(x + \frac{h}{2}, y + \frac{k_2}{2}\right) = 0.2 \cdot f(0.1; 1.0918) = 0.1817,$$

$$k_4 = h \cdot f(x + h, y + k_3) = 0.2 \cdot f(0.2; 1.1817) = 0.1686.$$

Hence we have

$$\Delta y = \frac{1}{6}(0.2 + 0.3672 + 0.3634 + 0.1686) = 0.1832.$$

This means that $y_1 = 1 + 0.1832 = 1.1832$ for $x = 0.2$. We use a similar technique to find y_2 and so on. The calculation procedure follows the scheme given below:

i	x	y	$f(x, y)$	$k_i = h \cdot f(x, y)$	Δy
1	0	1	1	0.2	
2	0.1	1.1	0.0918	0.1838	0.1832
3	0.1	1.0918	0.0908	0.1817	
4	0.3	1.1817	0.0843	0.1686	
1	0.2	1.1832	0.8451	0.1690	
2	0.3	1.2677	0.7944	0.1589	1.1584
3	0.3	1.2626	0.7874	0.1575	
4	0.4	1.3407	0.7440	0.1488	
1	0.4	1.3416	0.7453	0.1491	
2					
3					
4					
1					

Note that all the five digits in the numbers $y_1 = 1.1832$ and $y_2 = 1.3416$ are correct if we compare the result obtained with the exact solution $y = \sqrt{2x + 1}$.

1164. Integrate by the Runge-Kutta method the equation $x^2y' - xy = 1$ subject to the initial condition $y(1) = 0$ in the interval $[1, 2]$; h being equal to 0.2 (the exact solution is $y = (x^2 - 1)/(2x)$).

Solution. Here $f(x, y) = \dfrac{y}{x} + \dfrac{1}{x^2}$. We find the following numbers:

$$k_1 = h \cdot f(x, y) = 0.2 \cdot \left(\frac{0}{1} + \frac{1}{1^2}\right) = 0.2,$$

$$k_2 = h \cdot f\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right) = 0.2 \cdot \left(\frac{0.1}{1.1} + \frac{1}{1.1^2}\right) = 0.18,$$

$$k_3 = h \cdot f\left(x + \frac{h}{2}, y + \frac{k_2}{2}\right) = 0.2 \cdot \left(\frac{0.09}{1.1} + \frac{1}{1.1^2}\right) = 0.18,$$

$$k_4 = h \cdot f(x + h, y + k_3) = 0.2 \cdot \left(\frac{0.18}{1.2} + \frac{1}{1.2^2}\right) = 0.17.$$

Consequently,

$$\Delta y_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.18, \quad \text{i.e.}$$
$$y_1 = y_0 + \Delta y_0 = 0 + 0.18 = 0.18.$$

In a similar manner we find

$$k_1 = h \cdot f(x, y) = 0.2\left(\frac{0.18}{1.2} + \frac{1}{1.2^2}\right) = 0.17,$$

$$k_2 = h \cdot f\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right) = 0.2\left(\frac{0.26}{1.3} + \frac{1}{1.3^2}\right) = 0.15,$$

$$k_3 = h \cdot f\left(x + \frac{h}{2}, y + \frac{k_2}{2}\right) = 0.2\left(\frac{0.25}{1.3} + \frac{1}{1.3^2}\right) = 0.15,$$

$$k_4 = h \cdot f(x + h, y + k_3) = 0.2\left(\frac{0.33}{1.4} + \frac{1}{1.4^2}\right) = 0.14.$$

Consequently,

$$\Delta y_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.15, \quad \text{i.e.}$$

$$y_2 = y_1 + \Delta y_1 = 0.18 + 0.15 = 0.33 \text{ and so on.}$$

1165. Use the Runge-Kutta method to integrate the equation $4y' = y^2 + 4x^2$, $y(0) = 1$ in the interval $[0, 1]$; h being equal to 0.1. Perform the calculation with three correct digits.

1166. By the Runge-Kutta method integrate the equation $y' = x/y + 0.5y$, $y(0) = 1$ in the interval $[0, 1]$; $h = 0.1$. Perform the calculation with three correct digits.

9.6.3. The Adams method. Suppose we have to integrate the equation $y' = f(x, y)$, $y(x_0) = y_0$. One of the difference methods of approximate solution of the problem is the *Adams method*. Having assumed a certain step of variation of the argument h, we use some method, proceeding from the initial data $y(x_0) = y_0$, to find the following three values of the desired function $y(x)$:

$$y_1 = y(x_1) = y(x_0 + h), \quad y_2 = y(x_0 + 2h), \quad y_3 = y(x_0 + 3h)$$

(these three values can be obtained by any method ensuring the desired accuracy: by expanding the solution into a power series, by the Runge-Kutta method, etc., but not by Euler's method because of its insufficient accuracy). With the aid of the numbers x_0, x_1, x_2, x_3 and y_0, y_1, y_2, y_3 we compute the values

$$q_0 = h \cdot y_0' = h \cdot f(x_0, y_0), \quad q_1 = h \cdot f(x_1, y_1),$$
$$q_2 = h \cdot f(x_2, y_2), \quad q_3 = h \cdot f(x_3, y_3).$$

Next we compile a table of the finite differences of the quantities y and q:

x	y	Δy	q	Δq	$\Delta^2 q$	$\Delta^3 q$
x_0	y_0		q_0			
		Δy_0		Δq_0		
x_1	y_1		q_1		$\Delta^2 q_0$	
		Δy_1		Δq_1		$\Delta^3 q_0$
x_2	y_2		q_2		$\Delta^2 q_1$	
		Δy_2		Δq_2		
x_3	y_3		q_3			
...	...	...	...	...	...	...

Knowing the numbers in the lower sloping line of values, we use the Adams formula to obtain

$$\Delta y_3 = q_3 + \frac{1}{2} \Delta q_2 + \frac{5}{12} \Delta^2 q_1 + \frac{3}{8} \Delta^3 q_0,$$

and then we find the value $y_4 = y_3 + \Delta y_3$. Knowing now y_4, we compute $q_4 = h \cdot f(x_4, y_4)$, after which we can write the following sloping row:

$$\Delta q_3 = q_4 - q_3, \quad \Delta^2 q_2 = \Delta q_3 - \Delta q_2, \quad \Delta^3 q_1 = \Delta^2 q_2 - \Delta^2 q_1.$$

The new sloping row enables us to calculate by the Adams formula the value

$$\Delta y_4 = q_4 + \frac{1}{2} \Delta q_3 + \frac{5}{12} \Delta^2 q_2 + \frac{3}{8} \Delta^3 q_1,$$

and, consequently, $y_5 = y_4 + \Delta y_4$, and so on.

1167. Using Adams' method, find the value $y(0.4)$ with an accuracy to within 0.01 for the differential equation $y' = x^2 + y^2$; $y(0) = -1$.

Solution. Let us find the first four terms of the expansion of the solution of the given equation into Taylor's series in the neighbourhood of the point $x = 0$:

$$y(x) = y(0) + y'(0) \cdot x + \frac{1}{2} y''(0) \cdot x^2 + \frac{1}{6} y'''(0) \cdot x^3 + \ldots .$$

By the hypothesis, $y(0) = -1$; we find the values $y'(0)$, $y''(0)$ and $y'''(0)$ by a successive differentiation of the given equation:

$$y' = x^2 + y^2; \quad y'(0) = 0^2 + (-1)^2 = 1,$$

$$y'' = 2x + 2yy'; \quad y''(0) = 0 + 2 \cdot (-1) \cdot 1 = -2,$$

$$y''' = 2 + 2y'^2 + 2yy''; \quad y'''(0) = 2 + 2 \cdot (-1)^2 + 2 \cdot (-1)(-2) = 8.$$

Thus we have

$$y(x) \approx -1 + x - x^2 + \frac{4}{3} x^3 + \ldots .$$

We calculate $y(x)$ at the points $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$ with one extra (third) digit: $y_1 = -0.909$, $y_2 = -0.829$, $y_3 = 0.754$ and compile a table:

x	y	Δy	q	Δq	$\Delta^2 q$	$\Delta^3 q$
0	-1		0.1			
		0.091		-0.017		
0.1	-0.909		0.083		0.006	
		0.080		-0.011		-0.002
0.2	-0.829		0.072		0.004	
		0.075		-0.007		
0.3	-0.754		0.065			
0.4						

Then we have

$$\Delta y_3 = q_3 + \frac{1}{2} \Delta q_2 + \frac{5}{12} \Delta^2 q_1 + \frac{3}{8} \Delta^3 q_0$$

$$= 0.065 + \frac{1}{2}(-0.007) + \frac{5}{12} \cdot 0.004 + \frac{3}{8} \cdot (-0.002) = 0.062.$$

Consequently, $y_4 = y_3 + \Delta y_3 \approx -0.754 + 0.062 = -0.692 \approx -0.69$.

1168. Using Adams' method, find the value $y(0.5)$ for the differential equation $y' = x + y$, $y(0) = 1$; h being equal to 0.1. Carry out the calculations with an accuracy to within 0.001, retain two digits in the result.

1169. Using Adams' method, find the value $y(0.4)$ for the differential equation $y' = x^2 + y^2$, $y(0) = 0$; $h = 0.1$. Carry out the calculations with the same number of digits as in the preceding example.

9.7. Picard's Iterative Method

Picard's iterative method is one of the analytic methods of approximate solution of differential equations. As applied to the first-order differential equation

$$y' = f(x, y) \tag{1}$$

with the initial condition $y(x_0) = y_0$, it consists in constructing the desired solution $y = y(x)$ for $x \geqslant x_0$ (or $x \leqslant x_0$). Integrating the right-hand and left-hand sides of Eq. (1) in the limits from x_0 to x, we get

$$y(x) - y(x_0) = \int_{x_0}^{x} f(t, y)dt,$$

or

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y)dt. \tag{2}$$

It is assumed that in a certain neighbourhood of the point $(x_0; y_0)$ Eq. (1) satisfies the conditions of the existence and uniqueness theorem (Cauchy's theorem), that is, it is supposed that $f(x, y)$ is a continuous function of its arguments and $\left|\frac{\partial f}{\partial y}\right| < K$.

To find the successive approximations, we replace in Eq. (2) the unknown function y by the given value y_0; we obtain the first approximation

$$y = y_0 + \int_{x_0}^{x} f(t, y_0)dt.$$

Next we substitute into Eq. (2) the function y_1 we have found for the unknown function y and obtain the second approximation

$$y_2 = y_0 + \int_{x_0}^{x} f(t, y_1)dt.$$

All the subsequent approximations are constructed by the formula

$$y_n = y_0 + \int_{x_0}^{x} f(t, y_{n-1})dt \quad (n = 1, 2, \ldots).$$

Thus we have

$$y(x) \approx y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1})dt.$$

We can prove that $\lim\limits_{n \to \infty} y_n(x) = y(x)$.

The error is estimated by means of the inequality

$$|y(x) - y_n(x)| \leq \frac{M(Kc)^n}{K \cdot n!},$$

where $|f(x, y)| \leq M$, $|x - x_0| < a \leq \infty$, $|y - y_0| < b \leq \infty$, $c = \min(a, b/M)$.

Picard's approximations give a sequence of the *lower* functions, i.e.

$$y_0 < y_1 < y_2 < \ldots < y_n < y(x),$$

if $\frac{\partial f}{\partial y} > 0$ and $f(x, y_0) > 0$, and a sequence of the *upper* functions, i.e.

$$y_0 > y_1 > y_2 > \ldots > y_n > y(x)$$

if $\frac{\partial f}{\partial y} > 0$ and $f(x, y_0) < 0$. Thus, for $\frac{\partial f}{\partial y} > 0$ the Picard iterations form a *one-sided sequence of approximations*, and for $\frac{\partial f}{\partial y} < 0$, a *two-sided sequence*.

1170. Find the approximate solution of the equation $y' = x + y^2$ satisfying the initial condition $y(0) = 1$.

Solution. As the initial approximation we take $y_0 = y(0) = 1$. Then the first approximation is

$$y_1(x) = 1 + \int_0^x (t + 1)dt = 1 + x + \frac{1}{2}x^2.$$

By analogy we get the second approximation

$$y_2(x) = 1 + \int_0^x \left[t + \left(1 + t + \frac{1}{2}t^2\right)^2\right]dt$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5,$$

....

1171. What sequence of Picard's approximations expresses the solution of the equation $y' = x + y$ satisfying the initial condition $y(0) = 0$ for $x \geq 0$?

Solution. As the initial approximation we take $y_0 = y(0) = 0$. Then

$$y_1 = y_0 + \int_0^x (t + y_0)dt = \int_0^x t\,dt = \frac{1}{2}x^2,$$

$$y_2 = \int_0^x \left(t + \frac{1}{2}t^2\right)dt = \frac{1}{2}x^2 + \frac{1}{6}x^3 = \frac{x^2}{2!} + \frac{x^3}{3!},$$

$$y_3 = \int_0^x \left(t + \frac{t^2}{2!} + \frac{t^3}{3!}\right) dt = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!},$$

....

$$y_n(x) = \int_0^x \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots + \frac{t^n}{n!}\right) dt$$

$$= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^{n+1}}{(n+1)!}.$$

Here $f(x, y_0) = x + y_0 \geqslant 0$ and $\frac{\partial f}{\partial y} = 1 > 0$. Consequently, the Picard approximations form a sequence of the lower functions.

In the given case, the true analytic expression of $y(x)$ has the form

$$y(x) = \lim_{n\to\infty} y_n(x) = \lim_{n\to\infty} \left(\frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n+1}}{(n+1)!}\right)$$

$$= \lim_{n\to\infty} \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n+1}}{(n+1)!}\right) - (x + 1),$$

or

$$y(x) = e^x - x - 1.$$

1172. Find three successive approximations of the equation $y' = x^2 + y^2$ satisfying the initial condition $y(0) = 0$, taking $y = 0$ as the initial approximation.

1173. Find the approximate solution of the equation $y' + y \cosh x = 0$ satisfying the initial condition $y(0) = 1$.

1174. Find the approximate solution and determine the character of the Picard approximations of the equation $y' = x - y$; the initial condition is $y(0) = 1$, $x \geqslant 0$.

1175. Find the approximate solution and determine the character of the Picard approximations of the equation $y' = y \cos x$; the initial condition being $y(0) = 1$, $-2 < x < 2$.

1176. Find the approximate solution of the equation $y' = 2xy \cos(x^2)$, satisfying the initial condition $y(0) = 1, 0 < x < 1, 0 < y < 2$. Determine the character of the Picard approximations.

9.8. The Simplest Methods of Processing Experimental Data

9.8.1. Graphical method. Suppose the experimental data are tabulated. We draw a curve through the points defined by the table or through points close to them and from the shape of the curve we choose the form of the empirical formula. The simplest case is considered to be the one for which the experimental data lead to the points lying approximately on the straight line $y = a_0 + a_1 x$ or on the curves

whose equations $S = At^{\alpha}$ and $S = Ae^{\alpha t}$ can be reduced to a linear function by means of a change of variables. Solving this problem by a graphical method, we plot the points on the coordinate grid (with a uniform or logarithmic scale) and draw a straight line approximately through these points so that it should lie as close to each plotted point as possible, and then we take two arbitrary points on that line (lying rather far from each other) and substitute their coordinates into the relation $y = a_0 + a_1 x$. The two equations thus obtained yield the values of a_0 and a_1.

1177. The stationary distribution of temperature in an insulated thin bar is described by the linear function $u = a_0 + a_1 x$. Determine the constants a_0 and a_1 if the following table of temperatures measured at the respective points of the bar is given:

x	0	2	6	8	10	14	16	20
u	32	29.2	23.3	19.9	17.2	11.3	7.8	2

Solution. We construct the points corresponding to the given table and see that the straight line passes through the points (0; 32) and (20; 2). Substituting their coordinates into the equation $u = a_0 + a_1 x$, we get

$$\begin{cases} a_0 + 0 \cdot a_1 = 32, \\ a_0 + 20a_1 = 2; \end{cases} \quad a_0 = 32, \quad a_1 = -1.5,$$

From this we obtain the desired relation $u = 32 - 1.5x$.

We can judge whether the formula is in good agreement with the tabular data from the value of the sum of the diviations δ and the sum of the squares of the deviations δ^2 of the values of the function calculated by the formula from the tabular data. In the present example $\delta = -1.5x + 32 - u$. Consequently,

$\delta_1 = -1.5 \cdot 0 + 32 - 32 = 0$; $\delta_2 = -1.5 \cdot 2 + 32 - 29.2 = -0.2$;

$\delta_3 = -1.5 \cdot 6 + 32 - 23.3 = -0.3$; $\delta_4 = -1.5 \cdot 8 + 32 - 19.9 = -0.1$;

$\delta_5 = -1.5 \cdot 10 + 32 - 17.2 = -0.2$; $\delta_6 = -1.5 \cdot 14 + 32 - 11.3 = -0.3$;

$\delta_7 = -1.5 \cdot 16 + 32 - 7.8 = 0.2$; $\delta_8 = -1.5 \cdot 20 + 32 - 2 = 0$;

$$\sum_{i=1}^{8} \delta_i = -0.7; \quad \sum_{i=1}^{8} \delta_i^2 = 0.31.$$

1178. The tabular data

t	1	2	3	4	5	6	7
S	2.31	2.58	2.77	2.93	3.06	3.16	3.26

correspond to the formula $S = At^{\alpha}$. Find the values of A and α.

Solution. Taking logarithms from the equality $S = At^{\alpha}$, we get $\log S = \log A + \alpha \cdot \log t$; setting $\log S = y$, $\log t = x$, $\log A = a_0$, $\alpha = a_1$, we get $y = a_0 + a_1 x$. The graph of the linear equation obtained is a straight line the parameters of whose equation can be found by taking two points on that straight line, say, ($\log 1$; $\log 2.31$) and ($\log 7$; $\log 3.26$). Substituting the coordinates of those points into the equation $y = \log A + \alpha x$, we get

$$\begin{cases} \log 2.31 = \log A + \alpha \cdot \log 1, \\ \log 3.26 = \log A + \alpha \cdot \log 7, \end{cases} \quad \text{or} \quad \begin{cases} \log A = 0.364, \\ \log A + 0.845\alpha = 0.513. \end{cases}$$

Hence $\log A = 0.364$, $A = 2.312$; $\alpha = 0.149/0.845 = 0.176$. Consequently, $S = 2.312t^{0.176}$.

1179. The tabular data

x	19.1	25.0	30.1	36.0	40.0	45.1	50.0
y	76.30	77.80	79.75	80.80	82.35	83.90	85.10

correspond to the formula $y = a_0 + a_1 x$. Find a_0 and a_1.

1180. The tabular data

t	1	2	3	4	5	6	7	8
S	15.3	20.5	27.4	36.6	49.1	65.6	87.8	117.6

correspond to the formula $S = Ae^{\alpha t}$. Find A and α.

9.8.2. The method of means. This method is based on the assumption that the most suitable curve is that for which the algebraic sum of the deviations is zero. To find the unknown constants in an empirical formula by this method, we first substitute into this formula all the pairs of the observed or measured values of x and y and obtain as many deviations as the number of pairs of the values (x; y) the table contains (the deviations are the vertical distances between the given points and the graph of the function). Then we group these deviations, forming as many groups as is the number of the unknown parameters in the empirical formula that must be found. Finally, equating to zero the sum of the deviations in each group, we get a system of linear equations with respect to the parameters.

1181. Use the method of means to find the formula of the form $S = At^{\alpha}$ corresponding to the table

t	273	283	288	293	313	333	353	373
S	29.4	33.3	35.2	37.2	45.8	55.2	65.6	77.3

Solution. Here the deviations have the form $\delta = At^{\alpha} - S$. Substituting the values of t and S taken from the table and equating the deviations to zero, we get a system of equations with respect to the parameters A and α, which is difficult to solve. Without any appreciable loss of accuracy, we can equate to zero the sum of the deviations of the logarithm of S, i.e. $\delta' = \log A + \alpha \log t - \log S$.

Then the deviations will be expressed by the formulas

$$\begin{aligned} \sigma_1' &= \log A + 2.4362\alpha - 1.4683, & \sigma_5' &= \log A + 2.4955\alpha - 1.6609,\\ \sigma_2' &= \log A + 2.4518\alpha - 1.5224, & \sigma_6' &= \log A + 2.5224\alpha - 1.7419,\\ \sigma_3' &= \log A + 2.4594\alpha - 1.5465, & \sigma_7' &= \log A + 2.5478\alpha - 1.8169,\\ \sigma_4' &= \log A + 2.4669\alpha - 1.5705, & \sigma_8' &= \log A + 2.5717\alpha - 1.8882. \end{aligned}$$

Equating to zero the sums of the deviations in these two groups, we get a system of equations for determining the parameters A and α:

$$\begin{cases} 4\log A + 9.8143\alpha = 6.1077,\\ 4\log A + 10.1374\alpha = 7.1079. \end{cases}$$

The solution of this system is $\alpha = 3.096$, $\log A = \bar{7}.9345$; hence $A = 8.5 \cdot 10^{-7}$. Thus we have $S = 8.6 \cdot 10^{-7} t^{3.096}$.

1182. Given the table

x	87.5	84.0	77.8	63.7	46.7	36.9
y	292	283	270	235	197	181

Find the parameters a_0, a_1, a_2 from the formula $y = a_0 + a_1x + a_2x^2$ corresponding to the given table.

1183. Given the table

t	53.92	26.36	14.00	6.99	4.28	2.75	1.85
S	6.86	14.70	28.83	60.40	101.9	163.3	250.3

corresponding to the formula $S = At^{\alpha}$. Find A and α.

9.8.3. Selecting the parameters by the method of least squares. (1) The following problem is often encountered in practical applications. Suppose two functionally related quantities x and y are associated with n pairs of known values $(x_1; y_1)$, $(x_2; y_2)$, . . ., $(x_n; y_n)$. It is required to determine, in the preassigned formula $y = f(x, \alpha_1, \alpha_2, \ldots, \alpha_m)$, m parameters $\alpha_1, \alpha_2, \ldots, \alpha_m$ $(m < n)$ so that the known n pairs of the values of x and y would suit the formula in the best way.

Proceeding from the principles of the probability theory, we can consider as the best the values $\alpha_1, \alpha_2, \ldots, \alpha_m$ which turn into the minimum the sum

$$\sum_{k=1}^{k=n} [f(x_k, \alpha_1, \alpha_2, \ldots, \alpha_m) - y_k]^2,$$

(that is, the sum of the squares of deviations of the values of y, calculated by the formula, from the preassigned values); this explains the name of the *method of least squares*.

This condition yields a system of m equations which are used to determine $\alpha_1, \alpha_2, \ldots, \alpha_m$:

$$\sum_{k=1}^{k=n} [f(x_k, \alpha_1, \alpha_2, \ldots, \alpha_m) - y_k] \cdot \frac{\partial f(x_k, \alpha_1, \alpha_2, \ldots, \alpha_m)}{\partial \alpha_j} = 0 \qquad (1)$$

$$(j = 1, 2, \ldots, m).$$

In practical applications, to the detriment of the rigorousness of the solution, it is sometimes necessary to reduce the given formula $y = f(x, \alpha_1, \alpha_2, \ldots, \alpha_m)$ to the form which makes system (1) less difficult to solve (see below, the selection of the parameters in the formulas $y = Ae^{cx}$ and $y = Ax^q$).

Special cases. (a) $y = a_0x^m + a_1x^{m-1} + \ldots + a_m$ ($m + 1$ parameters $a_0, a_1, \ldots, a_m$; $n > m + 1$).

System (1) assumes the form

$$\left\{\begin{array}{l} a_0 \cdot \sum_{k=1}^{k=n} x_k^m + a_1 \cdot \sum_{k=1}^{k=n} x_k^{m-1} + \ldots + n \cdot a_m = \sum_{k=1}^{k=n} y_k, \\ a_0 \cdot \sum_{k=1}^{k=n} x_k^{m+1} + a_1 \cdot \sum_{k=1}^{k=n} x_k^m + \ldots + a_m \cdot \sum_{k=1}^{k=n} x_k = \sum_{k=1}^{k=n} x_k y_k, \\ a_0 \cdot \sum_{k=1}^{k=n} x_k^{m+2} + a_1 \cdot \sum_{k=1}^{k=n} x_k^{m+1} + \ldots + a_m \cdot \sum_{k=1}^{k=n} x_k^2 = \sum_{k=1}^{k=n} x_k^2 y_k, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_0 \cdot \sum_{k=1}^{k=n} x_k^{2m} + a_1 \cdot \sum_{k=1}^{k=n} x_k^{2m-1} + \ldots + a_m \cdot \sum_{k=1}^{k=n} x_k^m = \sum_{k=1}^{k=n} x_k^m y_k. \end{array}\right. \qquad (2)$$

This system of $m + 1$ equations with $m + 1$ unknowns always has a unique solution since its determinant is nonzero.

To determine the coefficients in system (2), it is convenient to compile an auxiliary table of the form

k	x_k	x_k^2	x_k^3	$\ldots$	x_k^{2m}	y_k	$x_k y_k$	$x_k^2 y_k$	$\ldots$	$x_k^m y_k$
1	x_1	x_1^2	x_1^3	$\ldots$	x_1^{2m}	y_1	$x_1 y_1$	$x_1^2 y_1$	$\ldots$	$x_1^m y_1$
2	x_2	x_2^2	x_2^3	$\ldots$	x_2^{2m}	y_2	$x_2 y_2$	$x_2^2 y_2$	$\ldots$	$x_2^m y_2$
.	..	..	..	..	..	..	..	..	..	..
.	..	..	..	..	..	..	..	..	..	..
n	x_n	x_n^2	x_n^3	$\ldots$	x_n^{2m}	y_n	$x_n y_n$	$x_n^2 y_n$	$\ldots$	$x_n^m y_n$
Σ										

The last row contains the sum of the elements of each column which are precisely the coefficients in system (2).

System (2) is usually solved by Gauss' method.

(b) $y = Ae^{cx}$.

To simplify system (1), we first take logarithms of this formula relating x and y and replace it by the formula

$$\log y = \log A + c \cdot \log e \cdot x.$$

In this case, system (1) assumes the form

$$\begin{cases} c \cdot \log e \cdot \sum_{k=1}^{k=n} x_k + n \cdot \log A = \sum_{k=1}^{k=n} \log y_k, \\ c \cdot \log e \cdot \sum_{k=1}^{k=n} x_k^2 + \log A \cdot \sum_{k=1}^{k=n} x_k = \sum_{k=1}^{k=n} x_k \cdot \log y_k. \end{cases} \quad (3)$$

The auxiliary table is of the form

k	x_k	x_k^2	$\log y_k$	$x_k \cdot \log y_k$
1	x_1	x_1^2	$\log y_1$	$x_1 \cdot \log y_1$
2	x_2	x_2^2	$\log y_2$	$x_2 \cdot \log y_2$
..	..	..	..	....
n	x_n	x_n^2	$\log y_n$	$x_n \cdot \log y_n$
Σ				

Then we determine c and $\log A$ from system (3).

(c) $y = Ax^q$.

Here we again first take logarithms of this formula and replace it by the following formula:

$$\log y = \log A + q \cdot \log x.$$

Now system (1) assumes the form

$$\begin{cases} q \cdot \sum_{k=1}^{k=n} \log x_k + n \cdot \log A = \sum_{k=1}^{k=n} \log y_k, \\ q \cdot \sum_{k=1}^{k=n} \log^2 x_k + \log A \cdot \sum_{k=1}^{k=n} \log x_k = \sum_{k=1}^{k=n} \log x_k \cdot \log y_k. \end{cases} \quad (4)$$

The auxiliary table undergoes a respective change.

(2) It is often necessary to choose the best way to replace some given function $y = f(x)$ on the interval $[a, b]$ by an mth-degree polynomial: $y \approx a_0x^m + a_1x^{m-1} + \ldots + a_m$. In that case, the application of the method of least squares helps in finding the coefficients $a_0, a_1, \ldots, a_m$ from the condition of the minimum of the integral

$$\int_a^b [\varphi(x) - f(x)]^2 dx = \int_a^b [a_0x^m + a_1x^{m-1} + \ldots + a_m - f(x)]^2 dx.$$

The necessary conditions for the minimum of that integral lead to a system of $m + 1$ equations with $m + 1$ unknowns $a_0, a_1, a_2, \ldots, a_m$, which is used to determine all these coefficients:

$$\begin{cases} \int_a^b [a_0x^m + a_1x^{m-1} + \ldots + a_m - f(x)] \cdot x^m dx = 0, \\ \int_a^b [a_0x^m + a_1x^{m-1} + \ldots + a_m - f(x)] \cdot x^{m-1} dx = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \int_a^b [a_0x^m + a_1x^{m-1} + \ldots + a_m - f(x)] dx = 0. \end{cases} \quad (5)$$

1184. Use the method of least squares to choose the quadratic function $\varphi(x) = a_0x^2 + a_1x + a_2$ for the given values of x and y:

x	7	8	9	10	11	12	13
y	7.4	8.4	9.1	9.4	9.5	9.5	9.4

Solution. We compile a table:

k	x_k	x_k^2	x_k^3	x_k^4	y_k	x_ky_k	$x_k^2y_k$
1	7	49	343	2 401	7.4	51.8	362.6
2	8	64	512	4 196	8.4	67.2	537.6
3	9	81	729	6 561	9.1	81.9	737.1
4	10	100	1000	10 000	9.4	94.0	940.0
5	11	121	1331	14 641	9.5	104.5	1149.5
6	12	144	1728	10 736	9.5	114.0	1368.0
7	13	169	2197	28 561	9.4	122.2	1588.6
Σ	70	728	7840	87 096	62.7	635.6	6683.4

Hence we have a system of equations

$$\begin{cases} 728a_0 + 70a_1 + 7a_2 = 62.7 \\ 7840a_0 + 728a_1 + 70a_2 = 635.6, \\ 87096a_0 + 7840a_1 + 728a_2 = 6683.4. \end{cases}$$

Solving this system we get $a_0 = -0.04$, $a_1 = 1.10$, $a_2 = 2.12$. Thus, the desired quadratic function has the form $\varphi(x) = -0.04x^2 + 1.10x + 2.12$.

1185. Use the method of least squares to choose the power function $S = At^q$ proceeding from the following tabular data:

t	1	2	3	4	5
S	7.1	27.8	62.1	110	161

Solution. We compile a table:

k	$x_k = \log t_k$	x_k^2	$y_k = \log S_k$	$x_k y_k$
1	0.0000	0.0000	0.8513	0.0000
2	0.3010	0.0906	1.4440	0.4346
3	0.4771	0.2276	1.7931	0.8555
4	0.6021	0.3625	2.0414	1.2291
5	0.6990	0.4886	2.2068	1.5425
Σ	2.0792	1.1693	8.3366	4.0637

Thus we obtain a system of equations

$$\begin{cases} 2.0792q + 5\log A = 8.3366, \\ 1.1693q + 2.0792\log A = 4.0637. \end{cases}$$

Hence $q = 1.958$, $\log A = 0.8532$, i.e. $A = 7.132$. Consequently, the desired power function has the form $S = 7.132t^{1.958}$.

1186. Use the method of least squares to choose the exponential function $S = Ae^{ct}$ proceeding from the following tabular data:

t	0	2	4	6	8	10	12
S	1280	635	324	162	76	43	19

Solution. We compile a table:

k	t	t^2	$y = \log S$	ty
1	0	0	3.1071	0.0000
2	2	4	2.8028	5.6056
3	4	16	2.5105	10.0420
4	6	36	2.2095	13.2570
5	8	64	1.8808	15.0464
6	10	100	1.6335	16.3350
7	12	144	1.2787	15.3444
Σ	42	364	15.4230	75.6304

and get a system of equations

$$\begin{cases} 42x \cdot \log e + \ \ 7\log A = 15.4230, \\ 364c \cdot \log e + 42\log A = 75.6304 \end{cases}$$

i.e. $c \cdot \log e = -0.1509$, $\log A = 3.1087$. Consequently, $A = 1284$ and $c = -0.347$. Thus, the desired exponential function has the form $S = 1284e^{-0.347t}$.

In the following problems use the method of least squares to choose the functions of the assigned form proceeding from the given tabular data.

1187. Find the linear function:

(1)

x	1	2	3	4	5	6
y	2	4.9	7.9	11.1	14.1	17

(2)

x	1	4	9	16	25
y	0.1	3	8.1	14.9	23.9

(3)

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
y	3.02	2.81	2.57	2.39	2.18	1.99	1.81	1.85

1188. Find the quadratic function:

(1)

x	7	8	9	10	11	12	13
y	3.1	4.9	5.3	5.8	6.1	6.4	5.9

(2)

x	0.78	1.56	2.34	3.12	3.81
y	2.50	1.20	1.12	2.25	4.28

(3)

x	−3	−2	−1	0	1	2	3
y	−0.71	−0.01	0.51	0.82	0.88	0.81	0.49

1189. Find the power function $S = At^q$:

t	1	2	3	4	5
S	7.1	15.2	48.1	96.3	150.1

1190. Find the exponential function $S = Ae^{ct}$:

(1)

t	2.2	2.7	3.5	4.1
S	67	60	53	50

(2)

t	1	3	5	7	9	11
S	0.75	1.81	5.34	10.86	24.52	59.00

1191. Find the best approximation to the function $f(x) = \sin(\pi x/2)$ in the interval $0 \leqslant x \leqslant 1$ by a third-degree polynomial.

Solution. To find the coefficients in the function $\varphi(x) = a_0x^3 + a_1x^2 + a_2x + a_3$, we derive a system of equations of form (5):

$$\begin{cases} \int_0^1 \left(a_0x^3 + a_1x^2 + a_2x + a_3 - \sin\dfrac{\pi x}{2}\right)x^3\,dx = 0, \\ \int_0^1 \left(a_0x^3 + a_1x^2 + a_2x + a_3 - \sin\dfrac{\pi x}{2}\right)x^2\,dx = 0 \\ \int_0^1 \left(a_0x^3 + a_1x^2 + a_2x + a_3 - \sin\dfrac{\pi x}{2}\right)x\,dx = 0 \\ \int_0^1 \left(a_0x^3 + a_1x^2 + a_2x + a_3 - \sin\dfrac{\pi x}{2}\right)dx = 0. \end{cases}$$

Integration yields

$$\begin{cases} \dfrac{1}{7}a_0 + \dfrac{1}{6}a_1 + \dfrac{1}{5}a_2 + \dfrac{1}{4}a_3 = \dfrac{12}{\pi^2} - \dfrac{96}{\pi^4}, \\ \dfrac{1}{6}a_0 + \dfrac{1}{5}a_1 + \dfrac{1}{4}a_2 + \dfrac{1}{3}a_3 = \dfrac{8}{\pi^2} - \dfrac{16}{\pi^3}, \\ \dfrac{1}{5}a_0 + \dfrac{1}{4}a_1 + \dfrac{1}{3}a_2 + \dfrac{1}{2}a_3 = \dfrac{4}{\pi^2}, \\ \dfrac{1}{4}a_0 + \dfrac{1}{3}a_1 + \dfrac{1}{2}a_2 + a_3 = \dfrac{2}{\pi}. \end{cases}$$

Solving the last system, we find $a_0 = -0.40$, $a_1 = -0.24$, $a_2 = 1.64$, $a_3 = -0.05$. Consequently,

$$\varphi(x) = -0.4x^3 - 0.24x^2 + 1.64x - 0.05.$$

Check up: if $x = 1/3$, then $f(1/3) = 0.50$, $\varphi(1/3) = 0.51$.

1192. Find the best approximation to the function $f(x) = \ln(4 + x)$ by a second-degree polynomial for $0 \leqslant x \leqslant 1$.

1193. Find the best approximation to the function $f(x) = 1/(1 + x)$ by a third-degree polynomial for $0 < x < 1$.

Chapter 10

Fundamentals of Calculus of Variations

10.1. Introduction

10.1.1. Functional. A function is one of the principal concepts of mathematical analysis. In the simplest case, the notion of functional relationship can be stated as follows. Let C be any set of real numbers. If to every number x of the set C there corresponds a number y, then we say that there is defined a function $y = f(x)$ on the set C. The set C is called the domain of definition of the function f.

In many cases, however, the notion of a function is insufficient. Thus, for instance, the electromagnetic field intensity at a given point caused by a current flowing through the conductor depends on the shape of the curve the conductor forms. The concept of a functional is a direct and natural generalization of the concept of a function and includes it as a special case.

Let C be a set of some objects, which may be numbers, points of space, curves, functions, surfaces and the like. If to every element x of C there corresponds a certain real number y, then we say that there is defined a functional $y = I(x)$ on the set C. If C is a set of numbers x, then the functional $y = I(x)$ is nothing other than a function of one argument. When C is a set of pairs of numbers (x_1, x_2), then the functional is a function $y = I(x_1, x_2)$ of two arguments, and so on. Considered in the calculus of variations are functionals whose domain of definition C are the sets of functions $y(x)$.

1194. $I[y(x)] = \int\limits_0^1 [y(x)]^2\,dx$. If we substitute for $y(x)$ various concrete functions, say, $y_1(x) = x$, $y_2(x) = e^x$, $y_3(x) = \sqrt{1 + x^2}$, then we obtain, respectively, $I(y_1) = \dfrac{1}{3}$, $I(y_2) = \dfrac{1}{2}(e^2 - 1)$, $I(y_3) = \dfrac{4}{3}$.

The calculus of variations is concerned with the problem of seeking the least and greatest values of the functionals defined on the sets of curves and surfaces. Here, as in the example given above, the functionals are defined by some definite integrals.

Functions from the domain of definition C of the given functional I will be called *functions admissible for comparison* or simply *admissible functions.*

10.1.2. Classes of functions and neighbourhoods. In what follows we shall use the following classes of functions defined on some interval $[x_0, x_1]$:

(1) $C[x_0, x_1]$, **a class of continuous functions;**

(2) $C^{(1)}[x_0, x_1]$, **a class of smooth functions, that is, functions possessing continuous first derivatives;**

(3) $C^{(m)}[x_0, x_1]$, a class of functions possessing continuous mth derivatives.

For the functions entering into the classes enumerated above we must introduce the notion of a distance. Indeed, if $y = y_1(x)$ and $y = y_2(x)$ are functions belonging to the class $C[x_0, x_1]$, then the distance between them is the number $\rho_0 = \rho_0(y_1, y_2) = \max\limits_{x_0 \leqslant x \leqslant x_1} |y_1(x) - y_2(x)|$ (the distance of order zero). If $y = y_1(x)$ and $y = y_2(x)$ enter into the class $C^{(1)}[x_0, x_1]$, then the distance between them can be introduced as follows: $\rho_1 = \rho_1(y_1, y_2) = \max\limits_{x_0 \leqslant x \leqslant x_1} |y_1(x) - y_2(x)| + \max\limits_{x_0 \leqslant x \leqslant x_1} |y_1'(x) - y_2'(x)|$ (the distance of order zero). By analogy, we can introduce the distance of order m between the functions belonging to the class $C^{(m)}[x_0, x_1]$.

1195. Find the distance of order zero between the functions $y = x^2$ and $y = x$ on the interval $[0, 1]$.

Solution. $\rho_0 = \max\limits_{x_0 \leqslant x \leqslant x_1} |x^2 - x|$. At the points of the interval $[0, 1]$ the function $y = x^2 - x$ assumes zero values. Let us verify whether it possesses any extrema on the interval $(0, 1)$. We have $y' = 2x - 1 = 0$ when $x = \frac{1}{2}$. In this case, $y''\left(\frac{1}{2}\right) = 2 > 0$. This means that at the point $x = \frac{1}{2}$ the function being investigated attains its minimum equal to $-\frac{1}{4}$. Therefore, $|x^2 - x|$ assumes at the point $x = \frac{1}{2}$ a value which is the greatest on the interval $[0, 1]$ and equals $\frac{1}{4}$ and $\rho = \frac{1}{4}$.

Suppose the number ε is positive. The ε-neighbourhood of order zero of the function $\bar{y}(x)$ belonging to the class $C[x_0, x_1]$ is the set of all functions $y(x)$ of that class such that $\rho_0(y, \bar{y}) < \varepsilon$. In graphical form this means that the curves $y = y(x)$ and $y = \bar{y}(x)$ defined on the interval $[x_0, x_1]$ are close with respect to their ordinates.

The ε-neighbourhood of the first order of the function $\bar{y}(x)$ belonging to the class $C^{(1)}[x_0, x_1]$ is the set of all functions $y(x)$ of that class such that $\rho_1(y, y) < \varepsilon$. In graphical form this means that on the interval $[x_0, x_1]$ the curves $y = y(x)$ and $y = \bar{y}(x)$ are close both with respect to their ordinates and with respect to the slopes of the tangents at the points with the same abscissas. By analogy, we can introduce the ε-neighbourhood of order m of the function $\bar{y}(x)$ belonging to the class $C^{(m)}[x_0, x_1]$.

1196. Find the neighbourhoods of the function $\bar{y}(x) \equiv 0$ on the interval $[0, 2\pi]$ which contain the functions forming the sequence $y_n(x) = \frac{\cos nx}{n+1}$, $n = 1, 2, \ldots$.

Solution. Since $|y_n(x) - y(x)| = \left|\frac{\cos nx}{n+1}\right| \leqslant \frac{1}{n+1}$ and $\lim\limits_{n \to \infty} \frac{1}{n+1} = 0$,

the functions $y_n(x)$ belong to any ε-neighbourhood of order zero of the function $\bar{y}(x) \equiv 0$ if n is sufficiently large. It is no longer true for the ε-neighbourhood of order zero, as is shown by the equalities

$$|y_n'(x) - \bar{y}'(x)| = \left| -\frac{n}{n+1} \sin nx \right| = \frac{n}{n+1}, \quad \text{if}$$

$$x = \frac{\pi}{2n}, \quad \text{and} \quad \lim_{n \to \infty} \frac{n}{n+1} = 1.$$

In such cases we say that there is proximity of order zero between the function $\bar{y}(x)$ and the sequence $\{y_n(x)\}_{n=1}^{0}$.

10.1.3. Extrema of functionals. Suppose C is a class of admissible functions of the functional I. We say that in this class the functional I possesses an absolute minimum (maximum) provided by the function $\bar{y}(x)$ belonging to the class C is for any function $y(x)$ of that class there holds the inequality

$$I[y(x)] \geqslant [I\bar{y}(x)] \quad (I[y(x)] \leqslant I[\bar{y}(x)]). \tag{1}$$

The functional I is said to possess a relative minimum (maximum) in class C, provided by the function $\bar{y}(x)$ of class C, if there is a ε-neighbourhood of the function $\bar{y}(x)$ such that inequality (1) is satisfied for any function $y(x)$ of class C from the ε-neighbourhood.

The relative minimum (maximum) is called *strong* if inequality (1) holds for all admissible function $y(x)$ belonging to a certain ε-neighbourhood of order zero of the function $\bar{y}(x)$. The relative minimum (maximum) is called *weak* if inequality (1) is satisfied for all admissible functions $y(x)$ lying in the ε-heighbourhood of order unity of the function $\bar{y}(x)$.

Thus, every absolute extremum is a strong or a weak relative extremum. Every strong relative extremum is at the same time a weak extremum, whereas a weak relative extremum is not, in general, strong.

10.1.4. The notion of variation of a functional. Let us consider, on the example of the functional $I_1[y(x)] = \int_0^1 y^2\,dx$, how its value changes with a small variation of the function $y(x)$ on which it depends. Suppose we first substitute a certain function $y(x)$ into the right-hand side of functional, and then a new function $y(x) + \delta y(x)$, where $\delta y(x)$, a so-called variation of $y(x)$, is an arbitrary function assuming small values. For instance, we may first have $y = x^2$ and then $y = x^2 + \alpha \cdot x(1-x)$, where the constant α is small. Then, the value of the functional changes, too, and becomes equal to

$$\int_0^1 (y + \delta y)^2\,dx = \int_0^1 y^2\,dx + 2\int_0^1 y \cdot \delta y\,dx + \int_0^1 (\delta y)^2\,dx.$$

Thus, in the example being considered the increment of the functional is $\Delta I_1 =$

$= 2 \int_0^1 y \cdot \delta y \, dx + \int_0^1 (\delta y)^2 \, dx$. If we fix the value of the function $y(x)$ and change its variation δy, we see that the first term on the right-hand side is linear with respect to δy and the second term is quadratic. We shall call the linear term in the increment of the functional the *variation* of the functional and denote it by δI_1, i.e. $\delta I_1 = 2 \int_0^1 y \cdot \delta y \, dx$. Thus, $\Delta I_1 \approx \delta I_1$ with an accuracy to within the terms of the higher order of smallness.

Let us now consider the case of the general functional of the form

$$I = \int_{x_0}^{x_1} F(x, y, y') \, dx,$$

where F is some known function of three variables. If we substitute into the integral, instead of $y(x)$, a function $y(x) + \alpha \cdot \eta(x)$, where $\eta(x)$ is some fixed function of class $C^{(1)}[x_0, x_1]$ such that $\eta(x_0) = \eta(x_1) = 0$, we obtain a function of the parameter

$$I(\alpha) = \int_{x_0}^{x_1} F[x, y(x) + \alpha \cdot \eta(x), y'(x) + \alpha \cdot \eta'(x)] \, dx.$$

Let us write its formal expansion into the Maclaurin series:

$$I(\alpha) = I(0) + \frac{f}{1!} \cdot I'(0) + \frac{f^2}{2!} I''(0) + \ldots .$$

The expressions $\alpha \cdot I'(0)$ and $\alpha^2 \cdot \frac{I''(0)}{2}$ obtained here are called the *first and the second variation* of the functional I and are symbolized as δI and $\delta^2 I$. For the functional considered in the example given above, we have $\delta I = \alpha \cdot I_1'(0) = 2 \int_0^1 y(x) \, \alpha \cdot \eta(x) \, dx$. This value coincides with that given above in the case of the variation of the function $y(x)$ being equal to $\delta y = \alpha \cdot \eta(x)$.

10.1.5. Principal lemmas.

Lemma 1 (Lagrange's). *If $f(x)$ is a function continuous on the interval $[x_0, x_1]$ and if $\int_{x_0}^{x_1} f(x)\,\eta(x) \, dx = 0$ for any m times continuously differentiable function $\eta(x)$, equal to zero at the end points of the interval $[x_0, x_1]$ together with all its kth derivatives ($k \leqslant m$), then $f(x) = 0$.*

Proof. Let us assume the contrary. Then $f(\xi) \neq 0$ at some interior point ξ of the interval $[x_0, x_1]$. Suppose, for the sake of definiteness, that $f(\xi) > 0$. By virtue of

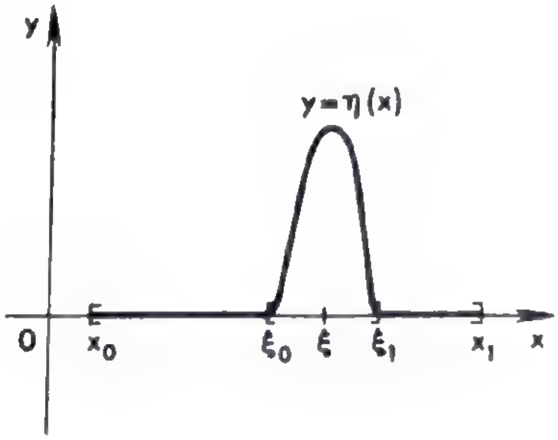


Fig. 85

the continuity of $f(x)$, we can indicate a subinterval $[\xi_0, \xi_1] \subset [x_0, x_1]$ in which $f(x) > 0$. Let us now define $\eta(x)$ as follows (Fig. 85):

$$\eta(x) = \begin{cases} 0 & \text{outside the interval } [\xi_0, \xi_1], \\ [(\xi_1 - x)(x - \xi_0)]^{m+1} & \text{for } \xi_0 \leqslant x \leqslant \xi_1. \end{cases}$$

It is easy to verify that the function $\eta(x)$ is m times continuously differentiable on the interval $[x_0, x_1]$ and vanishes at the end points of the interval $[x_0, x_1]$ together with its derivatives of order m inclusive. Therefore,

$$0 = \int_{x_0}^{x_1} f(x)\,\eta(x)\,dx = \int_{\xi_0}^{\xi_1} f(x)[(\xi_1 - x)(x - \xi_0)]^{m+1}\,dx > 0.$$

The contradiction obtained proves the lemma.

Generalization of Lemma 1. Evidently, Lemma 1 can be extended to integrals of the form $\int_{x_0}^{x_1} [\eta_1 f_1 + \eta_2 f_2 + \eta_3 f_3]\,dx$, where f_1, f_2, f_3 are functions continuous on the interval $[x_0, x_1]$, and η_1, η_2, η_3 are functions satisfying the same conditions as $\eta(x)$. For that integral to be equal to zero for various functions η_1, η_2, η_3, satisfying the indicated conditions, it is necessary that the functions f_1, f_2, f_3 should be identically zero.

Lemma 2 (Ostrogradsky's). *If $f(x, y)$ is a function continuous in the domain D and if $\iint_D f(x, y)\eta(x, y)\,dx\,dy = 0$ for every function $\eta(x, y)$, continuous in D together with its first-order partial derivatives and equal to zero along the contour Γ bounding the domain D, then $f(x, y) \equiv 0$.*

Proof. Reasoning as in the proof of the first lemma, we assume that at some point (ξ, ζ) within the domain D the function $f(x, y) > 0$. Then it is aslo positive in some circle with centre at (ξ, ζ) and radius $\rho > 0$ lying in the interior of D. We define the function $\eta(x, y)$ as follows:

$$\eta(x, y) = \begin{cases} 0 & \text{for} \quad (x - \xi)^2 + (y - \zeta)^2 > \rho^2, \\ [(x - \xi)^2 + (y - \zeta)^2 - \rho^2]^2 & \text{for} \quad (x - \xi)^2 + (y - \zeta)^2 \leqslant \rho^2. \end{cases}$$

It is easy to verify that $\eta(x, y)$ satisfies the conditions of the lemma, and the integral reduces to the integral along the indicated circle of the continuous positive function and is positive, in contradiction to the lemma.

10.2. The Necessary Condition for an Extremum of a Functional $I = \int_{x_0}^{x_1} F(x, y, y')\,dx$.

10.2.1. Euler's equation. Let us consider the functional $I[y(x)] = \int_{x_0}^{x_1} F(x, y, y')dx$, where F is a given function of three variables, continuous together with its partial derivatives with respect to all the variables up to the second order inclusive for all points with the coordinates x, y belonging to a certain plane domain D and for all finite values of y'. We shall define the set C of the admissible functions as follows: every function $y(x)$ belongs to class $C^{(1)}[x_0, x_1]$, the curve $y = y(x)$ lying entirely in the domain D and $y(x_0) = y_0, y(x_1) = y_1$, where y_0 and y_1 are some constants.

To find whether the given functional possesses an extremum, we take any function $\eta(x)$, satisfying the conditions of Lagrange's lemma. We form a function $y(x, \alpha) = y(x) + \alpha \cdot \eta(x)$, where α is a small numerical parameter (Fig. 86). This function satisfies the same boundary conditions as $y(x)$.

Substituting it into the functional, we obtain the function α:

$$I(\alpha) = \int_{x_0}^{x_1} F[x, y(x) + \alpha \cdot \eta(x), y'(x) + \alpha \cdot \eta'(x)]\,dx.$$

For any given positive ε the function $y(x) + \alpha \cdot \eta(x)$ is in the ε-neighbourhood (even of the first order) of the curve $y(x)$ for all the values of the parameter α sufficiently close to zero. Consequently, if $y(x)$ provides an extremum for the functional I, then the function $I(\alpha)$ must possess an extremum for $\alpha = 0$ and, therefore, its derivative must vanish for $\alpha = 0$. Differentiating under the integral sign, we get

$$I'(0) = \int_{x_0}^{x_1} [F_y(x, y, y')\,\eta(x) + F_{y'}(x, y, y')\,\eta'(x)]\,dx.$$

Next we perform integration by parts in the second summand. Setting $u = F_{y'}$, and $dv = \eta'(x)\,dx$, we get

$$\int_{x_0}^{x_1} F_{y'}(x, y, y')\,\eta'(x)\,dx = F_{y'} \cdot \eta(x)\Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta(x) \cdot \frac{d}{dx}(F_{y'})\,dx.$$

Here the term outside the integral is zero since by the hypothesis $\eta(x)$ vanishes at

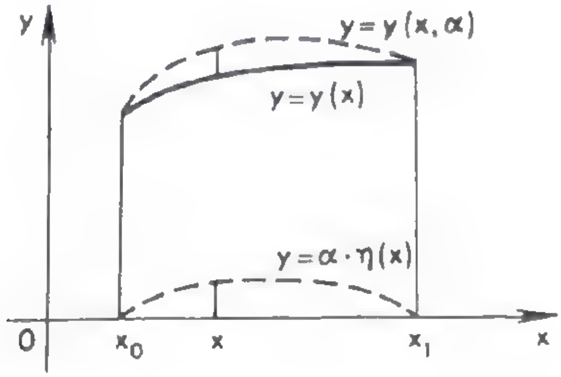


Fig. 86

the end points of the interval $[x_0, x_1]$. Consequently,

$$I'(0) = \int_{x_0}^{x_1} \eta(x)\left[F_y - \frac{d}{dx}(F_{y'})\right]dx = 0.$$

Applying Lagrange's lemma, we can assert that the curve $y(x)$, providing an extremum for the integral, must satisfy the differential equation

$$F_y - \frac{d}{dx}(F_{y'}) = 0$$

or, written out,

$$y''\cdot F_{y'y'} + y'\cdot F_{yy'} + F_{xy'} - F_y = 0.$$

This equation was obtained by Euler and is called *Euler's equation*. It is a differential equation of the second order with respect to the unknown function $y(x)$. The general solution of this equation includes two arbitrary constants C_1 and C_2 which should be defined by the boundary conditions $y(x_0) = y_0$ and $y(x_1) = y_1$. The integral curves of Euler's equation are called *extremals*. For an extremal to pass through two points, $M(x_0, y_0)$ and $N(x_1, y_1)$, we must choose the constants C_1 and C_2 such that $\varphi(x_0, C_1, C_2) = y_0$ and $\varphi(x_1, C_1, C_2) = y_1$, where $y = \varphi(x, C_1, C_2)$ is a general solution of Euler's equation.

10.2.2. Special cases of integrability of Euler's equation.

Case 1. The function F does not depend on y', i.e. $F = F(x, y)$. Then the terms containing partial derivatives with respect to y' in Euler's equation are equal to zero and the equation assumes the form $\dfrac{\partial F}{\partial y} = 0$. This equation is not differentiable with respect to the unknown function $y(x)$. It defines one or several functions which, in general, do not satisfy the boundary conditions $y(x_0) = y_0$ and $y(x_1) = y_1$. Consequently, the variational problem in question does not have a solution in a general case. Only in special cases there can be found a curve $y = y(x)$ which passes through the points $M(x_0, y_0)$ and $N(x_1, y_1)$ and is a solution of the functional equation $\dfrac{\partial F}{\partial y} = 0$.

1197. $I[y(x)] = \int_0^1 (x \sin y + \cos y)\, dx,\ y(0) = 0,\ y(1) = \frac{\pi}{4}.$

Solution. Here $F = x \sin y + \cos y$; Euler's equation has the form $\frac{\partial F}{\partial y} = x \cos y - \sin y = 0, \Rightarrow y = \arctan x$. This equation of the extremal satisfies the boundary conditions.

1198. $I[y(x)] = \int_1^e (xe^y - ye^x)\, dx,\ y(1) = 1,\ y(e) = 1.$

Solution. $F = xe^y - ye^x$; Euler's equation is $\frac{\partial F}{\partial y} = xe^y - e^x$, $y = x - \ln x$. The solution obtained does not satisfy the boundary conditions.

Case 2. F is linearly dependent on y', i.e. $F = P(x, y) + y' \cdot Q(x, y)$. Euler's equation is of the form

$$y' \cdot \frac{\partial Q}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} - y' \cdot \frac{\partial Q}{\partial y} = 0, \quad \text{or} \quad \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0.$$

The equation obtained is not differential with respect to the unknown function $y(x)$ and, in general, does not have any solutions satisfying the imposed boundary conditions. Now if $\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}$ with respect to both variables x and y, then the expression $P dx + Q dy$ is a total differential and, therefore, the line integral

$$I[y(x)] = \int_{x_0}^{x_1} (P + y' \cdot Q)\, dx = \int_{(x_0, y_0)}^{(x_1, y_1)} P\, dx + Q\, dy$$

does not depend on the integration path. Consequently, the value of the functional I is constant on all the admissible curves, and variational problem becomes meaningless.

1199. $I[y(x)] = \int_\alpha^\beta [xy' + 1)e^y + x^2 - y^2 y']\, dx,\ y(\alpha) = a,\ y(\beta) = b.$

Solution. Here F is linearly dependent on y':

$$F = (xy' + 1)e^y + x^2 - y^2 y' = (xe^y - y^2)y' + (x^2 + e^y),$$

i.e. $P(x, y) = x^2 + e^y$, $Q(x, y) = xe^y - y^2$ and $\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}$. The expression $(x^2 + e^y)\, dx + (xe^y - y^2)\, dy$ is a total differential and, consequently, the integral does not depend on the path of integration:

$$I[y(x)] = \int_{(\alpha, a)}^{(\beta, b)} (x^2 + e^y)\, dx + (xe^y - y^2)\, dy$$

$$= \int_{(\alpha, a)}^{(\beta, b)} d(xe^y + \frac{1}{3}y^3 + \frac{1}{3}x^3) = (xe^y - \frac{1}{3}y^3 + \frac{1}{3}x^3)\Big|_{(\alpha, a)}^{(\beta, b)}$$

$$= \beta e^b - \alpha e^a + \frac{\beta^3 - \alpha^3}{3} + \frac{a^3 - b^3}{3}.$$

The value of the functional I is constant for all the curves $y(x)$ passing through the points (α, a) and (β, b), and the variational problem is meaningless.

Case 3. F depends solely on y', i.e. $F = F(y')$. In that case, Euler's equation is of the form $y''\cdot F_{y'y'} = 0$. In particular, we get an equation $y'' = 0$. Its general solution is $y = C_1x + C_2$. Here the extremals are straight lines.

1200. $I[y(x)] = \int_0^1 (y'^2 + y' + 1)\,dx$, $M(0, 1)$, $N(1, 2)$.

Solution. Here $F = y'^2 + y' + 1$, $F_{y'} = 2y' + 1$, $F_y = 0$, $F_{y'y'} = 2$. Euler's equation is of the form $2y'' = 0$, $\Rightarrow y = C_1x + C_2$. Let us find C_1 and C_2 from the condition of the extremal passing through the points M and N; $C_2 = 1$, $C_1 + C_2 = 2$; $\Rightarrow C_1 = 1$, $C_2 = 1$. Thus, the extremal is a straight line $y = x + 1$.

Case 4. F depends only on x and y', i.e. $F = F(x, y')$. Since in this case $\frac{\partial F}{\partial y} = 0$, Euler's equation is $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ and we immediately find its first integral: $\frac{\partial F(x, y')}{\partial y'} = C_1$. Solving this equation for y' and integrating, we find the general solution of Euler's equation.

1201. $I[y(x)] = \int_0^e (xy'^2 - 2y')\,dx$, $y(1) = 1$, $y(e) = 2$.

Solution.

$$F = xy'^2 - 2y'; \quad \frac{\partial F}{\partial y'} = 2xy' - 2 = C, \; y' = \frac{C_1 + 2}{2x}, \; y = \frac{1}{2}(C_1 + 2)\ln x + C_2.$$

Using the boundary conditions, we get $1 = C_2$, $2 = \frac{1}{2}C_1 + C_2 + 1$. Hence $C_1 = 0$, $C_2 = 1$. The extremal is the curve $y = \ln x + 1$.

Case 5. F depends only on y and y', i.e. $F = F(y, y')$. In this case Euler's equation is $y''.\, F_{y'y'} + y'\cdot F_{yy'} - F_y = 0$.

Its first integral is easy to find. Indeed, we consider the equality

$$\frac{d}{dx}(F - y'F_{y'}) = y'\cdot F_y + y''\cdot F_{y'} - y''\cdot F_{y'} - y'^2\cdot F_{yy'} - y'y''F_{y'y}$$
$$= -y'(y''\cdot F_{y'y'} + y'\cdot F_{yy'} - F_y).$$

If the function y satisfies Euler's equation, then the right-hand side vanishes and $F - y'\cdot \frac{\partial F}{\partial y'} = C_1$ provides the first integral of Euler's equation.

1202 (problem on the surface of revolution of the least area).
Among the plane smooth curves joining the points $A(x_0, y_0)$ and $B(x_1, y_1)$ find that one which generates the surface with the least area upon rotation around the Ox axis.

Solution. The area of the surface of revolution is expressed by the integral $S = 2\pi\int_{x_0}^{x_1} y\sqrt{1 + y'^2}\,dx$. Here $F = y\sqrt{1 + y'^2}$, and therefore Euler's equation has the first integral $y\sqrt{1 + y'^2} - \dfrac{y \cdot y'^2}{\sqrt{1 + y'^2}} = C_1$ or $y = C_1 \cdot \sqrt{1 + y'^2}$.

Setting $y' = \sinh t$, we find $y = C_1 \cosh t$. Hence $dx = \dfrac{dy}{y'} = \dfrac{C_1 \cdot \sinh t}{\sinh t}\,dt = C_1\,dt$, $x = C_1 t + C_2$. Consequently, the sought-for curve is a catenary $y = C_1 \cdot \cosh \dfrac{x - C_1}{C_1}$.

The problem has a solution in the case when the arbitrary constants can be determined from the system of equations $\cosh \dfrac{x_0 - C_2}{C_1} = \dfrac{y_0}{C_1}$, $\cosh \dfrac{x_1 - C_2}{C_2} = \dfrac{y_1}{C_1}$.

1203 (Brachistochrone problem). Among the curves joining two given points A and B find the path AB down which a freely started heavy particle will fall in the shortest time.

Solution. We draw a vertical plane through the points $A(x_0, 0)$ and $B(x_1, y_1)$ (Fig. 87). Since the heavy particle moves without initial velocity, we obtain, on the basis of the law of conservation of energy, that $\dfrac{mv^2}{2} = mgy$, $y(x_0) = 0$, where m is the mass of the particle, v is the velocity and g is the free fall acceleration.

Hence $v = \sqrt{2gy}$. On the other hand, $v = \dfrac{dS}{dt} = \sqrt{1 + y'^2}\,\dfrac{dx}{dt}$. Then, $dt = \dfrac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\,dx$. Consequently, $T(y) = \int_{x_0}^{x_1} \dfrac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\,dx$. For the given case,

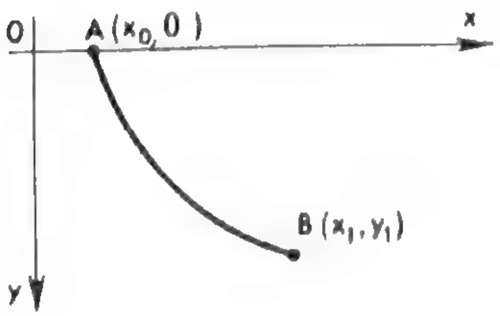


Fig. 87

Euler's equation has the first integral

$$\frac{\sqrt{1 + y'^2}}{\sqrt{y}} - y' \cdot \frac{y'}{\sqrt{y(1 + y'^2)}} = C_1 \quad \text{or} \quad y = \frac{1}{C_1^2(1 + y'^2)}.$$

Setting $y' = \cot\frac{t}{2}$, we have $y = \frac{1}{2C_1^2}(1 - \cos t)$. Consequently,

$$dx = \frac{dx}{y'} = \frac{\sin\frac{t}{2}\cos\frac{t}{2}}{\cot\frac{t}{2}\, C_1^2}\, dt \quad \text{or} \quad dx = \frac{1}{2C_1^2}(1 - \cos t)\, dt,$$

$$x = \frac{1}{2C_1^2}(t \sin t) + C_2.$$

Thus we have obtained parametric equations of a cycloid:

$$\begin{cases} x = \dfrac{1}{2C_1^2}(t - \sin t) + C_2, \\ y = \dfrac{1}{2C_1^2}(1 - \cos t). \end{cases}$$

The arbitrary constants C_1 and C_2 can be found from the condition of the curve passing through the points A and B.

Case 6. The function F depends on y alone, i.e. $F = F(y)$. Here Euler's equation has the form $\frac{\partial F}{\partial y} = 0$.

1204. $I[y(x)] = \int\limits_0^1 (2e^y - y^2)\, dx,\ y(0) = 1,\ y(1) = e.$

Solution. $F = 2e^y - y^2,\ \frac{\partial F}{\partial y} = 2e^y - 2y,\ \Rightarrow y = e^y.$

The last equation does not possess even numerical solutions. Indeed, if $y < 0$, then $e^y > 0$; if $y > 0$, then we use the expansion $e^y = 1 + y + \frac{y^2}{2!} + \ldots$. Therefore, there are no extremals.

Case 7. $F = p(x)\, y'^2 + q(x)\, y^2 + 2f(x) \cdot y$. For this case Euler's equation is $\frac{d}{dx}(py') - qy - f = 0$. We have thus established that the function $y(x)$, providing a minimum for the integral $I\,[y(x)] = \int\limits_{x_0}^{x_1} [p(x)\, y'^2 + q(x)y^2 + 2f(x)y]\, dx$, must necessarily satisfy the so-called adjoint differential equation of the second order $\frac{d}{dx}(py') - qy - f = 0$. The general solution of this equation includes two

arbitrary constants. Consequently, through two given points (x_0, y_0) and (x_1, y_1) we can, in general, draw one straight line satisfying the equation.

1205. $I[y(x)] = \int_0^{\ln 2} (y'^2 + 2y^2 + 2y)\, e^{-x} dx,\ y(0) = y(\ln 2) = 0.$

Solution. Here $F = (y'^2 + 2y^2 + 2y)\ e^{-x}$. Euler's equation is $\frac{d}{dx}(e^{-x}y') - 2e^{-x}y - e^{-x} = 0,\ y'' - y' - 2y = 1,\ \Rightarrow y = C_1e^{2x} + C_2e^{-x} - \frac{1}{2}.$

Using the boundary conditions, we get $C_1 + C_2 = \frac{1}{2},\ 4C_1 + \frac{1}{2}C_2 = \frac{1}{2},\ \Rightarrow$ $C_1 = \frac{1}{14},\ C_2 = \frac{3}{7}$. Consequently, $y = \frac{1}{14}e^{2x} + \frac{3}{7}e^{-x} - \frac{1}{2}$.

1206. $I[y(x)] = \int_0^1 (y \sinh x - y^2 \cosh x)\, dx,\ y_0 = 0,\ y_1 = 1.$

1207. $I[y(x)] = \int_{x_0}^{x_1} e^y(1 + xy')\, dx,\ y(x_0) = y_0,\ y(x_1) = y_1.$

1208. $I[y(x)] = \int_0^1 y'^2 dx,\ y(0) = 0,\ y(1) = 1.$

1209. $I[y(x)] = \int_0^1 (xy' - y'^2)\, dx,\ y(0) = 1,\ y(1) = \frac{1}{4}.$

1210. $I[y(x)] = \int_{-1}^{2} \frac{\sqrt{1 + y'^2}}{y} dx,\ y(-1) = 1,\ y(2) = 4.$

1211. $I[y(x)] = \int_{x_0}^{x_1} (y^2 + 1)\, dx,\ y(x_0) = y(x_1) = 0.$

1212. $I[y(x)] = \int_0^{3\pi/2} (y^2 - 2y'^2)\, e^{-x} dx,\ y(0) = 0,\ y\left(\frac{3}{2}\pi\right) = e^{3\pi/4}.$

10.3. Functionals Dependent on Higher-Order Derivatives

Let us now consider the case when the integral contains derivatives of the desired function of the order higher than the first:

$$I = \int_{x_0}^{x_1} F(x, y, y', \dots, y^{(n)})dx.$$

As before, we construct a close curve $y(x) + \alpha \cdot \eta(x)$, substitute it into the integral, differentiate with respect to α and set $\alpha = 0$. We obtain

$$I'(0) = \int_{x_0}^{x_1} [F_y\eta(x) + F_{y'}\eta'(x) + \ldots + F_{y^{(n)}}\eta^{(n)}(x)]\,dx.$$

We transform all the summands on the right-hand side, except for the first, performing integration by parts several times:

$$\int_{x_0}^{x_1} F_{y^{(k)}}\eta^{(k)}(x)\,dx = \left[F_{y^{(k)}}\eta^{(k-1)}(x) - \frac{d}{dx}F_{y^{(k)}}\eta^{(k-2)}(x) + \ldots\right.$$

$$\left.\ldots + (-1)^{(k-1)}\frac{d^{(k-1)}}{dx^{(k-1)}}F_{y^{(k)}}\eta(x)\right]_{x_2}^{x_1} + (-1)^k\int_{x_0}^{x_1}\frac{d^k}{dx^k}F_{y^{(k)}}\eta(x)\,dx.$$

We shall assume that $\eta(x)$ and its derivatives up to the order $n - 1$ vanish at the end points. Consequently, the terms outside the integral will vanish; equating $I'(0)$ to zero, we obtain the condition which, by virtue of the principal lemma, leads us to the Euler-Poisson equation $F_y - \frac{d}{dx}F_{y'} + \ldots + (-1)^n\frac{d^n}{dx^n}F_{y^{(n)}} = 0$. This is a differential equation of order $2n$. Its general solution contains $2n$ arbitrary constants and we must have $2n$ more boundary conditions consisting in the fact that we assign the values of the function and of its derivatives up to order $n - 1$ at the end points of the interval. It follows from these boundary conditions that analogous values for $\eta(x)$ must vanish.

1213. Among the functions of class $C^{(2)}$ satisfying the boundary conditions $y(0) = y(\pi) = 0, y'(0) = y'(\pi) = 1$, find the one which can provide an extremum for the functional

$$I[y(x)] = \int_0^{\pi}(16y^2 - y''^2 + x^2)\,dx.$$

Solution. The Euler-Poisson equation has the form $32y + (-1)^2 \times \frac{d^2}{dx^2} \times (-2y'') = 0$ or $y^{IV} - 16y = 0$. The general solution is $y = C_1e^{2x} + C_2e^{-1x} + C_3\cos 2x + C_4\sin 2x$. Using the boundary conditions, we get $C_1 = C_2 = C_3 = 0, C_4 = \frac{1}{2}$.

Consequently, $y = \frac{1}{2}\sin 2x$.

Find the extremals of the following functionals:

1214. $I = \frac{1}{2}\int_{x_0}^{x_1} y''^2 dx$, $y(x_0) = y(x_1) = 0$, $y'(x_0) = y'(x_1) = 0$.

1215. $I = \int_0^1 (y''^2 + 2y'^2 + y^2)\,dx, y(0) = y(1) = 0, y'(0) = 1, y'(1) - \sinh 1.$

10.4. Functionals Dependent on Two Functions of One Independent Variable

$$I = \int_{x_0}^{x_1} F(x, y, z, y', z')\,dx$$

In this problem it is necessary to find the curves $y = y(x)$ and $z = z(x)$ satisfying the conditions $y(x_0) = y_0, y(x_1) = y_1$ and $z(x_0) = z_0, z(x_1) = z_1$ which provide the least value for the integral I.

We proceed in the same way as in the simplest problem. As close curves we take $y(x) + \alpha \cdot \eta_1(x)$ and $z(x) + \alpha \cdot \eta_2(x)$, where $\eta_1(x)$ and $\eta_2(x)$ are arbitrary functions of class $C^{(1)}$ vanishing at the ends of the interval $[x_0, x_1]$.

We derive the variation of the functional, $\delta I = \alpha \cdot I'(0)$, and get

$$\delta I = \alpha \cdot \int_{x_0}^{x_1} \left\{ \eta_1(x)\left[F_y - \frac{d}{dx}(F_{y'})\right] + \eta_2(x)\left[F_z - \frac{d}{dx}(F_{z'})\right]\right\} dx.$$

Since $\delta I = 0$ for any $\eta_1(x)$ and $\eta_2(x)$, we can take $\eta_2(x) = 0$ and arbitrary $\eta_1(x)$ and verify, using the first lemma, that the multiplier in $\eta_1(x)$ is zero; conversely, we can take $\eta_1(x) = 0$ and arbitrary $\eta_2(x)$, and make sure that the multiplier in $\eta_2(x)$ is zero. Thus, on the interval $[x_0, x_1]$ the following conditions must be satisfied:

$$F_y - \frac{d}{dx}(F_{y'}) = F_z - \frac{d}{dx}(F_{z'}) = 0.$$

This system of equations with respect to the desired functions $y(x)$ and $z(x)$ plays the same role in this problem as Euler's equation for one unknown function $y(x)$.

1216. Find the extremals of the functional $I = \int_0^{\pi} (y'^2 - 2y^2 + 2yz - z'^2)\,dx$, if $y(0) = z(0) = 0$, $y(\pi) = z(\pi) = 1$.

Solution. Here the system of differential equations has the form $y'' + 2y - z = 0$, $z'' + y = 0$.

Eliminating z, we get the equation $y^{(IV)} + 2y'' + y = 0$, whose general solution is $y = C_1 \cos x + C_2 \sin x + x(C_3 \cos x + C_4 \sin x)$. By virtue of the boundary conditions $C_1 = 0$, $C_3 = -\frac{1}{\pi} \Rightarrow y = C_2 \sin x + C_4 x \sin x - \frac{x}{\pi}\cos x$. For z we get $z = C_2 \sin x + C_4(2\cos x + x \sin x) + \frac{1}{\pi}(2 \sin x - x \cos x)$. The constants C_2 and C_4 can be found from the boundary conditions for z: $C_4 = 0$, C_2 is arbitrary. Then we have $z = C_2 \sin x + \frac{1}{\pi}(2 \sin x - x \cos x)$. The family of extremals has the form $y = C_2 \sin x - \frac{x}{\pi}\cos x$, $z = C_2 \sin x + \frac{1}{\pi}(2 \sin x - x \cos x)$.

1217. $I = \int\limits_{1/2}^{1} (y'^2 - 2xyz')\,dx,\ y\left(\frac{1}{2}\right) = 2,\ z\left(\frac{1}{2}\right) = 15,\ y(1) = z(1) = 1.$

1218. $I = \int\limits_{1}^{2} (z'^2 - xy'z)\,dx,\ y(1) = z(1) = 1,\ y(2) = -\frac{1}{6},\ z(2) = \frac{1}{2}.$

10.5. Functionals Dependent on Functions of Two Independent Variables

$$I = \iint\limits_D F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) dx\,dy$$

Let us introduce the designations $\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q$. Assume that $F(x, y, z, p, q)$ is a function of its five arguments, continuous together with its derivatives up to the second order inclusive in some space domain R of the values of the variables x, y, z and for all finite p and q. Suppose that Γ is a closed space curve whose projection on the plane xOy is a simple closed contour C bounding the domain D. The equation of the surface S lying in the domain R and passing through the curve Γ will be taken in the form $z = f(x, y)$, where the function $f(x, y)$, continuous together with its partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, will be called *admissible.*

The double integral $I = \iint\limits_D F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) dx\,dy$ possesses a definite finite value for each admissible surface S. The problem is to find the surface $z = f(x, y)$ for which the integral possesses the least value as compared to the integrals taken over the close admissible surfaces $z = f(x, y) + \alpha \cdot \eta(x, y)$, where $\eta(x, y)$ is an arbitrary function continuous in the domain D together with its derivatives $\frac{\partial \eta}{\partial x}$ and $\frac{\partial \eta}{\partial y}$ and vanishing on the contour C. Then the function

$$I(\alpha) = \iint\limits_D F\left(x, y, f(x, y) + \alpha \cdot \eta(x, y), \frac{\partial f}{\partial x} + \alpha \cdot \frac{\partial \eta}{\partial x}, \frac{\partial f}{\partial y} + \alpha \cdot \frac{\partial \eta}{\partial y}\right) dx\,dy$$

must attain its minimum for $\alpha = 0$. In that case the first variation $\delta I = \alpha \cdot I'(0)$ must be zero. Differentiating $I(\alpha)$ under the integral sign and setting $\alpha = 0$, we find

$$\delta I = \alpha \cdot \left[\frac{dI}{d\alpha}\right]_{\alpha = 0} = \alpha \cdot \iint\limits_D \left[\frac{\partial F}{\partial z}\eta(x, y) + \frac{\partial F}{\partial p} \cdot \frac{\partial \eta}{\partial x} + \frac{\partial F}{\partial q} \cdot \frac{\partial \eta}{\partial y}\right] dx\,dy.$$

We transform the last two summands by Green's formula

$$\left(\iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int\limits_C P\,dx + Q\,dy\right);$$

$$\iint_D \left(\frac{\partial F}{\partial p} \cdot \frac{\partial \eta}{\partial x} + \frac{\partial F}{\partial q} \cdot \frac{\partial \eta}{\partial y} \right) dx\,dy = \iint_D \left[\frac{\partial}{\partial x} \left(\eta \cdot \frac{\partial F}{\partial p} \right) \right.$$

$$\left. + \frac{\partial}{\partial y} \left(\eta \cdot \frac{\partial F}{\partial q} \right) \right] dx\,dy - \iint_D \eta \left[\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial p} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial q} \right) \right] dx\,dy$$

$$= \int_C \left(\eta \cdot \frac{\partial F}{\partial p} dy - \eta \cdot \frac{\partial F}{\partial q} dx \right) - \iint_D \eta \left[\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial p} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial q} \right) \right] dx\,dy.$$

The contour integral obtained is equal to zero since by the hypothesis $\eta(x, y)$ vanishes on the contour C and, therefore, replacing the two last terms in the expression for δI by their new expression, we find

$$\delta I = \alpha \cdot \iint_D \eta(x, y) \cdot \left[\frac{\partial F}{\partial z} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial p} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial q} \right) \right] dx\,dy.$$

In this case, $\delta I = C$ for every $\eta(x, y)$ which is continuous together with its partial derivatives in the domain D and equal to zero on C; all the conditions of the second lemma are fulfilled for the expression $\frac{\partial F}{\partial z} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial p} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial q} \right)$.

Consequently, $\frac{\partial F}{\partial z} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial p} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial q} \right) = 0$ (the Euler Ostrogradsky equation). The reasonings presented above can be fully extended to the triple integral, with the only difference that the equation will contain one more term.

1219 (Plateau's problem). Find the surface with the least square passing through the given curve Γ in space.

Solution. The problem reduces to finding the minimum of the integral $\iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2}\, dx\,dy$. Here $F = \sqrt{1 + p^2 + q^2}$. For this case Euler-Ostrogradsky's equation assumes the form

$$\frac{\partial}{\partial x} \left(\frac{p}{\sqrt{1 + p^2 + q^2}} \right) + \frac{\partial}{\partial y} \left(\frac{q}{\sqrt{1 + p^2 + q^2}} \right) = 0.$$

Writing out this expression, we find

$$r(1 + q^2) + t\,(1 + p^2) - 2pqs = 0,$$

where $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x\,\partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$.

We have obtained an equation in partial derivatives defining the least surfaces. This equation describes the geometrical property of these surfaces by which the sum

of the main radii of curvature is equal to zero at every point of the surface. This sum is equal to

$$R_1 + R_2 = \frac{(1+q)\,r + (1+p^2)\,t - 2pqs}{qt - s^2} \cdot \sqrt{1 + p^2 + q^2}\,.$$

Indeed, if the function f satisfies the equation for the least surfaces obtained above, then $R_1 + R_2 = 0$.

1220. Write Euler-Ostrogradsy's equation for the functional

$$I = \iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2\right] dx\,dy.$$

1221. Write Euler-Ostrogradsky's equation for the functional

$$I = \iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 2z\,\varphi(x, y)\right] dx\,dy.$$

10.6. Parametric Form

In seeking the extremum of a functional, the requirement that the desired curve possess the explicit equation $y = y(x)$ may narrow the problem considerably. In a number of cases, it is expedient to take as admissible curves those which can be presented in parametric form: $x = x(t)$, $y = y(t)$, $t_0 \leqslant t \leqslant t_2$. In this case, the functional $I = \int_{x_0}^{x_1} F(x, y, y')\,dx$ discussed above assumes the form $I = \int_{t_0}^{t_1} F\left(x, y, \frac{\dot{y}}{\dot{x}}\right) \dot{x}\,dt$ where $\dot{x}$ and $\dot{y}$ are derivatives with respect to t, and t_0 and t_1 are the values of the parameter corresponding to the ends of the curves. Note that the integrand does not contain the independent variable t and is a homogeneous function of the first degree of $\dot{x}$ and $\dot{y}$.

Let us consider a certain integral in the general form:

$$I = \int_{t_0}^{t_1} F(x, y, \dot{x}, \dot{y})\,dt, \tag{1}$$

in which the integrand does not contain the independent variable t and is a homogeneous function of the first degree of $\dot{x}$ and $\dot{y}$, that is,

$$F(x, y, k \cdot \dot{x}, k \cdot \dot{y}) = k \cdot F(x, y, \dot{x}, \dot{y}). \tag{2}$$

We shall show that integral (1) does not change its form upon any change of the parameter t. Let us introduce, instead of t, some other parameter τ, setting $\tau = \tau(t)$, assuming that $\tau'(t) > 0$, so that τ increases with an increase in t. Since

$y_t' = y_\tau' \cdot \tau_t, x_t' = x_\tau' \cdot \tau_t', dt = t_\tau' d\tau = \dfrac{d\tau}{\tau_t'}$, we can transform integral (1), reducing it to the variable τ and obtain

$$I = \int_{\tau_0}^{\tau_1} F(x, y, \tau_t' \cdot x'_\tau, \tau'_t \cdot y'_\tau) \frac{d\tau}{\tau'_t}.$$

Using (2), we get

$$I = \int_{\tau_0}^{\tau_1} F(x, y, x'_\tau, y'_\tau)\, d\tau.$$

In parametric representation, the distance between the curve is defined irrespective of the choice of the parameter t, namely, the curve l_1 lies in the ε-neighbourhood of order zero of the curve l_2 if there is a mutually one-to-one and mutually continuous correspondence between l_1 and l_2 such that the distance between the respective points does not exceed ε. By analogy, we can define ε, the proximity of the first order.

Passing to derivation of the necessary condition for an extremum, we assume that some curve l, specified by the equations $x = x(t)$, $y = y(t)$, provides an extremum for the integral I. Let us take a curve close to l, specified by the equations $x = x(t) + d_1 \cdot \eta_1(t), y = y(t) + \alpha_2 \cdot \eta_2(t)$, and consider to be respective the points obtained at the same value of the parameter t. Substituting the equations of the close curve integral (1) and equating to zero the derivatives with respect to α_1 and α_2 for $\alpha_1 = \alpha_2 = 0$, we shall find, as before, that the functions $x(t)$ and $y(t)$ must satisfy the system of two Euler's equations for any choice of the parameter t:

$$F_x - \frac{d}{dt}F_{x'} = 0 \text{ and } F_y - \frac{d}{dt}F_{y'} = 0. \tag{3}$$

These equations do not contain the parameter itself in an explicit form. Besides, we can consider one of the functions, $x(t)$ or $y(t)$, to be arbitrary. Indeed, performing a change of the parameter, we get an equation of the curve l: $x = x[t(\tau)], y = y[t(\tau)]$. By virtue of the arbitrariness of the choice of the function $t(\tau)$, we can consider one of the functions, $x[t(\tau)]$ or $y[t(\tau)]$, to be an arbitrary function of τ. Consequently, we can expect that the two equations (3) will reduce to one. Let us show this.

It is known that if the function $F(x, y, \dot{x}, \dot{y})$ is homogeneous with respect to $\dot{x}$ and $\dot{y}$, then it satisfies the identity

$$F = \dot{x} \cdot F_{\dot{x}} + \dot{y} \cdot F_{\dot{y}}.$$

Differentiating, in turn, both sides of this identity with respect to the variables x, y, $\dot{x}$, and $\dot{y}$, we get

$$F_x = \dot{x} \cdot F_{x\dot{x}} + \dot{y} \cdot F_{x\dot{y}}; F_y = \dot{x} \cdot F_{y\dot{x}} + y \cdot F_{y\dot{y}}; \tag{4}$$
$$0 = \dot{x} \cdot F_{\dot{x}\dot{x}} + \dot{y} \cdot F_{\dot{x}\dot{y}}; 0 = \dot{x} \cdot F_{\dot{x}\dot{y}} + \dot{y} \cdot F_{\dot{y}\dot{y}}.$$

The last two equalities yield

$$\frac{F_{\dot{x}\dot{x}}}{\dot{y}^2} = \frac{F_{\dot{x}\dot{y}}}{-\dot{x}\dot{y}} = \frac{F_{\dot{y}\dot{y}}}{\dot{x}^2} = F_1(x, y, \dot{x}, \dot{y}), \tag{5}$$

where F_1 denotes the common value of the three relations written above. Returning to equations (3) and writing out in it the derivatives with respect to t, we obtain

$$F_x - \dot{x} \cdot F_{x\dot{x}} - \dot{y} \cdot F_{y\dot{x}} - \ddot{x} \cdot F_{\dot{x}\dot{x}} - \ddot{y} \cdot F_{\dot{x}\dot{y}} = 0,$$
$$F_y - \dot{x} \cdot F_{x\dot{y}} - \dot{y} \cdot F_{y\dot{y}} - \ddot{x} \cdot F_{\dot{x}\dot{y}} - \ddot{y} \cdot F_{\dot{y}\dot{y}} = 0.$$

Replacing $F_{\dot{x}\dot{x}}$, $F_{\dot{x}\dot{y}}$ and $F_{\dot{y}\dot{y}}$ in these equations using (5) and F_x, F_y using (4), we reduce them to the following form:

$$\dot{y}T = 0, \; \dot{x}T = 0.$$

Here we have introduced the designations $T = F_1(x, y, \dot{x}, \dot{y})\,(\ddot{x}\dot{y} - \dot{y}\ddot{x}) + F_{x\dot{y}} - F_{y\dot{x}}$.

We assume that $\dot{x}$ and $\dot{y}$ do not vanish simultaneously and so the last two equations indeed reduce to one equation:

$$T = F_1(x, y, \dot{x}, \dot{y})\,(\dot{x}\ddot{y} - \ddot{y}\dot{x}) + F_{x\dot{y}} - F_{y\dot{x}} = 0. \tag{6}$$

Recalling the expression for the radius of curvature of a plane curve, we rewrite (6) in a final form:

$$\frac{1}{R} = \frac{F_{x\dot{y}} - F_{y\dot{x}}}{(\dot{x}^2 + \dot{y}^2)^{3/2} F_1}. \tag{7}$$

1222. Find the extremals of the functional $I = \int\limits_{t_0}^{t_1} [\sqrt{\dot{x}^2 + \dot{y}^2} + a^2(x\dot{y} - \dot{x}y)]\, dt$.

(Here a is some positive number.)

Solution. Here $F = \sqrt{\dot{x}^2 + \dot{y}^2} + a^2(x\dot{y} - \dot{x}y)$ is a first-degree positive homogeneous function with respect to $\dot{x}$ and $\dot{y}$. We have

$$F_{x\dot{y}} = a^2, F_{\dot{x}y} = -a^2, F_1 = \frac{F_{\dot{x}\dot{x}}}{\dot{y}^2} = \frac{1}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

Therefore, Eq. (7) assumes the form $\frac{1}{R} = 2a^2$. Thus we see that the the extremals are curves with a constant radius of curvature, equal to $R = \frac{1}{2a^2}$, that is, are the arcs of circles with radius $\frac{1}{2a^2}$. In particular, they are full circles if $x(t_0) = x(t_1)$ and $y(t_0) = y(t_1)$. To find the equations of these circles, we must impose certain boundary conditions on the admissible curves.

10.7. Sufficient Conditions for an Extremum of a Functional

Up till now, when considering the problem on the extreme values of the functional $I = \int_{x_0}^{x_1} F(x, y, y')\,dx$, we only sought the necessary conditions for an extremum and found it to have the form $\delta I = 0$, but we did not concern ourselves with the question whether the extremals provided a maximum or a minimum for the functional I, that is, we did not study the sufficient conditions for an extremum. It has been shown that on the family of admissible curves $y = x(\alpha) = y(x) + \alpha \cdot \eta(x)$ the functional I turns into an ordinary function of the parameter α and, consequently, the condition $I'(\alpha)|_{\alpha=0} = 0$ is necessary for the functional I to attain its extremum on the function $y(x)$. By analogy with an extremum of an ordinary function we can assume that the function $y(x)$ will provide a minimum for the functional I if the inequality $I''(\alpha)|_{\alpha=0} > 0$ is satisfied (in other words, if $\delta^2 I > 0$), and a maximum if $I''(\alpha)|_{\alpha=0} < 0$ (i.e. $\delta^2 I < 0$).

Suppose the curve $y(x)$ is an extremal of the functional I. Since

$$I'(\alpha) = \int_{x_0}^{x_1} [F_y(x, y + \alpha \cdot \eta, y' + \alpha \cdot \eta')\eta + F_{y'}(x, y + \alpha \cdot \eta, y' + \alpha \cdot \eta')\eta']\,dx,$$

it follows that

$$I''(\alpha)|_{\alpha=0} = \int_{x_0}^{x_1} [F_{yy} \cdot \eta^2(x) + 2F_{yy'} \cdot \eta(x) \cdot \eta'(x) + F_{y'y'} \cdot \eta'^2(x)]\,dx.$$

Let us denote, for brevity, $F_{yy} = P$, $F_{yy'} = Q$, $F_{y'y'} = R$. Then the expected condition for the minimum of the functional will be expressed by the inequality

$$\int_{x_0}^{x_1} [P\eta^2(x) + 2Q\eta(x)\,\eta'(x) + R\eta'^2(x)]\,dx > 0.$$

We add an auxiliary integral

$$\int_{x_0}^{x_1} [\eta^2(x)\omega'(x) + 2\eta(x)\eta'(x)\omega(x)]\,dx$$

to the left-hand side of this inequality, where $\omega(x)$ is an arbitrary function of class $C^{(1)}[x_0, x_1]$. The value of this auxiliary integral is equal to zero since

$$\int_{x_0}^{x_1} [\eta^2\omega' + 2\eta\eta'\omega]\,dx = \int_{x_0}^{x_1} \frac{d}{dx}(\eta^2\omega), \text{ and } \eta(x_0) = \eta(x_1) = 0.$$

Then we get

$$\int_{x_0}^{x_1} [(P + \omega')\eta^2 + 2(Q + \omega)\eta\eta' + R\eta'^2]\, dx > 0.$$

Let us choose the function $\omega(x)$ such that $(Q + \omega)^2 = (P + \omega')R$, that is, we shall solve Riccati's equation for $\omega(x)$. Then we get

$$\int_{x_0}^{x_1} R\left[\eta' + \frac{Q + \omega}{R}\eta\right]^2 dx > 0.$$

The last inequality is satisfied if $R > 0$, i.e. $F_{y'y'} > 0$. The condition $F_{y'y'} > 0$ is called the strengthened Legendre condition. For the maximum the Legendre condition has the form $F_{y'y'} < 0$.

It is shown in the courses of calculus of variations that the Legendre condition $F_{y'y'} > 0$, in conjunction with some other conditions, provides a weak minimum for the integral I.

Answers

Chapter 1

6. $(e-1)(e^{\pi}-1)$. **7.** 5. **8.** 244/21. **10.** $\pi a^2/2$. **11.** $112\frac{8}{105}$. **12.** $5(2\ln 2-1)/8$.
13. $(\pi+1-2\sqrt{2})/4$. **14.** $-432/169$. **15.** 1. **16.** 26.

17. $\int\limits_{-1}^{0} dy \int\limits_{-2\sqrt{1+y}}^{2\sqrt{1+y}} f(x, y)dx + \int\limits_{0}^{8} dy \int\limits_{-2\sqrt{1+y}}^{2-y} f(x, y)dx.$

18. $\int\limits_{0}^{1} dy \int\limits_{e^y}^{e} f(x, y)dx$. **19.** $\int\limits_{1}^{2} dx \int\limits_{2-x}^{\sqrt{2x-x^2}} f(x, y)dy.$

20. $\int\limits_{0}^{1} dy \int\limits_{y}^{1} f(x, y)dx$. **21.** $\int\limits_{0}^{1/2} dy \int\limits_{\sqrt{1-2y}}^{\sqrt{1-y^2}} f(x, y)dx$
$+ \int\limits_{1/2}^{1} dy \int\limits_{0}^{\sqrt{1-y^2}} f(x, y)dx$. **22.** $\int\limits_{0}^{1} dy \int\limits_{\arcsin y}^{\pi-\arcsin y} f(x, y)dx.$

26. $2\pi^3$. **27.** $0.5\pi \ln 2$. **28.** $3\pi a^4/2$. **29.** 3π. **30.** $14\pi a^3/3$. **31.** 1/2. **32.** 0.5 ln 3.
37. 1/6 sq. units. **38.** 64/3 sq. units. **39.** $2\pi - 16/3$ sq. units. **40.** 5 sq. units.
41. 125/18 sq. units. **42.** 1/2 sq. units. **43.** 27/2 sq. units. **44.** 4/3 sq. units. **45.** $8-\pi$ sq. units.
46. 5π sq. units. **47.** $2\pi - 8/3$ sq. units. **51.** $8\pi - 32\sqrt{2}/3$ cu. units. **52.** 17/5 cu. units.
53. $\pi/4$ cu. units. **54.** 88/105 cu. units. **55.** 40/3 cu. units. **56.** 32/9 cu. units. **57.** 90 cu. units.
58. 12 cu. units. **59.** 79/60 cu. units. **60.** 4 cu. units. **61.** $a^2b/3$. **66.** $\pi(5\sqrt{5}-1)/24$ sq. units.
67. $16\pi/3$ sq. units. **68.** $2\pi\sqrt{2}$ sq. units. **69.** $5/6 + (\sqrt{2}/4)\cdot\ln(3+2\sqrt{2})$ sq. units.
70. π sq. units. **71.** 32 sq. units. **72.** $40\sqrt{2}/3$ sq. units. **77.** $C(45/28;\ 279/70)$. **78.** $\bar{x} = (5/6)a$,
$\bar{y} = 0$. **79.** $\bar{x} = (3/5)a$, $\bar{y} = (3/8)a$. **80.** $\bar{x} = \bar{y} = 128a/(105\pi)$. **81.** $\bar{x} = \bar{y} = 9/20$.
82. $\bar{x} = (6/5)p$, $\bar{y} = 0$. **83.** $\bar{x} = 1$, $\bar{y} = 4/(3\pi)$. **84.** 2.4. **85.** 8/3. **86.** 4096/105. **87.** $\pi ab^3/4$.
88. $(2/3)a^4k$, where k is proportionality factor. **89.** $2\pi r^2\delta$. **90.** $ab^3/12$.
95. $abc(a^2+b^2+c^2)/3$. **96.** 1/48. **97.** 1/364. **98.** $ab^2(10b-3a)/12$. **99.** $3\pi/2$.
100. $(1/5)\pi a^5(18\sqrt{3}-97/6)$. **101.** $16\pi/3$. **102.** $4\pi r^3/3$. **103.** $8\pi(2\sqrt{2}-1)/9$.
106. $\pi/6$ cu. units. **107.** $8(3\pi-4)/9$ cu. units. **108.** $3a^4/2$. **109.** $\bar{x} = \bar{y} = 2/5$, $\bar{z} = 7/30$.
110. $\bar{x} = \bar{y} = 3$, $\bar{z} = 45/32$. **111.** $(6/5;\ 12/5;\ 8/5)$. **112.** $2a^5/3$. **120.** $(\pi/2)\ln(\lambda+\sqrt{1+\lambda^2})$.
121. $(\pi/2)\ln(\lambda+\sqrt{1+\lambda^2})$. **122.** $(\pi/2)\ln(1+\lambda)$. **123.** $\pi\alpha$. **124.** $\ln(\beta/\alpha)$.
125. $(\pi/2)\ln(1+\lambda)$. **126.** $\ln(\lambda+1)/(\mu+1)$. **127.** $\sqrt{\pi}(\sqrt{\beta}-\sqrt{\alpha})$.
128. $\pi(\sqrt{1-\lambda^2}-1)$. **137.** 1.1645. **138.** -4.5781. **139.** 2.4240. **140.** $15\sqrt{\pi}/8$. **141.** 0.1225.
142. 0.8934. **143.** ∞. **144.** $-2\pi\sqrt{3}/7$. **145.** $\pi\sqrt{3}/5$. **146.** $-4\pi\sqrt{2}$. **147.** $4\pi\sqrt{2}/5$. **154.** 1/24.

155. $3\pi/16$. **156.** 8/105. **157.** $9\pi/4096$. **158.** $\dfrac{\sqrt{\pi}}{a}\cdot\dfrac{\Gamma\left(\frac{1}{a}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{a}\right)}$.

159. $\dfrac{\pi(2n-1)!!\,a^{2n+2}}{(2n+2)!!}$. **160.** $\dfrac{5\pi}{256}$.

161. $\dfrac{\pi}{b\sin(a\pi/b)}$. **162.** $\pi/\sin a\pi$. **163.** $2\pi/\sqrt{3}$.

164. $5\pi/32$. **165.** π. **166.** $1/364$. **167.** $\dfrac{\Gamma\left(\dfrac{n}{k}\right)\cdot\Gamma(m)}{k\cdot\Gamma\left(\dfrac{n}{k}+m\right)}$.

168. $\pi/\sqrt{2}$. **169.** $\dfrac{\pi}{n\sin(\pi/n)}$. **170.** $\pi/(2\sqrt{2})$.

171. $\pi/(2\sin n\pi)$. **172.** $\dfrac{\sqrt{\pi}}{2}\cdot\dfrac{\Gamma\left(\dfrac{n+1}{2}\right)}{\Gamma\left(\dfrac{n+2}{2}\right)}$.

Chapter 2

177. 67/6. **178.** 136/3. **179.** 2152/45. **180.** 4. **181.** 12. **182.** 17.5. **183.** π. **184.** 6/35. **185.** 2. **186.** $\bar{x}=(3\ln 2-1)/3$, $\bar{y}=(16\ln 2+15)/24$. **187.** $\bar{x}=2/5$, $\bar{y}=-1/5$, $\bar{z}=1/2$. **188.** $2a^2$. **189.** $(\sqrt{a^2+b^2}/ab)\arctan(2\pi b/a)$. **190.** $\sqrt{2}[\pi\sqrt{1+4\pi^2}+0.5\ln(2\pi+\sqrt{1+4\pi^2})]$. **191.** 1/6. **195.** $U=e^{x+y}+\sin(x-y)+2y+C$. **196.** $U=x-e^{x-y}+\sin x+\sin y+C$. **197.** $U=(1/3)x^3-x^2y^2+3x+(1/3)y^3+3y+C$. **198.** $U=x^2+y^2-(3/2)x^2y^2+2xy+C$. **199.** $U=\cosh x+x\cosh y+y+C$. **200.** $U=x\arcsin x-y\arcsin y+\sqrt{1-x^2}-\sqrt{1-y^2}-(1/2)x^2\ln y+C$. **201.** π^2 (in all cases). **202.** 0 (in both cases). **205.** $\iint\limits_D \dfrac{2y(x-1)}{x^2+y^2}\,dx\,dy$. **206.** 8. **209.** 1/3 sq. units. **210.** πab. **211.** 45/2 sq. units. **212.** 25/6 sq. units. **213.** $6\pi r^2$. **217.** $\bar{x}=\bar{y}=0$, $\bar{z}=(307-15\sqrt{5})/310$. **218.** $4\pi(1+6\sqrt{3})/15$. **219.** $(125\sqrt{5}-1)/420$. **228.** 1/2. **229.** 1. **231.** 0. **232.** $12\pi a^5/5$. **233.** 0. **234.** $4\pi abc$. **235.** $6\pi a^2h$. **236.** $\pi R^2h(3R^2+2h^2)/10$. **237.** $3u$. **238.** $x+y+z$. **239.** πR^2h. **240.** 2π. **241.** πab. **242.** $-[(y-z)\mathbf{i}+(z-x)\mathbf{j}+(x-y)\mathbf{k}]$. **243.** $2(\mathbf{j}-\mathbf{k})$.

Chapter 3

271. $\dfrac{1}{11}+\dfrac{2}{102}+\dfrac{3}{1003}+\dfrac{4}{10004}+\ldots$.

272. $\dfrac{1}{11}+\dfrac{1}{1111}+\dfrac{1}{111111}+\dfrac{1}{11111111}+\ldots$. **273.** $\dfrac{10^n}{2n+5}$.

274. $\dfrac{2n-1}{2n}$. **275.** $\dfrac{2^n}{n!}$. **276.** $\dfrac{(-1)^n}{2n+1}$. **277.** 1.

278. 1. **279.** 1/12. **280.** m. **283.** Diverges. **284.** Converges. **285.** Converges. **286.** Diverges. **287.** Converges. **288.** Diverges. **289.** Converges. **290.** Diverges. **291.** Converges. **292.** Diverges.

293. Diverges. **294.** Diverges. **295.** Is conditionally convergent. **296.** Is absolutely convergent. **297.** Diverges. **298.** Is conditionally convergent. **299.** Is absolutely convergent. **300.** Diverges. **301.** Diverges (compare to the series in the preceding example). **302.** Converges. **303.** Diverges. **304.** Converges. **305.** Converges. **306.** Diverges. **307.** Converges. **308.** Is absolutely convergent. **309.** Diverges. **310.** Is absolutely convergent. **311.** Diverges. **312.** Is absolutely convergent. **313.** Is conditionally convergent. **314.** $1 + \frac{1}{9} + \frac{1}{81} + \frac{1}{729} + \ldots$.

315. $1 - \frac{4}{1!} + \frac{4^2}{2!} - \frac{4^3}{3!} + \ldots$. **323.** Diverges at points $x = 1$ and $x = 2$, converges at point $x = 3$. **324.** Diverges at point $x = 1$, converges at point $x = 2$. **325.** $0 < x < +\infty$. **326.** $1 < x < +\infty$. **327.** $-\infty < x < +\infty$. **331.** Is uniformly convergent. **332.** Yes. **333.** Yes. **334.** No, the series diverges for any value of x. **346.** $-\infty < x < +\infty$. **347.** $3 \leqslant x < 5$. **348.** $1 < x < 3$. **349.** The series converges only at point $x = 0$. **350.** The series converges at any value of x. **351.** $-1 < x < 1$. **352.** $-2 \leqslant x < 2$. **353.** $-3 < x < 3$. **354.** $-1 < x < 3$. **355.** $-1 \leqslant x \leqslant 1$. **356.** $a/(a - x)^2$. **357.** $a \ln a/(a - x) - x$. **358.** $2a/(a - x)^3$.

359. $-2x/(1 + x^2)^2$. **365.** $1 + x \ln 3 + \frac{x^2 \ln^2 3}{2!} + \frac{x^3 \ln^3 3}{3!} + \ldots (-\infty < x < +\infty)$.

366. $1 - \frac{2x}{1!} + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \ldots (-\infty < x < +\infty)$.

367. $1 - \frac{2}{2!} x^2 + \frac{2^3}{4!} x^4 - \frac{2^5}{6!} x^6 + \ldots (-\infty < x < +\infty)$.

368. $\frac{2}{2!} x^2 + \frac{2^3}{4!} x^4 + \frac{2^5}{6!} x^6 + \frac{2^7}{8!} x^8 + \ldots$ $(-\infty < x < +\infty)$. **369.** $\ln a + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \ldots$ $(-a < x \leqslant a)$. **370.** $\sqrt{a}\left[1 + \frac{x}{2a} - \frac{x^2}{(2a)^2 \cdot 2!} + \frac{1 \cdot 3 \cdot x^3}{(2a)^3 \cdot 3!} - \frac{1 \cdot 3 \cdot 5 \cdot x^4}{(2a)^4 \cdot 4!} + \ldots\right] (-a < x \leqslant a)$.

371. $1 + \frac{2}{2!} \cdot x^4 + \frac{2^3}{4!} \cdot x^8 + \frac{2^5}{6!} \cdot x^{12} + \frac{2^7}{8!} \cdot x^{16} + \ldots (-\infty < x < +\infty)$. **384.** 2.71828. **385.** 0.60653. **386.** 0.1564. **387.** 1.0453. **388.** 1.0196. **389.** 5.196. **390.** −0.0202. **391.** 0.0953. **392.** 1.0986. **393.** 2.3026. **394.** 0.4636. **395.** 3.142. **400.** 1/3. **401.** 1. **402.** 0.1996. **403.** 0.102. **412.** $2, -2, 2\sqrt{2}, -\pi/4$. **413.** $z = 13(\cos 157°23' + i \sin 157°23')$. **414.** i. **415.** 1. **416.** $2 \cos 10° (\cos 10° + i \sin 10°)$. **417.** $-46 + 9i$. **418.** $(249/1025) - (68/1025)i$. **419.** $(5/169) + (12/169)i$. **420.** $5.831 [\cos (-30°58') + i \sin (-30°58')]$. **421.** $\sqrt{2}(\cos 135° + i \sin 135°)$. **422.** $(\sqrt{2}/2) + (\sqrt{2}/2)i$. **423.** $\cos (-150°) + i \sin (-150°) = -(\sqrt{3}/2) - (1/2)i$.

424. $\cos 22°30' + i \sin 22°30' = 0.9239 + 0.3827i$;
$\cos 112°30' + i \sin 112°30' = -0.3827 + 0.9239i$;
$\cos 202°30' + i \sin 202°30' = -0.9239 - 0.3827i$;

$\cos 292°30' + i \sin 292°30' = 0.3827 - 0.9239i$.
425. $w_0 = 2(\cos 15° + i \sin 15°) = 1.9318 + 0.5176i$;
$w_1 = 2(\cos 135° + i \sin 135°) = -\sqrt{2} + i\sqrt{2}$;
$w_2 = 2(\cos 255° + i \sin 255°) = -0.5176 - 1.9318i$.
426. $\cos 4\varphi = \cos^4 \varphi - 6 \cos^2 \varphi \sin^2 \varphi + \sin^4 \varphi$, $\sin 4\varphi = 4 \cos^3 \varphi \sin \varphi - 4 \cos \varphi \sin^3 \varphi$.
428. A circle of radius R with centre at point $z = c$. **429.** (1) A set of points of the circle with the boundary $|z - c| = R$; (2) a set of points of a plane lying outside the circle $|z - c| = R$.
430. (1) A set of points of a half-plane lying on the right of the imaginary axis; (2) a set of points of a half-plane lying under the real axis. **436.** $S = 1/4 - i$. **437.** Converges. **438.** Diverges. **440.** The series converges throughout the plane. **441.** The series converges only at point $z = 1 + i$. **446.** $2e^{\pi i/6}$. **447.** $e^{-\pi i/2}$. **448.** -1. **450.** $(3/4) \sin x - (1/4) \sin 3x$.

451. $i^i = e^{-\pi/2 + 2k\pi}$ $(k \in Z)$. **456.** $f(x) = -2 \sum\limits_{m=1}^{\infty} (-1)^m \dfrac{\sin mx}{m}$.

457. $$f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{m=0}^{\infty} \frac{\cos (2m+1)\pi x}{(2m+1)^2}.$$

458. $$f(x) = \frac{2}{\pi} \sinh \pi \cdot \left[\frac{1}{2} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2+1} (\cos mx - m \sin mx)\right].$$

459. $$f(x) = \sum_{m=1}^{\infty} (-1)^m \left(\frac{12}{m^3} - \frac{2\pi^2}{m}\right) \sin mx.$$

460. (1) $f(x) = \dfrac{8}{\pi} \sum\limits_{m=0}^{\infty} \dfrac{\cos (2m+1)x}{(2m+1)^2}$; (2) $f(x) = 2 \sum\limits_{m=0}^{\infty} \dfrac{\sin 2mx}{m}$.

461. $$f(x) = \frac{4h}{\pi} \sum_{m=0}^{\infty} \frac{\sin (2m+1)x}{(2m+1)}.$$

462. $$f(x) = \frac{5\pi}{4} - \frac{10}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) + $$
$$+ \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right).$$

463. $$f(x) = 2\pi\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right) -$$
$$- \frac{8}{\pi}\left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\right).$$

464. $$\frac{\pi}{4} - \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{\cos (2m+1)x}{(2m+1)^2} + \sum_{m=1}^{\infty} (-1)^m \frac{\sin mx}{m}.$$

465. $-\frac{4}{\pi}\left[\frac{\sin x}{2^2-1}+\frac{3\sin 3x}{2^2-3^2}+\frac{5\sin 5x}{2^2-5^2}+\ldots\right].$

466. $\frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{\sin m\pi x}{m}.$

467. $\frac{1}{2}-\frac{1}{\pi^2}\sum_{m=0}^{\infty}\frac{\cos(2m+1)\pi x}{(2m+1)^2}.$

471. $F(z)=\frac{1}{\sqrt{2\pi}}\cdot\frac{4}{1-4z^2}\cos\pi z.$

472. $F(z)=\frac{2i}{\sqrt{2\pi}}\cdot\frac{ze-\sin z-z\cos z}{e(1+z^2)}.$

473. $f_c(z)=\frac{\sin z-\sin(z/2)}{z}\cdot\sqrt{\frac{2}{\pi}},$

$f_s(z)=\frac{\cos(z/2)-\cos z}{z}\cdot\sqrt{\frac{2}{\pi}}.$

Chapter 4

481. $y=\arccos e^{Cx}$. **482.** $2e^{-y}(y+1)=x^2+1$. **483.** $(1+e^x)^3\tan y=8$.
484. $2\ln|\sin y|=e^{(x-1)^2}-1$. **485.** $y^3/3+\pi/4=\arctan e^x$.
486. $\ln|\tan y|=4(1-\cos x)$. **487.** $2^x-2^y=3/32$. **488.** $y=e^{\pm 1/(2\sqrt[4]{x}+1)}$.
489. $2(x-2)=\ln^2 y$. **490.** $\sqrt{1+x^2}+\sqrt{1+y^2}=C$.
491. $(1-\sqrt{1-x^2})(1-\sqrt{1-y^2})=C\,xy$.
492. $2\sin x+\ln|\tan(y/2)|=C$. **493.** $x^2+y\sin y+\cos y=C$.
494. $y=\ln\tan(e^x+\pi/4-1)$. **495.** $y=\ln\tan(\cosh x+C)$.
496. $y=a\sin(\arcsin(x/a)+C)$; the answer can be also written as $y\sqrt{a^2-x^2}-$ $-x\sqrt{a^2-y^2}=C_1$. **497.** $x+y+2\ln x-\ln y=2$. **498.** $3\arctan x^2+2\arctan y^3=\pi/2$.
499. $x+y-2\sqrt{x}+2\sqrt{y}+2\ln|(\sqrt{x}+1)(\sqrt{y}-1)|=C$.
500. $\sqrt{2}\sin x+\sin y-\cos y=0$.
501. $\tan(y/2)=C[\tan(y/2)+1][1-\tan(x/2)]$.
502. $(3/2)\ln(y^2+4)+\arctan(y/2)=\sqrt{x^2+4x+13}-$ $-\ln(x+2+\sqrt{x^2+4x+13})+C$.
503. $\tan x\tan y=1$. **504.** $y=\arctan C(1-e^x)^5$. **505.** $y=C/x$. **506.** $A_t=A_0e^{-kt}$.
507. (1) ≈ 56.5 g; (2) ≈ 7.84 h. **508.** ≈ 18.4 min.
509. $t=2\pi\tan^2\alpha(H^{5/2}-h^{5/2})/(5\sigma\omega\sqrt{2g})$;
$T=2\pi\tan^2\alpha\,H^{5/2}/(5\sigma\omega\sqrt{2g})\approx 844$ s ≈ 14.1 min. **510.** ≈ 4.6 min. **516.** $Cx=e^{\cos(y/x)}$.
517. $y^2=Cxe^{-y/x}$. **518.** $\ln x=(y/x)[\ln(y/x)-1]+C$. **519.** $y^2=4x^2\ln Cx$.
520. $y=x\arcsin x$. **521.** $1+\sin(y/x)=Cx\cos(y/x)$.
522. $\arctan(0.5y/x)-2\ln|x|=\pi/4$. **523.** $y^2=x^2\ln Cx^2$.
524. $\arctan(y/x)=\ln C\sqrt{x^2+y^2}$. **525.** $y=-x\ln|1-\ln x|$.
526. $(y/x)\cdot\arctan(y/x)=\ln C\sqrt{x^2+y^2}$. **527.** $x^5+10x^3y^2+5xy^4=1$.
528. $16xy=(y+4x-Cx^2)^2$. **529.** $\ln|y|-\cos(3x/y)=C$.
530. $y=\pm x\sqrt{C^2x^2-1}$. **531.** $y-1=C(x-1)$.

534. $3x + 2y - 4 + 2\ln|x + y - 1| = 0$. **535.** $x^2 + xy - y^2 - x + 3y = C$.
536. $x^2 + 2xy - y^2 - 4x + 8y = C$. **537.** $x^2 - y^2 + 2xy - 4x + 8y - 6 = 0$.
541. $(1/2)x^2 + x\sin y - \cos y = C$. **542.** $xy + e^x \sin y = C$. **543.** $(1/2)x^2 y + x\sin y = C$.
544. $(1/3)x^3 + xy^2 + xy + e^y = 1$. **545.** $ye^{x^2} + x\ln y = 1$.
546. $(1 + x)\sin y + (1 - y)\sin x = C$. **547.** $x^2 \ln y + 2y(x + 1) = C$.
548. $x^3 + 3y + 3x\sin y = C$. **549.** $ye^x + (1/2)y^2 = C$. **550.** $x^2 + y^2 + 2e^x \sin y = C$.
551. $x\ln y + y^2 \cos 5x = e^2$. **552.** $x\arcsin x + \sqrt{1 - x^2} +$
$+ x^2 y + y\arctan y - (1/2)\ln(1 + y^2) + y = C$. **553.** $x^3 y - \cos x - \sin y = C$.
554. $e^{x+y} + x^3 + y^4 = 1$. **555.** $x\tan y + y\cot x = C$. **556.** $\arctan(x/y) - xy + e^y = C$.
557. $y = Cx - \ln x - 1$; $\mu = 1/x^2$. **558.** $y = x(C - \sin x)$; $\mu = 1/x^2$. **559.** $x = y(C + y)$;
$\mu = 1/y^2$. **560.** $xy - \sqrt{1 - y^2} = C$; $\mu = 1/\sqrt{1 - y^2}$.
569. $y = x(\sin x + C)$. **570.** $y = e^{-x^2}(x^2/2 + C)$. **571.** $\cos x(x + C)/(1 + \sin x)$.
572. $y = a(x - 1)/x^n$. **573.** $y = \arctan x - 1 + Ce^{-\arctan x}$.
574. $y = e^{-\arcsin x} + \arcsin x - 1$. **575.** $y = \tan(x/2)[(1/2)x + (1/4)\sin 2x + C]$.
576. $y = (1/2)x^2 \ln x$. **577.** $y = -\cos x$. **578.** $x = Cy + y^2$.
579. $y = \cos 3x[1 - (2/3)\cos 3x]$. **580.** $x = Cy^2 - 1/y$. **581.** $x = Cy^2 + y^4/2$.
582. $y^{-1/3} = Cx^{2/3} - (3/7)x^3$. **583.** $y = (x - 1)/(C - x)$.
584. $y^{-1/2} - \tan x = (\ln\cos x + C)/x$. **585.** $y^{-4} = x^3(e^x + C)$.
586. $y = e^{-x}[(1/2)e^x + 1]^2$. **587.** $y = \sec x/(x^3 + 1)$. **588.** $x = 1/[y(y + C)]$.
589. $y = \sec^2 x/(\tan x - x + C)$. **590.** $x^2 + y^2 = e^{-y}$.
594. $x = p\sin p$, $y = (p^2 - 1)\sin p + p\cos p + C$. **595.** $x = e^p + C$, $y = e^p(p - 1)$,
or $y = (x - C)[\ln(x - C) - 1]$. **596.** $x = 2(\ln p - p)$, $y = 2p - p^2 + C$.
597. $x = \ln[(\sqrt{1 + p^2} - 1)/p] + p/\sqrt{1 + p^2} + C$, $y = p/\sqrt{1 + p^2}$.
598. $x = 2p + 3p^2$, $y = 2p^3 + p^2 + C$. **599.** $x = p(1 + e^p)$, $y = 0.5p^2 + (p^2 - p + 1)e^p + C$.
600. $x = e^{2p}(2p^2 - 2p + 1)$, $y = e^{2p}(2p^3 - 3p^2 + 3p - 1.5) + C$.
601. $x = 0.5\ln^2 p + \ln p + C$, $y = p\ln p$.
604. General solution is $y = Cx + \sqrt{b^2 + a^2C^2}$;
particular solution is $\begin{cases} x = -a^2 p/\sqrt{b^2 + a^2p^2}, \\ y = b^2/\sqrt{b^2 + a^2p^2}, \end{cases}$ or $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
605. General solution is $y = Cx - 1/C$;
particular solution is $\begin{cases} x = -1/p^2, \\ y = -2/p, \end{cases}$ or $y^2 = -4x$.
606. General solution is $y = Cx + C(1 - C)$; particular
solution is $\begin{cases} x = 2p - 1, \\ y = p^2, \end{cases}$ or $y = \dfrac{1}{4}(x + 1)^2$.
607. General solution is $y = Cx + C^2 + 1$; particular
solution is $\begin{cases} x = -2p, \\ y = 1 - p^2, \end{cases}$ or $y = 1 - \dfrac{x^2}{4}$.
608. General solution is
$\begin{cases} x = C(p + 1), \\ y = Cp^2/2, \end{cases}$ or $y = \dfrac{(x - C)^2}{2C}$; particular
solutions are $y = 0$, $y = -2x$. **610.** $y = (1/48)x^4 + (1/8)x^2 + (1/32)\cos 2x$.
611. $y = x\cos x - 3\sin x + x^2 + 2x$. **612.** $y = \ln\sin x + C_1x^2 + C_2x + C_3$.
613. $y = (1/3)\sin^3 x + C_1x + C_2$. **614.** $y = -(x + 3)e^{-x} + (3/2)x^2 + 3$.

617. $y = (3x^4 - 4x^3 - 36x^2 + 72x + 8)/24$. **618.** $y = (\arcsin x)^2 + C_1 \arcsin x + C_2$.
619. $y = \pm 4[(C_1 x + a^2)^{5/2} + C_2 x + C_3]/(15C_1^2)$.
620. $y = (1 + C_1^{-2}) \ln (1 + C_1 x) - C_1^{-1} x + C_2$.
621. $y = (x^3 - 3x^2 + 6x + 4)/6$. **624.** $0.5 \ln (2y + 3) = C_1 x + C_2$. **625.** $y = e^{2x}$.
626. $\pm (x + C_2) = a \ln [(y + C_1 + \sqrt{(y + C_1)^2 - a^2})/a]$,
or $y + C_1 = \pm a \cosh (x + C_2)/a$. **627.** $\ln y = C_1 e^x + C_2 e^{-x}$.
628. $y = e^{(x + C_2)/(x + C_1)}$. **629.** $\ln [C_1(y + 1) - 1] = C_1(x + C_2)$.
630. $x = \sqrt{y} - 0.5C_1 \ln (2\sqrt{y} + C_1) + C_2$.
633. $y = C_2 e^{C_1 x}$. **634.** $y = \sqrt{y^2 + C_1^2} + C_1^2 \ln (y + \sqrt{y^2 + C_1^2}) =$
$= \pm(-y^2 + 2C_1^2 x + 3C_2)$. **635.** $y = C_2 x + C_3 \pm 4(C_1 x + a^2)^{5/2}/(15C_1^2)$.
636. $y = -\ln |1 - x|$. **637.** $y = -a \ln \cos (x/a)$.
638. $y = 1 + \ln \sec x$. **639.** $s = \dfrac{m^2 g}{k^2}\left(e^{-kt/m} - 1\right) + \dfrac{mgt}{k}$.
642. $y = C_2 + (C_1 - C_2 x) \cot x$. **643.** $y = (1/2)x \ln^2 x + C_1 x \ln x + C_2 x$.
644. $y = C_1 \sin x + C_2 \sin^2 x$. **647.** Yes. **648.** Yes. **649.** No. **650.** Yes. **651.** Yes. **652.** No.
660. $y = C_1 e^{2x} + C_2 e^{-x}$. **661.** $y = C_1 \cos 5x + C_2 \sin 5x$. **662.** $y = C_1 + C_2 e^x$.
663. $y = (C_1 + C_2 x)e^{2x}$. **664.** $y = C_1 + C_2 x + C_3 e^x + C_4 x e^x$.
665. $y = (C_1 e^{xa\sqrt{2}/2} + C_2 e^{-xa\sqrt{2}/2}) \cos (xa\sqrt{2}/2) + (C_3 e^{xa\sqrt{2}/2} + C_4 e^{-xa\sqrt{2}/2}) \sin (xa\sqrt{2}/2)$.
666. $y = C_1 \cos x + C_2 \sin x + C_3 \cos 2x + C_4 \sin 2x$. **667.** $y = 4e^{-3x} - 3e^{-2x}$. **668.** $y =$
$= xe^{5x}$. **669.** $y = -(1/3)e^x \cos 3x$. **670.** $y = 2 \sin (x/3)$. **671.** $y = (5 - 2e^{-3x})/3$.
672. $y = \sqrt{2} \sin 3x$. **673.** $y = \sin x + (1/\sqrt{3}) \cos x$. **674.** $x = C_1 \cos \beta t + C_2 \sin \beta t$,
or $x = A \sin (\varphi_0 + \beta t)$, $\beta = \sqrt{a/m}$. **684.** $y = (e^{5x} + 22e^{3x} + e^x)/8$.
685. $y = 0.5x(x + 2)e^{4x}$. **686.** $y = e^{3x}(C_1 \cos 4x + C_2 \sin 4x) + (14 \cos x + 5 \sin x)/102$.
687. $y = -(11/8)\cos x + 4\sin x - (1/8)\cos 3x$. **688.** $y = C_1 e^{4x} + C_2 e^{2x} + (24x^2 + 52x + 41)/64$.
689. $y = 4e^{x/2} - x - 4$. **690.** $y = (1/8) \sin 2x - (1/4)(x \cos 2x - 1)$.
691. $y = C_1 + C_2 e^{4x} - (1/6)\cdot (2 \cosh 2x + \sinh 2x)$. **692.** $y = (1/16)(4x - \pi) \sin 2x$.
693. $y = C_1 e^{2x} + C_2 e^{-5x} + (1/144)(1 - 12x)e^{-2x}$.
694. $y = C_1 e^{\alpha x} + C_2 e^{\beta x} + x(ae^{\alpha x} - be^{\beta x})/(\alpha - \beta)$.
695. $y = C_1 e^x + C_2 e^{-x} - (1/2)x - (1/10)x \cos 2x + (2/25) \sin 2x$.
696. $y = C_1 e^{5x} + C_2 e^{4x} - (x^3/3 + x^2 + 2x)e^{4x}$. **697.** $y = x \cosh x$.
698. $y = C_1 e^{2x} + C_2 e^{-2x} + (1/4)x \sinh 2x$.
699. $y = e^{x \cos \varphi}[C_1 \cos (x \sin \varphi) + C_2 \sin (x \sin \varphi)] + \cos x$.
700. $y = e^x(C_1 \cos x + C_2 \sin x) - 0.5xe^x \cos x$.
701. $y = (1/8) \cos x - (1/8) \cos 3x - (1/6)x \sin 3x + (\pi/12) \sin 3x$.
704. $m\ddot{x} + ax = A \sin \omega t$, $x = C_1 \cos \beta t + C_2 \sin \beta t + [A/(a - m\omega^2)] \sin \omega t$,
if $\omega \neq \beta = \sqrt{a/m}$, and $x = C_1 \cos \beta t + C_2 \sin \beta t - [At/(2\beta m)] \cdot \cos \beta t$,
if $\omega = \beta = \sqrt{a/m}$. **705.** $y = C_1 \cos x + C_2 \sin x + (1/\sqrt{2}) \cos x \ln |\cos x +$
$+ \sqrt{\cos^2 x - 1/2}| + (1/\sqrt{2}) \sin x \arcsin (\sqrt{2} \sin x)$.
706. $y = C_1 e^{-2x} + C_2 e^{-3x} + (1/2)e^{-2x} \ln (1 + e^{2x}) - e^{-2x} + e^{-3x} \arctan e^x$.
707. $y = C_1 \cos 2x + C_2 \sin 2x + (1/4) \sin 2x \ln \tan 2x$.
708. $y = C_1 \cos (x/2) + C_2 \sin (x/2) + 2x \sin (x/2) + 4 \cos (x/2) \ln \cos (x/2)$.
709. $t = (3/\sqrt{g}) \ln (17 + 12\sqrt{2})s$. **713.** $y = x(C_1 \cos \ln x + C_2 \sin \ln x)$.
714. $y = C_1 x + C_2 x^3 + (1/9) \cdot (9 \ln^2 x + 24 \ln x + 26)$.
715. $y = C_1 \cos \ln x + C_2 \sin \ln x - (1/3) \sin (2 \ln x)$.
716. $y = (\ln^2 x + 2 \ln x + 2)/(2x)$. **717.** $y = (1/2)x^3 - (1/\ln 2)x^2 \ln x$.

719. $y = C_0\left(1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \ldots\right) =$
$= C_0 e^{-x^2/2}$ (solution exists throughout the number axis).

720. $y = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{4 \cdot n!} = \frac{1}{4} e^{-2x} - \frac{1}{4} + \frac{x}{2}$
(solution exists throughout the number axis).

721. $y = C_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2 \cdot 4 \cdot 6 \ldots 2n} + C_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{1 \cdot 3 \cdot 5 \ldots (2n+1)}$
(solution exists throughout the number axis).

722. $y = C_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{1 \cdot 3 \ldots (2n-1)} + C_1 \sum_{n=1}^{\infty} \frac{x^{2n+1}}{2 \cdot 4 \ldots 2n}$
(solution exists throughout the number axis).

723. $y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4 \cdot 5 \cdot 8 \cdot 9 \ldots 4n(4n+1)}$
(solution exists throughout the number axis).

726. $y = 1 + \frac{x}{1!} + \frac{3x^2}{2!} + \frac{17x^3}{3!} + \ldots$.

727. $y = \frac{x^2}{2!} + \frac{12x^5}{5!} + \ldots$.

728. $y = 1 + \frac{x}{1!} + \frac{x^3}{3!} + \frac{4x^4}{4!} + \ldots$.

729. $y = 4\left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right) +$
$+ 2(x-1) = 4e^{-x} + (2x - 1)$.

730. $y = 1 + x + \frac{3x^2}{2!} + \frac{8x^3}{3!} + \frac{34x^4}{4!} + \ldots$.

731. $y = x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{5}{24}x^4 - \frac{1}{24}x^5 - \ldots$ (for $x = 0$ successive derivatives are connected by the recurrent relation $y_0^{(n+2)} = -y_0^{(n)} + 2ny_0^{(n-1)}$).

734. $J_1(x) = \frac{x}{2}\left(1 - \frac{x^2}{2 \cdot 4} + \frac{x^4}{2 \cdot 4 \cdot 4 \cdot 6} - \ldots\right)$.

735. $y = \frac{2}{3\sqrt{x}}\left[C_1 x^{3/2}\left(1 - \frac{x^2}{2 \cdot 5} + \frac{x^4}{2 \cdot 4 \cdot 5 \cdot 7} - \ldots\right) + \right.$
$\left. + C_2 x^{-3/2}\left(1 + \frac{x^2}{2 \cdot 1} - \frac{x^4}{2 \cdot 4 \cdot 1} + \ldots\right)\right]$.

736. $y = C_1 \cdot \left(\frac{x}{2}\right)^{2/3} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k}}{k!\,\Gamma\left(\frac{2}{3} + k + 1\right)} +$

$$+ C_2\left(\frac{x}{2}\right)^{-2/3} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k}}{k!\Gamma\left(\frac{2}{3}+k+1\right)}.$$

740. $x = 2e^{3t} - e^t, y = 2e^{3t} + e^t$. **741.** $x = C_1 + C_2e^{7t} - (3/49)t(7t + 2)$, $y = -(2/3)C_1 + (1/2)C_2e^{7t} + (1/49)(14t^2 - 3t - 1)$.
742. $x = C_1e^t + C_2e^{-t} + (1/8)e^{3t}, y = -C_1e^t + C_2e^{-t} + (5/8)e^{3t}$.
743. $y = C_1 \cos x + C_2 \sin x + \sinh x, z = C_1 \sin x - C_2 \cos x + \sinh x$. **744.** $x = (1/4)(3e^t + 5e^{-t}) + (1/2)te^t - 1, y = (5/4)(e^t - e^{-t}) + (1/2)te^t - t$.
745. $x = (1/3)t + 2, y = (2/3)t + 4$. **746.** $x = (C_1 + C_2t)e^t + (1/2) \cos t$, $y = [C_2(1 - t) - C_1]e^t - 2 \cos t - (1/2) \sin t$.
747. $x = C_1e^{2t} + C_2e^{-2t}, y = 3C_1e^{2t} - C_2e^{-2t}$.
748. $x = t/3 + C_2/t^2, y = C_1e^t - t/3 - 2C_2/t^3$.
749. $x = -[(1 + C_1)t + C_2]^{-1}, y = -C_1[(1 + C_1)t + C_2]^{-1}$.
750. $x = C_1e^{-6t} + e^{2t} - (3/7)e^t, y = C_2e^{-t} - (4/5)C_1e^{-6t} + (9/14)e^t$. **751.** $x = e^{tm\sqrt{2}/2}[C_1 \cos(tm\sqrt{2}/2) + C_2 \sin(tm\sqrt{2}/2)] + e^{-tm\sqrt{2}/2}[C_3 \cos(tm\sqrt{2}/2) + C_4 \sin(tm\sqrt{2}/2)], y = e^{tm\sqrt{2}/2}[C_1 \sin(tm\sqrt{2}/2) - C_2 \cos(tm\sqrt{2}/2)] + e^{-tm\sqrt{2}/2}[C_4 \cos(tm\sqrt{2}/2) - C_3 \sin(tm\sqrt{2}/2)]$. **752.** $x^2 = C_1^2(2t + C_1)/(1 + C_2^2 + C_3^2)$, $y^2 = C_2^2(2t + C_1)/(1 + C_2^2 + C_3^2), z^2 = C_3^2(2t + C_1)/(1 + C_2^2 + C_3^2)$.
758. $x_1 = C_1e^{at} + C_2e^{-at}, x_2 = C_2e^{-at} - C_1e^{at}$.
759. $x_1 = 2C_1e^{3t} - 4C_2e^{-3t}, x_2 = C_1e^{3t} - C_2e^{-3t}$.
760. $x_1 = C_1e^{-t} + C_2e^{2t} + C_3e^{-2t}, x_2 = C_1e^{-t} + C_2e^{2t} - C_3e^{-2t}$, $x_3 = -C_1e^{-t} + 2C_2e^{2t}$. **761.** $x_1 = C_1e^t + C_2e^{2t} + C_3e^{-t}$, $x_2 = C_1e^t - 3C_3e^{-t}, x_3 = C_1e^t + C_2e^{2t} - 5C_3e^{-t}$.
762. $x_1 = e^{12t}(C_1 \cos 5t + C_2 \sin 5t), x_2 = e^{12t}(-C_1 \sin 5t + C_2 \cos 5t)$.
763. $x_1 = 2C_1 \cos t + 2C_2 \sin t, x_2 = (C_1 - C_2) \cos t + (C_1 + C_2) \sin t$. **764.** $x_1 = 2C_1e^{-t} + 2(4C_3 + C_2) \cos t - 2(4C_2 - C_3) \sin t, x_2 = -2C_1e^{-t} + 3(5C_2 + 3C_3) \cos t + 3(5C_3 - 3C_2) \sin t, x_3 = C_1e^{-t} + (7C_2 + 11C_3) \cos t + (7C_3 - 11C_2) \sin t$. **765.** $x = e^{at}(C_1t + C_2)$, $y = e^{at}(C_1t + C_2 - C_1)$. **766.** $x = -2e^t(C_1 \sin 2t - C_2 \cos 2t)$, $y = e^t(C_1 \cos 2t + C_2 \sin 2t)$. **767.** $x = e^t(C_1t + C_2), y = e^t(C_1 - 2C_2 - 2C_1t)$.

Chapter 5

773. $P(A) = 0$ (impossible event). **774.** 1/4. **775.** (1) 1; (2) 1/5; (3) 3/5. **776.** 499/1998. **777.** 1/406. **786.** (1) $a/(a + b + c)$; (2) $b/(a + b + c)$; (3) $c/(a + b + c)$; (4) $(a + b)/(a + b + c)$; (5) $(a + c)/(a + b + c)$; (6) $(b + c)/(a + b + c)$. **787.** $bd(a + b)^{-1}(c + d)^{-1}$. **788.** $p_1 + p - 2p_1p_2$. **789.** $1 - 3\alpha$. **790.** 1/3. **791.** ≈ 0.88. **792.** (1) 22/145; (2) 51/145; (3) 72/145. **793.** 0.7. **794.** 0.375. **799.** 7/64. **800.** 21/32. **801.** 4/9. **802.** 27/128. **806.** 15. **808.** No, the problem always has a solution since $(m_0 + q)/p - (m_0 - p)/p = (p + q)/p = 1/p > 1$. **808.** The first worker manufactured 114 articles, the second, 112 articles. **810.** 60. **815.** 1/3.

822.

x_i	0	1	2	3
p_i	0.343	0.441	0.189	0.027

823.

x_i	3	4	5	6	7
p_i	1/6	1/6	1/3	1/6	1/6

824. (1) $a = 1/\pi$; (2) $P(a/2 < X < a) = 1/3$. **825.** $P(\pi < X < \infty) = 1/4$.

826. $a = 5$; $F(x) = \begin{cases} 0, & \text{if } x < 0; \\ 0.5(1 - \cos x), & \text{if } 0 \leqslant x \leqslant \pi; \\ 1, & \text{if } x > \pi. \end{cases}$

827. $F(x) = \begin{cases} 0, & \text{if } x < 0; \\ 5/6, & \text{if } 0 \leqslant x \leqslant 1; \\ 1, & \text{if } x > 1. \end{cases}$

830.

x_i	0	1	2	3	4	5
p_i	0.010	0.077	0.230	0.346	0.259	0.078

$M(X) = 3.00$; $D(X) = 1.20$

831. $\lambda = 1/4$; $M(X) = 16/15$; $\sigma_x = \sqrt{44/225} = 0.44$. **834.** $\bar{M} = 20$. **835.** $a = 0.75$; $\bar{M} = \mu = 3$. **838.** $P(3 < X < 5) = (5 - 3) \cdot (1/6) = 1/3$. **839.** 2/5. **842.** 0.000055. **843.** $M(X = m/n) = p$; $D(X) = pq/n$. **852.** 4.4%; the result obtained does not depend on the numerical value of m. **853.** 0.34; 0.14; 0.02. **857.** $\alpha_1 = 4$; $\alpha_2 = 20$; $\alpha_3 = 116.8$; $\alpha_4 = 752$; $\mu_1 = 0$; $\mu_2 = 4$; $\mu_3 = 4.8$; $\mu_4 = 35.2$; $S_k = 0{,}6$; $E_x = -0.8$. **858.** $\alpha_1 = 1$; $\alpha_2 = 7/6$; $\alpha_3 = 3/2$; $\alpha_4 = 31/15$; $\mu_1 = 0$; $\mu_2 = 1/6$; $\mu_3 = 0$; $\mu_4 = 1/15$; $S_k = 0$; $E_x = -0.6$. **859.** $\lambda = 1/2$; $E_x = 3$.

864. $P\left(\left|\dfrac{m}{10\,000} - \dfrac{1}{6}\right| < 0.01\right) \geqslant \dfrac{31}{36}$.

865. $P\left(\left|\dfrac{m}{50} - \dfrac{1}{2}\right| < \dfrac{1}{5}\right) \geqslant \dfrac{7}{8}$. **866.** 3/4. **869.** 0.954. **870.** 61.

878. (1) $\lambda = 1/20$; (2) $m_x = 22$, $m_y = 41$; (3) $\sigma_x^2 = 56$, $\sigma_y^2 = 259$; (4) $r_{xy} = 0.56$. **879.** (1) $a = 24$; (2) $m_x = m_y = 2/5$; (3) $\sigma_x^2 = \sigma_y^2 = 1/25$; (4) $r_{xy} = -2/3$. **880.** (1) $a = \sqrt[4]{2/\pi}$; (2) $m_x = m_y = 0$; (3) $\sigma_x^2 = \sigma_y^2 = 1/(3\sqrt{2\pi})$; (4) $r_{xy} = 0$. **883.** $r_{xy} = 0.664$; $\bar{y}_x = 3.64x - 0.15$; $\bar{x}_y = 0.12y + 1.24$. **884.** $r_{xy} = 0.321$; $\bar{y}_x = 1.21x - 2.45$; $\bar{x}_y = 0.085y + 10.58$. **891.** $\bar{x} = 10.005$; $D(X) = 0.010475$; $\sigma(X) = 0.1023$. **892.** $\bar{y} = 10.64$; $D(Y) = 34.97$. **896.** $\alpha_1 = 6.32$; $\alpha_2 = 44.64$; $\alpha_3 = 340.16$; $\alpha_4 = 2743.68$; $\mu_1 = 0$; $\mu_2 = 4.6976$; $\mu_3 = -1.3425$; $\mu_4 = 56.422$; $S_k(X) = -0.132$; $E(X) = -0.442$. **898.** $M(X) = 4.13$; $D(X) = 9.07$;

$$f(X) = \begin{cases} 0; & \text{if } x < -1.09; \\ 0.096, & \text{if } -1.09 \leqslant x \leqslant 9.35; \\ 0, & \text{if } x > 9.35. \end{cases}$$

900. $M(x) = 5.06$; $D(X) = 5.01$. **902.** $M(X) = 8.02$, $D(X) = 8.23$, $\sigma(X) \approx 2.87$, $f(x) = 1/(2.87\sqrt{2\pi})e^{-(x-8.02)^2/(2\cdot 2.87^2)}$.

Chapter 6

914. $z = xy + \varphi(x) + \psi(y)$. **915.** $z = x\varphi_1(y) + \varphi_2(y) + y\varphi_3(x) + \varphi_4(x)$. **919.** $\tan(z/2) = \tan(x/2) \times \psi\left(\dfrac{\tan(y/2)}{\tan(x/2)}\right)$. **920.** $z^2 = x^2 + \psi(y^2 - x^2)$. **921.** Paraboloid of revolution $z = x^2 + y^2$. **925.** $\dfrac{\partial^2 z}{\partial \eta^2} = 0$, $\xi = y/x$, $\eta = y$. **926.** $\dfrac{\partial^2 z}{\partial\xi\partial\eta} - \dfrac{\partial z}{\partial \xi} = 0$, $\xi = x+y$, $\eta = 3x+y$.

927. $\dfrac{\partial^2 z}{\partial \xi^2} + \dfrac{\partial^2 z}{\partial \eta^2} + \dfrac{1}{2}\left(\dfrac{1}{\xi}\cdot\dfrac{\partial z}{\partial \xi} + \dfrac{1}{\eta}\cdot\dfrac{\partial z}{\partial \eta}\right) = 0$,

$\xi = y^2$, $\eta = x^2$. **931.** $u = x(1 - t)$. **932.** $u = (\cos x \sin at)/a$. **933.** $u = -\sin x$. **937.** $u(x, t) =$

$$= -(0.9/\pi^2). \sum_{k=1}^{\infty} (1/k^2)\sin(2\pi k/3) \quad \sin(k\pi x/3)\cdot\cos(k\pi at/3).$$

938. $u =$

$$= (96h/\pi^5)\sum_{k=0}^{\infty} 1/[(2k+1)^5]\cos(2k+1)\pi at \quad \sin(2k+1)\pi x.$$

939. $u(x, t) = \dfrac{4hl^2}{\pi^2 a} \times$

$$\times \sum_{k=1}^{\infty} \frac{1}{k^2}\,\frac{\sin(\pi k/2)\cos(k\pi h/l)}{l^2 - k^2h^2}\sin(k\pi x/l)\sin(k\pi at/l).$$

943.
$$u(x, t) = \frac{1}{2}\left[\left(1 + \frac{x}{l}\right)\Phi\left(\frac{x+l}{2\sqrt{t}}\right) - 2\cdot\frac{x}{l}\cdot\Phi\left(\frac{x}{2\sqrt{t}}\right) - \left(1 - \frac{x}{l}\right)\times \Phi\left(\frac{x-l}{2\sqrt{t}}\right)\right] + \frac{1}{l}\sqrt{\frac{t}{\pi}}\left[e^{-(x+l)^2/(4t)} - 2e^{-x^2/(4t)} + e^{-(x-l)^2/(4t)}\right].$$

944.
$$u(x, t) = \frac{u_0}{2}\cdot\left[\Phi\left(\frac{x+l}{2a\sqrt{t}}\right) - \Phi\left(\frac{x-l}{2a\sqrt{t}}\right)\right].$$

945.
$$u(x, t) = \frac{8c}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^3}e^{-(2n+1)^2\pi^2a^2t/l^2}\cdot\sin\frac{(2n+1)\pi x}{l}.$$

947. $u = u_a + (u_b - u_a)\cdot[\ln(r/a) : \ln(b/a)]$. **949.** $u(r, \Theta) = (8/3)\sinh(\ln r)\cdot\sin\Theta$.

Chapter 7

956. (1) $w = i$; (2) $w = -e^{\pi}$; (3) $w = ei$. **957.** $(1+i)/2$; i; $(3-2i)/13$. **959.** $(1/2)\ln 2 + (2k\pi - \pi/4)i$, $k \in Z$. **961.** $z = \pm i\ln(2+\sqrt{3})$. **962.** $i\ln(1\pm\sqrt{2})$. **963.** $1.1752i$. **964.** $0.772 + 1.018i$. **965.** (1) $e^{\cos 1}[\cos(\sin 1) + i\sin(\sin 1)]$; (2) $\cos e + i\sin e$. **971.** No. **972.** $f'(z) = 3z^2$. **973.** $f'(z) = \cos z$. **974.** $\varphi(y) = ay + C_1$, $\psi(x) = -ax + C_2$, $f(z) = Az + C$, $A = -ai$, $C = C_1 + C_2$. **975.** $\lambda = -1$, $f(z) = -iz$. **976.** $a = 0$. **977.** $f(z) = z^2 + C$. **978.** $f(z) = -\cos z + C$. **985.** (a) $u = 4 - v^2/16$, $u = v^2/4 - 1$; (b) $v = (u^2-1)/2$; (c) $u = 1$, $v = 0$. **986.** $u = x\cos\varphi - y\sin\varphi$; $v = x\sin\varphi + y\cos\varphi$ is the transformation of coordinates upon rotation of the axes. **987.** $u = (v/2)^{2/3} - (v/2)^{4/3}$. **989.** (a) $\alpha = -\dfrac{\pi}{2}$, $k = 6$; (b) $\alpha = 0$, $k = \dfrac{1}{4}$; (c) $\alpha = 0$, $k = e$. **990.** (a) $|z| = \dfrac{1}{2}$; (b) $|z-1| = \dfrac{1}{2}$. **991.** (a) $\operatorname{Re} z = 0$; (b) $\arg z = -\dfrac{\pi}{2}$. **999.** (a) $1+i$; (b) $-(1+i)/3$; (c) 0; (d) 0. **1000.** $2\pi i$. **1001.** $2\pi i(a+b)$. **1010.** The domain of convergence is $1 < |z| < 2$. **1011.** The series diverges at all points of the plane. **1012.** (1) $f(z) = -z^2 - z^3 - z^4 - \ldots$; (2) $f(z) = z + 1 + \dfrac{1}{z} + \dfrac{1}{z^2} + \ldots$. **1013.** $\dfrac{1}{3!z^2} + \dfrac{1}{5!} + \dfrac{z^2}{7!} + \dfrac{z^4}{9!} + \ldots$. **1014.** ± 1 are first-order poles; $\pm i$ are second-order poles. **1015.** $f(z) = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \ldots$; the domain of convergence is $|z-1| < 1$. **1016.** $f(z) = + \dfrac{1}{2!} + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \ldots$; the series converges throughout the plane. **1017.** $f(z) = \sum\limits_{n=1}^{\infty} \times \times \dfrac{(\ln 2)^n}{n}(z^n + z^{-n})$. **1027.** $2i$. **1028.** 1; -1. **1029.** -1. **1030.** 1. **1031.** $2\pi a^2$. **1032.** $2\pi i$. **1033.** 0. **1034.** $2\pi/(3-i)$. **1035.** $3\pi/8$.

Chapter 8

1041. $\bar{f}(p) \dfrac{2}{p(p^2+4)}$. **1042.** $\bar{f}(p) = \dfrac{p(p^2+2p+3)}{(p-1)(p^2-2p+5)}$. **1043.** $\bar{f}(p) = \dfrac{p}{p^2-b^2}$.

1044. $\bar{f}(p) = \dfrac{a(p^2-a^2-b^2)}{p[(p-a)^2+b^2][(p+a)^2+b^2]}$.

1045. $\bar{f}(p) = -\dfrac{b(p^2+a^2-b^2)}{[(p-a)^2+b^2][(p+a)^2+b^2]}$.

1046. $\bar{f}(p) = -\dfrac{p(p^2-a^2+b^2)}{[(p-a)^2+b^2][(p+a)^2+b^2]}$.

1047. $\bar{f}(p) = -\dfrac{2pb}{(p^2-b^2)^2}$. **1053.** $f(t) = 1/4 - (1/3)\cos t + (1/12)\cos 2t$.

1054. $f(t) = -(1/3)e^t + (1/4)e^{2t} + (1/12)e^{-2t}$. **1055.** $f(t) = 1 - 2e^t + e^{3t}$. **1056.** $f(t) = 1/4 -$

$-$ (1/3)cosh t + (1/2)cosh $2t$. **1057.** $f(t) = \dfrac{t^k}{k!} - \dfrac{a^k \cdot t^{2k}}{(2k!)} + \dfrac{a^{2k} \cdot t^{3k}}{(3k!)} - \dots$.

1061. $1 - \cos t \div \dfrac{1}{p^2} \cdot \dfrac{p}{p^2+1} = \dfrac{1}{p(p^2+1)}$. **1062.** $f(t) = \int\limits_0^t \cos(t - \tau)\cos\tau\, d\tau =$

$= \dfrac{1}{2}(\sin t + t\cos t)$. **1063.** $(1 - 2p) \cdot \bar{y}(p)$. **1064.** $(p^3 - p^2 + 2p - 2) \cdot \bar{y}(p) - p - 1$.
1065. $\dfrac{\bar{y}(p)}{p} \cdot (p^2 - 1)$. **1072.** $y = e^{2t}$. **1073.** $y = \sinh t$. **1074.** $y = 0$. **1075.** $y = (1/3)te^t -$
$- (7/9)e^t - (2/9)e^{-2t}$. **1076.** $y = -(5/2)e^t + 4e^{2t} - (3/2)e^{3t}$. **1077.** $x = (5/2)e^{2t} - (1/2)e^{-2t}$, $y = (5/2)e^{2t} - (1/2)e^{-2t}$. **1078.** $x = (6/5)e^{5t} - (1/5)e^{-5t}, y = (3/5)e^{5t} + (2/5)e^{-5t}$. **1079.** $y(t) =$
$= 1$. **1080.** $y(t) = t$. **1083.** $2e^t - 4t - 3$. **1084.** $-1/6 + (1/2)e^t - (1/2)e^{2t} + (1/6)e^{3t}$. **1085.** $(1/8) \times$
$\times (2t^2 - 6t + 3)e^t - (1/24)e^{-t} + (2/3) \times \sin(t\sqrt{3}/2 + \pi/6)$. **1090.** $u(x, t) = A\cos(n\pi at/l) \times$
$\times \cos(n\pi x/l)$. **1091.** $u(x, t) = B\sin(n\pi at/l)\sin(n\pi at/l)$. **1092.** $u(x, t) = A\,\mathrm{Erf}(\alpha x/2\sqrt{t}))$.

Chapter 9

1098. (0, 1), (2, 3), (6, 7). **1099.** $(-4, -3)$, (0, 1), (3, 4). **1100.** 1.94. **1101.** 2.09. **1102.** 0.33; 1.30. **1103.** -1.15. **1104.** 1.11. **1105.** 0.42. **1106.** 3.62. **1107.** -0.56. **1108.** 1.27. **1114.** $\xi =$ $= 1.70997$. **1115.** $\xi = 1.23429$. **1118.** 2.214. **1119.** 1.37973. **1120.** -1.4142. **1123.** $y = -(2x^3 -$ $- 15x^2 + 25x - 9)/3$. **1124.** $y = 0.2(x^3 - 13x^2 - 69x - 92)$. **1125.** $y = 2x - 1$.

1128.

x	6.5	6.6	6.7	6.8	6.9	7.0
$\log x$	0.8129	0.8195	0.8261	0.8325	0.8388	0.8451

1129. 39.0625. **1130.** $f(x) = x^3 + x^2 + x + 1$. **1135.** 0.5000. **1136.** 1.16912; $\delta_S = -0.000004$; the exact value of the integral is $4(\sqrt{2} - 1) - 2\ln[(2\sqrt{2} + 1)/3] \approx 1.16912 \dots$. **1137.** $|\delta_t| \leqslant$ $\leqslant (h^2/12) \cdot (b - a)M_1 \approx 0.007$ (M_1 is the greatest value of $|f''(x)|$ in the integration interval). Therefore, the calculation must be carried out with three decimal digits (to obtain two correct digits): $I \approx 1.35$. **1138.** 0.69. **1139.** 0.24. **1140.** 0.75. **1141.** 0.67. **1147.** 183; 552. **1153.** The exact value is $I = 62.572$; (1) $I = 62.673$; $\delta = 0.12\%$; (2) $I = 62.730$; $\delta = 0.03\%$, (3) $I = 66.509$; $\delta = 5.99\%$. **1154.** The exact value is $I = 0.747$; (1) $I = 0.746$; $\delta = 0.13\%$; (2) $I = 0.800$; $\delta = 7.1\%$.

1158.

x	0	0.1	0.2	0.3
y	1	1.2	1.45	1.78

1159.

x	0.1	0.2	0.3	0.4
y	0	0.001	0.005	0.014

1160.

x	2	2.1	2.2	2.3	2.4
y	4	5.8	9.44	18.78	54.86

1161.

x	0	0.2	0.4	0.6	0.8	1.0	1.2
y	1	1.1	1.18	1.24	1.27	1.27	1.24

1162.

t	1	1.2	1.4	1.6	1.8	2.0
x	1	1	1.07	1.17	1.30	1.45
y	1	1.4	1.8	2.21	2.63	3.06

1165.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	-1	-0.975	-0.949	-0.921	-0.888	-0.842	-0.802	-0.744	-0.675	-0.593	-0.495

1166.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0.
y	1	1.05	1.12	1.20	1.29	1.39	1.50	1.62	1.75	1.89	2.03

1168. 1.78. **1169.** 0.02. **1172.** $y_1 = (1/3)x^3$, $y_2 = (1/3)x^3 + (1/63)x^7$, $y_3 = (1/3)x^3 + (1/63)x^7 + (2/2079)x^{11} + (1/59535)x^{15}$. **1173.** $y = e^{-\sinh x}$. **1174.** $y_n = 1 - x + 2\left[\frac{x^2}{2!} - \frac{x^3}{3!} + \ldots + (-1)^n \cdot \frac{x^n}{n!}\right] + (-1)^{n+1} \cdot \frac{x^{n+1}}{(n+1)!}$ ($n = 3, 4, 5, \ldots$); a two-sided sequence. **1175.** $y_n(x) = \sum_{m=0}^{n} \frac{\sin^m x}{m!}$; the true solution is $y(x) = e^{\sin x}$; the sequence of lower functions. **1176.** $y_n(x) = \sum_{m=0}^{n} \frac{\sin^m(x^2)}{m!}$; the true solution is $y(x) = e^{\sin(x^2)}$; the sequence of lower functions. **1179.** $y = 0.279x + 71.14$. **1180.** $S = 11.58e^{0.2898t}$. **1182.** $y = 111.7 + 1.663x + 0.00437x^2$. **1183.** $S = 33.02t^{1.065}$. **1187.** (1) $y = 3.023x - 1.08$; (2) $y = 0.992x - 0.909$; (3) $y = -1.802x + 2.958$. **1188.** (1) $y = -0.145x^2 + 3.324x - 12.794$; (2) $y = 1.009x^2 - 4.043y + 5.045$; (3) $y = -0.102x^2 + 0.200x + 0.806$. **1189.** $S = 5.7t^{1.97}$. **1190.** (1) $S = 92e^{-0.15t}$; (2) $S = 0.49e^{0.44t}$. **1192.** $\varphi(x) = -0.2723x^2 + 0.5003x + 1.3424$. **1193.** $\varphi(x) = 0.670x^3 - 0.728x^2 - 0.350x + 0.943$.

Chapter 10

1206. No solution. **1207.** $I = x_1e^{y_1} - x_0e^{y_0}$. The intergral is independent of the integration path. **1208.** $y = x$. **1209.** $y = \frac{1}{4}x^2 - x + 1$. **1210.** $y = \sqrt{8 + 6x - x^2}$. **1211.** $y = 0$. **1212.** $y = \sqrt{2}e^{x/2} \sin^{x/2}$. **1214.** $y(x) \equiv 0$. **1215.** $y = (1 - x)\sinh x$. **1217.** $y = \frac{1}{x}, z = \frac{2}{x^3} - 1$. **1218.** $y = \frac{4}{3x^3} - \frac{1}{3}$, $z = \frac{1}{x}$. **1220.** $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$. **1221.** $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \varphi(x, y)$.

Appendix

Table 1

The Value of the Gamma-Function $\Gamma(p)$ (for $1 \leqslant p \leqslant 2$)

p	$\Gamma(p)$	p	$\Gamma(p)$	p	$\Gamma(p)$	p	$\Gamma(p)$
1.00	1.0000	1.25	0.9064	1.50	0.8862	1.75	0.9191
1.01	0.9943	1.26	9044	1.51	8866	1.76	9214
1.02	9888	1.27	9025	1.52	8870	1.77	9238
1.03	9835	1.28	9007	1.53	8876	1.78	9262
1.04	9784	1.29	8990	1.54	8882	1.79	9288
1.05	9735	1.30	8975	1.55	8889	1.80	9314
1.06	9687	1.31	8960	1.56	8896	1.81	9341
1.07	9642	1.32	8946	1.57	8905	1.82	9368
1.08	9597	1.33	8934	1.58	8914	1.83	9397
1.09	9555	1.34	8922	1.59	8924	1.84	9426
1.10	9514	1.35	8912	1.60	8935	1.85	9456
1.11	9474	1.36	8902	1.61	8947	1.86	9487
1.12	9436	1.37	8893	1.62	8959	1.87	9518
1.13	9399	1.38	8885	1.63	8972	1.88	9551
1.14	9364	1.39	8879	1.64	8986	1.89	9584
1.15	9330	1.40	8873	1.65	9001	1.90	9618
1.16	9298	1.41	8868	1.66	9017	1.91	9652
1.17	9267	1.42	8864	1.67	9033	1.92	9688
1.18	9237	1.43	8860	1.68	9050	1.93	9724
1.19	9209	1.44	8858	1.69	9068	1.94	9761
1.20	9182	1.45	8857	1.70	9086	1.95	9799
1.21	9156	1.46	8856	1.71	9106	1.96	9837
1.22	9131	1.47	8856	1.72	9126	1.97	9877
1.23	9108	1.48	8857	1.73	9147	1.98	9917
1.24	9085	1.49	8859	1.74	9168	1.99	9958
						2.00	1.0000

Table 2

The Values of the Functions

$$\Phi(x) = \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \text{ and } \bar{\Phi}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt$$

x	$\Phi(x)$	$\bar{\Phi}(x)$	x	$\Phi(x)$	$\bar{\Phi}(x)$	x	$\Phi(x)$	$\bar{\Phi}(x)$	x	$\Phi(x)$	$\bar{\Phi}(x)$
0.00	0.0000	0.0000	0.60	0.6039	0.2257	1.20	0.9103	0.3849	1.80	0.9891	0.4641
02	0226	0080	62	6194	2324	22	9155	3888	82	9899	4656
04	0451	0160	64	6346	2389	24	9205	3925	84	9907	4671
06	0676	0239	66	6494	2454	26	9252	3962	86	9915	4686
08	0.901	0319	68	6638	2517	28	9297	3997	88	9922	4699
0.10	1125	0398	0.70	6778	2580	1.30	9340	4032	1.90	9928	4713
12	1348	0478	72	6914	2642	32	9381	4066	92	9934	4726
14	1569	0557	74	7047	2703	34	9419	4099	94	9939	4738
16	1790	0636	76	7175	2764	36	9456	4131	96	9944	4750
18	2009	0714	78	7300	2823	38	9490	4162	98	9949	4761
0.20	2227	0793	0.80	7421	2881	1.40	9523	4192	2.00	9953	4772
22	2443	0871	82	7538	2939	42	9554	4222	05	9963	4798
24	2657	0948	84	7651	2995	44	9583	4251	10	9970	4821
26	2869	1026	86	7761	3051	46	9610	4279	15	9976	4842
28	3079	1103	88	7867	3106	48	9636	4306	20	9981	4860
0.30	3286	1179	0.90	7969	3159	1.50	9661	4332	2.25	9985	4877
32	3491	1255	92	8068	3212	52	9684	4357	30	9988	4892
34	3694	1331	94	8163	3264	54	9706	4382	35	9991	4906
36	3893	1406	96	8254	3315	56	9726	4406	40	9993	4918
38	4090	1480	98	8342	3365	58	9745	4429	45	9995	4928
0.40	4284	1554	1.00	8427	3413	1.60	9763	4452	2.50	9996	4938
42	4475	1628	02	8508	3461	62	9780	4474	60	9998	4953
44	4662	1700	04	8586	3508	64	9796	4495	70	9999	4965
46	4847	1772	06	8661	3554	66	9811	4515	80	9999	4974
48	5027	1844	08	8733	3599	68	9825	4535	2.90	0.9999	4981
0.50	5205	1915	1.10	8802	3643	1.70	9838	4554	3.00	1.0000	4986
52	5379	1985	12	8868	3686	72	9850	4573	20	1.0000	4993
54	5549	2054	14	8931	3729	74	9861	4591	40	1.0000	4996
56	5716	2123	16	8991	3770	76	9872	4608	60	1.0000	4998
0.58	0.5879	0.2190	1.18	0.9048	0.3810	1.78	0.9882	0.4625	3.80	1.0000	0.4999

Table 3

The Value of the Function $z_u = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$

u	0	1	2	3	4	5	6	7	8	9
0.0	0.3989	3989	3989	3988	3986	3984	3982	3980	3977	3973
0.1	3970	3965	3961	3956	3951	3945	3939	3932	3925	3918
0.2	3910	3902	3894	3885	3876	3867	3857	3847	3836	3825
0.3	3814	3802	3790	3778	3765	3752	3739	3726	3712	3697
0.4	3683	3668	3653	3637	3621	3605	3589	3572	3555	3538
0.5	3521	3503	3485	3467	3448	3429	3410	3391	3372	3352
0.6	3332	3312	3292	3271	3251	3230	3209	3187	3166	3144
0.7	3123	3101	3079	3056	3034	3011	2989	2966	2943	2920
0.8	2897	2874	2850	2827	2803	2780	2756	2732	2709	2685
0.9	2661	2637	2613	2589	2565	2541	2516	2492	2468	2444
1.0	2420	2396	2371	2347	2323	2299	2275	2251	2227	2203
1.1	2179	2155	2131	2107	2083	2059	2036	2012	1989	1965
1.2	1942	1919	1895	1872	1849	1826	1804	1781	1758	1736
1.3	1714	1691	1669	1647	1626	1604	1582	1561	1539	1518
1.4	1497	1476	1456	1435	1415	1394	1374	1354	1334	1315
1.5	1295	1276	1257	1238	1219	1200	1182	1163	1145	1127
1.6	1109	1092	1074	1057	1040	1023	1006	0989	0973	0957
1.7	0940	0925	0909	0893	0878	0863	0848	0833	0818	0804
1.8	0790	0775	0761	0748	0734	0721	0707	0694	0681	0669
1.9	0656	0644	0632	0620	0608	0596	0584	0573	0562	0551
2.0	0540	0529	0519	0508	0498	0488	0478	0468	0459	0449
2.1	0440	0431	0422	0413	0404	0395	0387	0379	0371	0363
2.2	0355	0347	0339	0332	0325	0317	0310	0303	0297	0290
2.3	0283	0277	0270	0264	0258	0252	0246	0241	0235	0229
2.4	0224	0219	0213	0208	0203	0198	0194	0189	0184	0180
2.5	0175	0171	0167	0163	0158	0154	0151	0147	0143	0139
2.6	0136	0132	0129	0126	0122	0119	0116	0113	0110	0107
2.7	0104	0101	0099	0096	0093	0091	0088	0086	0084	0081
2.8	0079	0077	0075	0073	0071	0069	0067	0065	0063	0061
2.9	0060	0058	0056	0055	0053	0051	0050	0048	0047	0046
3.0	0.0044	0043	0042	0040	0039	0038	0037	0036	0035	0034

Table 4

The Probability Values for the Criterion χ^2

χ^2 \ r	1	2	3	4	5	6	7	8
1	0.3173	0.6065	0.8013	0.9098	0.9626	0.9856	0.9948	0.9982
2	1574	3679	5724	7358	8491	9197	9598	9810
3	0833	2231	3916	5578	7000	8088	8850	9344
4	0455	1353	2615	4060	5494	6767	7798	8571
5	0254	0821	1718	2873	4159	5438	6600	7576
6	0143	0498	1116	1991	3062	4232	5398	6472
7	0081	0302	0719	1359	2206	3208	4289	5366
8	0047	0183	0460	0916	1562	2381	3326	4335
9	0027	0111	0293	0611	1091	1736	2527	3423
10	0016	0067	0186	0404	0752	1247	1886	2650
11	0009	0041	0117	0266	0514	0884	1386	2017
12	0005	0025	0074	0174	0348	0620	1006	1512
13	0003	0015	0046	0113	0234	0430	0721	1119
14	0002	0009	0029	0073	0156	0296	0512	0818
15	0001	0006	0018	0047	0104	0203	0360	0591
16	0001	0003	0011	0030	0068	0138	0251	0424
17	0000	0002	0007	0019	0045	0093	0174	0301
18		0001	0004	0012	0029	0062	0120	0212
19		0001	0003	0008	0019	0042	0082	0149
20		0000	0002	0005	0013	0028	0056	0103
21			0001	0003	0008	0018	0038	0071
22			0001	0002	0005	0012	0025	0049
23			0000	0001	0003	0008	0017	0034
24				0001	0002	0005	0011	0023
25				0001	0001	0003	0008	0016
26				0000	0001	0002	0005	0010
27					0001	0001	0003	0007
28					0000	0001	0002	0005
29						0001	0001	0003
30						0000	0001	0002

Part 1 covers the folloving themes: analytic geometry on a plane and in space (with elements of vector algebra); fundamentals of linear algebra; introduction to analysis; differential calculus of functions of one or several variables; indefinite and definite integrals; and elements of linear programming.
Each section begins with a brief theoretical introduction. Typical problems are followed by detailed solutions.